

The Frame of the p -Adic Numbers and a p -Adic Version of the Stone-Weierstrass Theorem in Pointfree Topology

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The connection between topology and lattice theory began to be exploited after the work of Marshall Stone. The fact that the lattice of open sets of a topological space contains plenty of information about the topological space indicates that a complete lattice, satisfying the distributive law $a \wedge \bigvee S = \{a \wedge s \mid s \in S\}$, deserves to be studied as a “generalized topological space”. In this sense, frames (locales) generalize the notion of topological spaces and frame homomorphisms (localic maps) generalize the notion of continuous functions; that is, pointfree topology is an abstract lattice approach to topology. The algebraic nature of a frame allows its definition by generators and relations. Joyal [5] used this to introduced the frame of the real numbers; the idea is to take the set of open intervals with rational endpoints for the basic generators. Later, Banaschewski [1] studied this frame with a particular emphasis on the pointfree extension of the ring of continuous real functions and provided a pointfree version of the Stone-Weierstrass Theorem. We are interested in the field of the p -adic numbers \mathbb{Q}_p and the ring of continuous p -adic functions. \mathbb{Q}_p is the completion of \mathbb{Q} with respect to the p -adic absolute value $|\cdot|_p$, which satisfies $|x + y|_p \leq \max\{|x|_p, |y|_p\}$ (i.e., it is nonarchimedean, see [3] and [4]). In particular, \mathbb{Q}_p is 0-dimensional, completely regular, and locally compact. In [2], Dieudonné proved that the ring $\mathbb{Q}_p[X]$ of polynomials with coefficients in \mathbb{Q}_p is dense in the ring $\mathcal{C}(F, \mathbb{Q}_p)$ of continuous functions defined on a compact subset F of \mathbb{Q}_p with values in \mathbb{Q}_p , and Kaplansky [6] extended this result by proving that if \mathbb{F} is a nonarchimedean valued field and X is a compact Hausdorff space, then any unitary subalgebra \mathcal{A} of $\mathcal{C}(X, \mathbb{F})$ which separates points is uniformly dense in $\mathcal{C}(X, \mathbb{F})$. We define the frame of \mathbb{Q}_p and we give a p -adic version of the Stone-Weierstrass theorem in pointfree topology.

To specify the frame of \mathbb{Q}_p by generators and relations, we consider the fact that the open balls centered at rational numbers generate the open subsets of \mathbb{Q}_p and thus we think of them as the basic generators; we consider the (lattice) properties of these balls to determine the relations these elements must satisfy. Let $\mathcal{L}(\mathbb{Q}_p)$ be the frame generated by the elements $B_r(a)$, where $a \in \mathbb{Q}$ and $r \in |\mathbb{Q}| := \{p^{-n}, n \in \mathbb{Z}\}$, subject to the following relations:

$$(Q1) \ B_s(b) \leq B_r(a) \text{ whenever } |a - b|_p < r \text{ and } s \leq r,$$

$$(Q2) \ B_r(a) \wedge B_s(b) = 0 \text{ whenever } |a - b|_p \geq r \vee s,$$

$$(Q3) \ 1 = \bigvee \{B_r(a) : a \in \mathbb{Q}, r \in |\mathbb{Q}|\},$$

$$(Q4) \ B_r(a) = \bigvee \{B_s(b) : |a - b|_p < r, s < r, b \in \mathbb{Q}, s \in |\mathbb{Q}|\}.$$

We prove that $\mathcal{L}(\mathbb{Q}_p)$ is the pointfree counterpart of \mathbb{Q}_p ; that is, $\mathcal{L}(\mathbb{Q}_p)$ is a spatial frame whose space of points is homeomorphic to \mathbb{Q}_p . In particular, we show with pointfree arguments that $\mathcal{L}(\mathbb{Q}_p)$ is 0-dimensional, completely regular, and continuous.

