On Kripke completeness of modal and superintuitionistic predicate logics with equality

Valentin Shehtman\(^1\) and Dmitry Skvortsov\(^2\)

\(^1\) National Research University Higher School of Economics, Moscow, Russia
vshehtman@gmail.com
\(^2\) Federal Research Center for Computer Science and Control (FRCCSC); All-Russian Institute of Scientific and Technical Information, VINITI
skvortsovdd@yandex.ru

We consider first-order normal modal and superintuitionistic predicate logics in a signature with only predicate letters and perhaps with equality. A logic is defined in a standard way, as a certain set of formulas, cf. [2], sec. 2.6.

Every logic \(L\) without equality has the minimal extension \(L^=\) with equality ([2], sec. 2.14.). It is well-known that completeness of \(L\) in the standard Kripke semantics does not imply the completeness of \(L^=\). So there is a natural question — how to axiomatize the logic with equality characterized by Kripke frames for \(L\). As we show, quite often (but not always) this is done by the extensions \(L^{=d} := L^= + DE\) in the intuitionistic case and \(L^{=c} := L^= + CE\) in the modal case, where

\[
DE := \forall x \forall y (x = y \lor \neg(x = y)) \text{ (the axiom of decidable equality)},
\]
\[
CE := \forall x \forall y (\neg(x = y) \lor x = y) \text{ (the axiom of closed equality)}.
\]

Here we deal with two kinds of semantics: the semantics of predicate Kripke frames (\(K\)) and the semantics of Kripke frames with equality (\(KE\)) (equivalent to the semantics of Kripke sheaves); cf. [2], sections 3.2, 3.5, 3.6. Recall that a predicate Kripke frame (PKF) over a propositional Kripke frame \(F = (W, R)\) is a pair \((F, D)\), where \(D = \{D_u\}_{u \in W}\) is a family of non-empty expanding domains \((uRv \implies D_u \subseteq D_v)\). A predicate Kripke frame with equality (KFE) is a triple \((F, D, =)\), where \((F, D)\) is a PKF and \(= = (\leq_u)_{u \in W}\) is a family of expanding equivalence relations \(\leq_u \subseteq D_u \times D_u (uRv \implies \leq_u \subseteq \leq_v)\). The notions of validity in these semantics are standard. The set of formulas valid in a PKF or a KFE \(F\) is called the logic of \(F\) (modal or superintuitionistic) and denoted by \(ML(F)\) or \(IL(F)\), or by \(ML^=(F)\) or \(IL^=(F)\) for logics with equality.

The logics of a class of frames \(\mathcal{C}\) are \(ML^=(\mathcal{C}) := \bigcap\{ML^=(F) \mid F \in \mathcal{C}\}\), \(IL^=(\mathcal{C}) := \bigcap\{IL^=(F) \mid F \in \mathcal{C}\}\); these logics are called Kripke (\(K\)-) complete if \(\mathcal{C}\) is a class of PKFs, Kripke sheaf (\(KE\)-) complete if \(\mathcal{C}\) is a class of KFEs.

Note that a KFE \((W, R, D, =)\) validates \(CE\) iff its reflexive transitive closure \((W, R^*, D, =)\) validates \(DE\) iff

\[
\forall u, v \in W \forall a, b \in D_u (uR^*v \land a \succeq_v b \implies a \succeq_u b).
\]

So \(CE\) and \(DE\) are obviously valid in every PKF, since a PKF can be regarded as a KFE, in which \(\succeq_u\) are the identity relations.

Usually \(KE\)-completeness transfers from \(L\) to \(L^=\) and \(L^{=d}\) (or \(L^{=c}\)); cf. [2], theorems 3.8.3, 3.8.4, 3.8.7, 3.8.8 for the details.

**Proposition 1.** (1) Suppose \(F \models CE\) is a KFE over a propositional frame \(F, F^*\) is the reflexive transitive closure of \(F\) and one of the following conditions holds: (i) \(F^*\) is an \(S4\)-tree; (ii) \(F^*\) is directed; (iii) \(F\) has a constant domain.

Then there exists a PKF \(F'\) such that \(ML^=(F') = ML^=(F)\).
(2) The same holds for the intuitionistic case and $F \vDash DE$.

Hence we obtain

**Theorem 1.** (1) Suppose $L$ is a $K$-complete modal predicate logic of one of the following types:
(i) $L$ is complete w.r.t. frames over trees; (ii) $L \vdash \forall x \Box P(x) \supset \Box \forall x P(x)$ (the Barcan formula). Then $L^{ac}$ is $K$-complete.

(2) Suppose $L$ is a $K$-complete superintuitionistic predicate logic of one of the following types:
(i) $L$ is complete w.r.t. frames over trees; (ii) $L \vdash J (= \neg p \lor \neg \neg p)$; (iii) $L \vdash CD (= \forall x (P(x) \lor q) \supset \forall x P(x) \lor q)$. Then $L^{sd}$ is $K$-complete.

**Remark.** Recall that $L = QH + CD + J$ is Kripke incomplete [1]. We do not know if $L^{sd}$ is Kripke complete in this case.

However, not every KFE validating $DE$ is equivalent to a PKF. This allows us to construct Kripke complete logics $L$, for which $L^{sd}$ is Kripke incomplete.

Consider the weak De Morgan law

$$J_2 := \neg(p_0 \land p_1 \land p_2) \supset \neg(p_0 \land p_1) \lor \neg(p_0 \land p_2) \lor \neg(p_1 \land p_2),$$

and the frame $F_0 := (W_0, \leq)$, with $W_0 := \{u_0\} \cup \{u_{ij} \mid i, j \in \{1, 2\}\}$, which is a poset with the root $u_0$ and $(u_{ij} < u_{i'j'})$ iff $(i < i')$. Then $F_0$ validates $J_2$, but not $J$. $IL(\mathcal{KF}_0)$ denotes the superintuitionistic logic of all PKFs over $F_0$ (which coincides with the logic of all KFEs over $F_0$).

**Theorem 2.** Let $L$ be a predicate logic such that $QH + J_2 \subseteq L \subseteq IL(\mathcal{KF}_0)$. Then the logic $L^{sd}$ is Kripke incomplete.

We do not know if the segment mentioned in Theorem 2 contains finitely axiomatizable Kripke complete logics.

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**References**
