First-order cologic for profinite structures

Alex Kruckman

Indiana University
Bloomington, IN, USA
akruckma@indiana.edu

1 Logic for LFP Categories

The domain of a first-order structure M is a (typically infinite) set. First-order logic provides a finitary syntax for describing properties of M by way of how M is constructed from finite pieces, i.e. as the directed colimit of all finite subsets of M. Explicitly, a first-order formula describes a property of a tuple of elements of M, and quantifiers allow us to explore how this tuple can be expanded to larger finite tuples. This perspective on the expressive power of first-order logic is elegantly captured by the Ehrenfeucht–Fraïssé game.

A locally finitely presentable (LFP) category is one in which every object is a directed colimit of objects which are finitary in a precise sense. In direct analogy with ordinary first-order logic for Set, we develop a logic for describing properties of an object M in an LFP category (possibly expanded by extra "finitary" structure) by way of how M is constructed from finitary pieces.

To be more precise, an object x in a category $\mathcal{D}$ is called finitely presentable if the functor $\mathrm{Hom}_{\mathcal{D}}(x, -)$ preserves directed colimits. The category $\mathcal{D}$ is called locally finitely presentable if it is cocomplete, every object is a directed colimit of finitely presentable objects, and the full subcategory $\mathcal{C}$ of finitely presentable objects is essentially small. We call $\mathcal{D}$ the category of *domains* and $\mathcal{C}$ the category of *variable contexts*, and we fix a set $\mathcal{A}$ of isomorphism representatives for the objects of $\mathcal{C}$, called *arities*.

Then a *signature* $\mathcal{L}$ consists of a set of relation symbols with associated arities from $\mathcal{A}$, together with a finitary endofunctor $F\colon \mathcal{D} \to \mathcal{D}$, and an *$\mathcal{L}$-structure* is an object M in $\mathcal{D}$, given with an F-algebra structure $\eta\colon F(M) \to M$, and interpretations of the relation symbols: given an arity $n \in \mathcal{A}$ and an object $M \in \mathcal{D}$, an n-tuple from M is just an arrow $n \to M$, and an n-ary relation is a subset of $\mathrm{Hom}(n, M)$.

We can now describe the logic $\mathrm{FO}(\mathcal{D}, \mathcal{L})$: For an arity n and a variable context x, an n-term in x is a map $n \to T(x)$, the *term algebra* (i.e. free F-algebra) on x. An atomic formula is an equality between two n-terms or an n-ary relation symbol applied to an n-term. General formulas are built from atomic formulas by ordinary Boolean combinations and by quantifiers: for each arrow $f\colon x \to y$ between contexts, we associate a universal and existential quantifier $\exists_f$ and $\forall_f$ which quantify over extensions of x-tuples to y-tuples, respecting f. Of course there is a completely natural semantics for evaluation of terms and satisfaction of formulas in L-structures.

2 The first-order translation

To each M in $\mathcal{D}$, we associate the finite-limit preserving presheaf $\mathrm{Hom}_{\mathcal{D}}(-, M)\colon \mathcal{A}^{\mathrm{op}} \to \mathsf{Set}$. In fact, by Gabriel-Ulmer duality (see [1]), $\mathcal{D}$ is equivalent to the category $\mathrm{Lex}(\mathcal{A}^{\mathrm{op}}, \mathsf{Set})$ of finite-limit preserving presheaves on $\mathcal{A}$. Such presheaves can be viewed as models for a certain (ordinary) first-order theory, in a language with a sort for each object in $\mathcal{A}$. Extending this equivalence from objects of $\mathcal{D}$ to $\mathcal{L}$-structures, we obtain the following theorem.

Theorem 1. *For every LFP category $\mathcal{D}$ and signature $\mathcal{L}$, there is an ordinary multi-sorted first-order signature* $\mathrm{PS}(\mathcal{D}, \mathcal{L})$ *and a theory* T_{PS} *in this signature, so that the category of $\mathcal{L}$-structures is equivalent to the category of models of* T_{PS}*. Further, there is an explicit satisfaction-preserving translation from formulas in* $\mathrm{FO}(\mathcal{D}, \mathcal{L})$ *to first-order* $\mathrm{PS}(\mathcal{D}, \mathcal{L})$*-formulas.*

This interpretation of $\mathrm{FO}(\mathcal{D}, \mathcal{L})$ in ordinary first-order logic allows us to easily import theorems and notions (compactness, Löwenheim-Skolem, interpretability, stability, etc.) from first-order model theory.

3 Cologic

Whenever $\mathcal{B}$ is a category with finite limits, the category $\mathsf{pro}{-}\mathcal{B}$ (the formal completion of $\mathcal{B}$ under codirected limits) is co-LFP, i.e. $(\mathsf{pro}{-}\mathcal{B})^{\mathrm{op}}$ is LFP. Then the logic $\mathrm{FO}((\mathsf{pro}{-}\mathcal{B})^{\mathrm{op}}, \mathcal{L})$ expresses properties of "cotuples" from an object M, i.e. maps $M \to x$, where $x \in \mathcal{B}$. For example, a cotuple from a Stone space S (an object of $\mathsf{Stone} = \mathsf{pro}{-}\mathsf{FinSet}$) is a continuous map from S to a finite discrete space, or equivalently a partition of M into clopen sets. And a cotuple from a profinite group G (an object of $\mathsf{pro}{-}\mathsf{FinGrp}$) is a group homomorphism from G to a finite group.

These logics provide a unified framework for the model theory of profinite structures, with connections to several independent bodies of work. I will mention a few:

1. Projective (or Dual) Fraïssé theory, as developed by Irwin and Solecki [3] and recently reformulated in terms of corelations by Panagiotopoulos [5]. The dual ultrahomogeneity exhibited by projective Fraïssé limits can be expressed by $\forall\exists$ sentences in the logic $\mathrm{FO}(\mathsf{Stone}, \mathcal{L})$.

2. The "cologic" of profinite groups (e.g. Galois groups), developed by Cherlin, van den Dries, and Macintryre [2] and by Chatzidakis, which plays an important role in the model theory of PAC fields. This logic is presented in a multi-sorted first-order framework, which is essentially equivalent to the first-order translation of Theorem 1, applied to $\mathrm{FO}(\mathsf{pro}{-}\mathsf{FinGrp}, \emptyset)$.

3. The theory of coalgebraic logic, in the special case of cofinitary functors on Stone spaces (see, e.g. [4]), is exactly the theory of equationally defined classes in $\mathrm{FO}(\mathsf{Stone}, \mathcal{L})$, since $\mathcal{L}$-structures are coalgebras for cofinitary functors. This theory has connections to modal logic; for example, when the functor F is the Vietoris functor, $\mathrm{FO}(\mathsf{Stone}, \mathcal{L})$ embeds modal logics on descriptive general frames.

References

[1] J. Adamek and J. Rosicky. *Locally Presentable and Accessible Categories:*. London Mathematical Society Lecture Note Series. Cambridge University Press, 1994.

[2] Gregory Chernin, Lou van den Dries, and Angus Macintyre. The elementary theory of regularly closed fields, 1981.

[3] Trevor Irwin and Sławomir Solecki. Projective Fraïssé limits and the pseudo-arc. *Trans. Amer. Math. Soc.*, 358(7):3077–3096, 2006.

[4] Clemens Kupke, Alexander Kurz, and Yde Venema. Stone coalgebras. *Theoret. Comput. Sci.*, 327(1-2):109–134, 2004.

[5] Aristotelis Panagiotopoulos. Compact spaces as quotients of projective fraisse limits, 2016.