Neighborhood-Kripke product of modal logics *

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There are many ways to combine modal logics. Some of them are syntactical and some semantical. The simplest syntactical combination is the fusion. The fusion of two unimodal logics L_1 and L_2 is the minimal bimodal logic containing axioms from L_1 rewritten with $\Box_1$ and axioms from L_2 rewritten with $\Box_2$. Notation: $\mathsf{L}_1 * \mathsf{L}_2$.

The product of two modal logics is a semantical way to combine logics. The product of two modal logics is the logic of the class of all products of semantical structures of the corresponding logics. Such construction based on the product of Kripke frames was introduced by Shehtman in 1978 [10]. Later in 2006 van Benthem et al. [1] introduced a similar construction based on product of topological spaces[1].

Neighborhood semantics is a generalization of the Kripke semantics and the topological semantics. It was introduced independently by Dana Scott [9] and Richard Montague [7]. The product of neighborhood frames was introduced by Sano in [8]. The product of topological spaces from [1] is a particular case of the product of the neighborhood product for $\mathsf{S4}$-frames. Several paper was studying neighborhood products [5, 6].

A recent paper by Kremer proposed a mixed space-frame product and proved a general completeness result for $\mathsf{S4}$ and Horn axiomatized extensions of logic D.

In this work we generalize Kremer's results to neighborhood-Kripke frames product.

A *(normal) neighborhood frame* (or an n-frame) is a pair $\mathfrak{X} = (X, \tau)$, where X is a nonempty set and $\tau : X \to 2^{2^X}$ such that $\tau(x)$ is a filter on X for any x. The function τ is called the *neighborhood function* of $\mathfrak{X}$, and sets from $\tau(x)$ are called *neighborhoods of* x.

A *Kripke frame* is a pair tuple (X, R), where X is a non-empty set and $R \subseteq X \times X$ is a relation on X.

A *valuation* on a Kripke (n-) frame is a function $V : PV \to 2^X$. For a Kripke (n-) frame $\mathfrak{X}$ and a valuation V pair $(\mathfrak{X}, V)$ is called a Kripke (neighborhood) model.

The truth for models define in the usual way see [2] and [3].

A *neighborhood-Krike frame* is a triple (X, τ, R) such that (X, τ) is a n-frame and (X, R) is a Kripke frame. The notion of truth uses neighborhood structure for $\Box_1$ and Kripke structure for $\Box_2$. For n-frame $\mathfrak{X}_1 = (X_1, \tau_1)$ and Kripke frame $F_2 = (X_2, R_2)$ the product of them is a neighborhood-Krike frame $\mathfrak{X}_1 \times F_2 = (X_1 \times X_2, \tau_1', R_2')$ such that for $(x, y) \in X_1 \times X_2$

$$\begin{aligned} &U \in \tau_1'(x, y) \iff \exists V \in \tau_1(x)\,(V \times \{y\} \subseteq U) \\ &R_2'(x, y) = \{(x, y') \mid yR_2y'\} \end{aligned}$$

A logic of a frame or a class of frames is all the formulas that are true at all points in all models of these frames.

A logic L is called an *PTC-logic* if it can be axiomatized by closed formulas and formulas of the type $\Diamond^m \Box p \to \Box^n p$, $n, m \geq 0$. (see [4]).

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[1] "Product of topological spaces" is a well-known notion in Topology but it is different from what we use here (for details see [1])

A logic L is called an *HTC-logic* (from Horn preTransitive Closed logic) if it can be axiomatized by closed formulas and formulas of the type $\Box p \to \Box^n p$, $n \geq 0$. These formulas correspond to universal strict Horn sentences (see [4]).

For two normal modal logics L_1 and L_2 the nk-product of them is the logic of all products of n-frames of logic L_1 and Kripke frames of logic L_2. Notation: $\mathsf{L}_1 \times_{nk} \mathsf{L}_2$.

Our main result is the following

Theorem 1. *For any HTC logic L_1 and PTC logic L_2*

$$\begin{aligned} \mathsf{L}_1 \times_{nk} \mathsf{L}_2 &= \mathsf{L}_1 * \mathsf{L}_2 + com_{12} + chr + \Delta_1 + \Delta_2, \text{ where} \\ com_{12} &= \Box_1 \Box_2 p \to \Box_2 \Box_1 p, \\ chr &= \Diamond_1 \Box_2 p \to \Box_2 \Diamond_1 p, \\ \Delta_1 &= \{\phi \to \Box_2 \phi \mid \phi \text{ is closed and } \Box_2\text{-free}\}, \\ \Delta_2 &= \{\psi \to \Box_1 \psi \mid \psi \text{ is closed and } \Box_1\text{-free}\}. \end{aligned}$$

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