As in the real case, from the well-known adjunction between frames and topological spaces (see, e.g., [7]), we have a natural isomorphism $\mathbf{Top}(X, \mathbb{Q}_p) \cong \mathbf{Frm}(\mathcal{L}(\mathbb{Q}_p), \Omega(X))$, for a topological space X . This provides a natural extension of the classical notion of a continuous p -adic

function: a *continuous p -adic function* on a frame L is a frame homomorphism $\mathcal{L}(\mathbb{Q}_p) \rightarrow L$. We denote the set of all continuous p -adic functions on a frame L with $\mathcal{C}_p(L)$, and we show that it is a \mathbb{Q}_p -algebra under the following operations:

$$\begin{aligned} (f + g)(B_r(a)) &= \bigvee \left\{ f(B_{s_1}(b_1)) \wedge g(B_{s_2}(b_2)) \mid B_{s_1 \vee s_2}(b_1 + b_2) \subseteq B_r(a) \right\} \\ (f \cdot g)(B_r(a)) &= \bigvee \left\{ f(B_{s_1}(b_1)) \wedge g(B_{s_2}(b_2)) \mid B_t(b_1 \cdot b_2) \subseteq B_r(a) \right\}, \end{aligned}$$

where $t = \max \{p^{-1}rs, s|a|_p, r|b|_p\}$.

If X is compact Hausdorff and $f \in \mathcal{C}(X, \mathbb{Q}_p)$, then $\|f\| = \sup\{|f(x)|_p\}$ defines a nonarchimedean norm on $\mathcal{C}(X, \mathbb{Q}_p)$. In our case, we show that if L is a compact regular frame, then $\|h\| = \inf \{p^{-n} : n \in \mathbb{Z}, h(B_{p^{-n+1}}(0)) = 1\}$ defines a nonarchimedean norm on $\mathcal{C}_p(L)$. Recall that if X is compact Hausdorff then X is 0-dimensional iff $\mathcal{C}(X, \mathbb{Q}_p)$ separates points, thus we assume that X is 0-dimensional; in the pointfree context, we assume that L is a compact 0-dimensional frame. Additionally, if X is compact Hausdorff and 0-dimensional, then each $f \in \mathcal{C}(X, \mathbb{Q}_p)$ can be approximated by a linear combination of \mathbb{Q}_p -characteristic functions of clopen subsets. Thus, if \mathcal{A} is a unitary subalgebra of $\mathcal{C}(X, \mathbb{Q}_p)$ such that its closure contains these \mathbb{Q}_p -characteristic functions, then \mathcal{A} is dense in $\mathcal{C}(X, \mathbb{Q}_p)$. It can be shown (see [6]) that this is the case whenever \mathcal{A} separates points. Therefore, we extend the notion of a \mathbb{Q}_p -characteristic function of a clopen subset, showing that if u is a complemented element (with complement u') in L , then the function $\chi_u : \mathcal{L}(\mathbb{Q}_p) \rightarrow L$ defined on generators by

$$\chi_u(B_r(a)) = \begin{cases} 1 & \text{if } |a|_p < r \text{ and } |1 - a|_p < r, \\ u & \text{if } |a|_p \geq r \text{ and } |1 - a|_p < r, \\ u' & \text{if } |a|_p < r \text{ and } |1 - a|_p \geq r, \\ 0 & \text{otherwise,} \end{cases}$$

is a frame homomorphism. We show that these elements are precisely the idempotents in $\mathcal{C}_p(L)$ and we extend the notion of a subalgebra in $\mathcal{C}(X, \mathbb{Q}_p)$ that separates points to the pointfree context as follows: Given a compact 0-dimensional frame L , we say that a unitary subalgebra \mathcal{A} of $\mathcal{C}_p(L)$ *separates points* if \mathcal{A} contains the idempotents of $\mathcal{C}_p(L)$.

Finally, we provide the following pointfree version of the Stone-Weierstrass Theorem: Let L be a compact 0-dimensional (regular) frame and let \mathcal{A} be a unitary subalgebra of $\mathcal{C}_p(L)$ which separates points, then \mathcal{A} is uniformly dense in $\mathcal{C}_p(L)$.

References

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