On Kripke completeness of modal and superintuitionistic predicate logics with equality

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We consider first-order normal modal and superintuitionistic predicate logics in a signature with only predicate letters and perhaps with equality. A logic is defined in a standard way, as a certain set of formulas, cf. [2], sec. 2.6.

Every logic L without equality has the minimal extension $L^=$ with equality ([2], sec. 2.14.). It is well-known that completeness of L in the standard Kripke semantics does not imply the completeness of $L^=$. So there is a natural question — how to axiomatize the logic with equality characterized by Kripke frames for L. As we show, quite often (but not always) this is done by the extensions $L^{=d} := L^= + DE$ in the intuitionistic case and $L^{=c} := L^= + CE$ in the modal case, where

$$DE := \forall x \forall y (x = y \vee \neg(x = y)) \text{ (the axiom of decidable equality),}$$
$$CE := \forall x \forall y (\Diamond(x = y) \supset x = y) \text{ (the axiom of closed equality).}$$

Here we deal with two kinds of semantics: the semantics of predicate Kripke frames ($\mathcal{K}$) and the semantics of Kripke frames with equality ($\mathcal{KE}$) (equivalent to the semantics of Kripke sheaves); cf. [2], sections 3.2, 3.5, 3.6. Recall that a *predicate Kripke frame* (PKF) over a propositional Kripke frame $F = (W, R)$ is a pair (F, D), where $D = (D_u)_{u \in W}$ is a family of non-empty expanding domains (uRv implies $D_u \subseteq D_v$). A *predicate Kripke frame with equality* (KFE) is a triple $(F, D, \asymp)$, where (F, D) is a PKF and $\asymp = (\asymp_u)_{u \in W}$ is a family of expanding equivalence relations $\asymp_u \subseteq D_u \times D_u$ (uRv implies $\asymp_u \subseteq \asymp_v$). The notions of validity in these semantics are standard. The set of formulas valid in a PKF or a KFE $\mathbf{F}$ is called the *logic of* $\mathbf{F}$ (modal or superintuitionistic) and denoted by $\mathbf{ML}(\mathbf{F})$ or $\mathbf{IL}(\mathbf{F})$, or by $\mathbf{ML}^=(\mathbf{F})$ or $\mathbf{IL}^=(\mathbf{F})$ for logics with equality.

The logics of a class of frames $\mathcal{C}$ are $\mathbf{ML}^{(=)}(\mathcal{C}) := \bigcap\{\mathbf{ML}^{(=)}(\mathbf{F}) \mid \mathbf{F} \in \mathcal{C}\}$, $\mathbf{IL}^{(=)}(\mathcal{C}) := \bigcap\{\mathbf{IL}^{(=)}(\mathbf{F}) \mid \mathbf{F} \in \mathcal{C}\}$; these logics are called *Kripke ($\mathcal{K}$-) complete* if $\mathcal{C}$ is a class of PKFs, *Kripke sheaf ($\mathcal{KE}$-) complete* if $\mathcal{C}$ is a class of KFEs.

Note that a KFE $(W, R, D, \asymp)$ validates CE iff its reflexive transitive closure $(W, R^*, D, \asymp)$ validates DE iff

$$\forall u, v \in W \, \forall a, b \in D_u \, (uR^*v \,\&\, a \asymp_v b \Rightarrow a \asymp_u b).$$

So CE and DE are obviously valid in every PKF, since a PKF can be regarded as a KFE, in which $\asymp_u$ are the identity relations.

Usually $\mathcal{KE}$-completeness transfers from L to $L^=$ and $L^{=d}$ (or $L^{=c}$); cf. [2], theorems 3.8.3, 3.8.4, 3.8.7, 3.8.8 for the details.

Proposition 1. (1) Suppose $\mathbf{F} \models CE$ is a KFE over a propositional frame F, F^* is the reflexive transitive closure of F and one of the following conditions holds: (i) F^* is an $\mathbf{S4}$-tree; (ii) F^* is directed; (iii) $\mathbf{F}$ has a constant domain.

Then there exists a PKF $\mathbf{F}'$ such that $\mathbf{ML}^=(\mathbf{F}') = \mathbf{ML}^=(\mathbf{F})$.

(2) The same holds for the intuitionistic case and $\mathbf{F} \Vdash DE$.

Hence we obtain

Theorem 1. (1) Suppose L is a $\mathcal{K}$-complete modal predicate logic of one of the following types: (i) L is complete w.r.t. frames over trees; (ii) $L \vdash \Diamond\Box p \supset \Box\Diamond p$; (iii) $L \vdash \forall x \Box P(x) \supset \Box \forall x P(x)$ (the Barcan formula). Then $L^{=c}$ is $\mathcal{K}$-complete.

(2) Suppose L is a $\mathcal{K}$-complete superintuitionistic predicate logic of one of the following types: (i) L is complete w.r.t. frames over trees; (ii) $L \vdash J$ ($= \neg p \vee \neg\neg p$); (iii) $L \vdash CD$ ($= \forall x(P(x) \vee q) \supset \forall x P(x) \vee q$). Then $L^{=d}$ is $\mathcal{K}$-complete.

Remark. Recall that $L = \mathbf{QH} + CD + J$ is Kripke incomplete [1]. We do not know if $L^{=d}$ is Kripke complete in this case.

However, not every KFE validating DE is equivalent to a PKF. This allows us to construct Kripke complete logics L, for which $L^{=d}$ is Kripke incomplete.

Consider the weak De Morgan law

$$J_2 := \neg(p_0 \wedge p_1 \wedge p_2) \supset \neg(p_0 \wedge p_1) \vee \neg(p_0 \wedge p_2) \vee \neg(p_1 \wedge p_2),$$

and the frame $F_0 := (W_0, \leq)$, with $W_0 := \{u_0\} \cup \{u_{ij} \mid i, j \in \{1, 2\}\}$, which is a poset with the root u_0 and $(u_{ij} < u_{i'j'})$ iff $(i < i')$. Then F_0 validates J_2, but not J. $\mathbf{IL}(\mathcal{K}F_0)$ denotes the superintuitionistic logic of all PKFs over F_0 (which coincides with the logic of all KFEs over F_0).

Theorem 2. Let L be a predicate logic such that $\mathbf{QH} + J_2 \subseteq L \subseteq \mathbf{IL}(\mathcal{K}F_0)$. Then the logic $L^{=d}$ is Kripke incomplete.

We do not know if the segment mentioned in Theorem 2 contains finitely axiomatizable Kripke complete logics.

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