First-order cologic for profinite structures

Alex Kruckman
Indiana University
Bloomington, IN, USA
akruckma@indiana.edu

1 Logic for LFP Categories

The domain of a first-order structure \( M \) is a (typically infinite) set. First-order logic provides a finitary syntax for describing properties of \( M \) by way of how \( M \) is constructed from finite pieces, i.e. as the directed colimit of all finite subsets of \( M \). Explicitly, a first-order formula describes a property of a tuple of elements of \( M \), and quantifiers allow us to explore how this tuple can be expanded to larger finite tuples. This perspective on the expressive power of first-order logic is elegantly captured by the Ehrenfeucht–Fraïssé game.

A locally finitely presentable (LFP) category is one in which every object is a directed colimit of objects which are finitary in a precise sense. In direct analogy with ordinary first-order logic for \( \text{Set} \), we develop a logic for describing properties of an object \( M \) in an LFP category (possibly expanded by extra “finitary” structure) by way of how \( M \) is constructed from finitary pieces.

To be more precise, an object \( x \) in a category \( D \) is called finitely presentable if the functor \( \text{Hom}_D(x, -) \) preserves directed colimits. The category \( D \) is called locally finitely presentable if it is cocomplete, every object is a directed colimit of finitely presentable objects, and the full subcategory \( C \) of finitely presentable objects is essentially small. We call \( D \) the category of domains and \( C \) the category of variable contexts, and we fix a set \( A \) of isomorphism representatives for the objects of \( C \), called arities.

Then a signature \( L \) consists of a set of relation symbols with associated arities from \( A \), together with a finitary endofunctor \( F : D \to D \), and an \( L \)-structure is an object \( M \) in \( D \), given with an \( F \)-algebra structure \( \eta : F(M) \to M \), and interpretations of the relation symbols: given an arity \( n \in A \) and an object \( M \in D \), an \( n \)-tuple from \( M \) is just an arrow \( n \to M \), and an \( n \)-ary relation is a subset of \( \text{Hom}(n, M) \).

We can now describe the logic \( \text{FO}(D, L) \): For an arity \( n \) and a variable context \( x \), an \( n \)-term in \( x \) is a map \( n \to T(x) \), the term algebra (i.e. free \( F \)-algebra) on \( x \). An atomic formula is an equality between two \( n \)-terms or an \( n \)-ary relation symbol applied to an \( n \)-term. General formulas are built from atomic formulas by ordinary Boolean combinations and by quantifiers: for each arrow \( f : x \to y \) between contexts, we associate a universal and existential quantifier \( \exists_f \) and \( \forall_f \) which quantify over extensions of \( x \)-tuples to \( y \)-tuples, respecting \( f \). Of course there is a completely natural semantics for evaluation of terms and satisfaction of formulas in \( L \)-structures.

2 The first-order translation

To each \( M \) in \( D \), we associate the finite-limit preserving presheaf \( \text{Hom}_D(-, M) : A^{\text{op}} \to \text{Set} \). In fact, by Gabriel-Ulmer duality (see [1]), \( D \) is equivalent to the category \( \text{Lex}(A^{\text{op}}, \text{Set}) \) of finite-limit preserving presheaves on \( A \). Such presheaves can be viewed as models for a certain (ordinary) first-order theory, in a language with a sort for each object in \( A \). Extending this equivalence from objects of \( D \) to \( L \)-structures, we obtain the following theorem.
Theorem 1. For every LFP category $D$ and signature $L$, there is an ordinary multi-sorted first-order signature $PS(D, L)$ and a theory $T_{PS}$ in this signature, so that the category of $L$-structures is equivalent to the category of models of $T_{PS}$. Further, there is an explicit satisfaction-preserving translation from formulas in $FO(D, L)$ to first-order $PS(D, L)$-formulas.

This interpretation of $FO(D, L)$ in ordinary first-order logic allows us to easily import theorems and notions (compactness, Löwenheim-Skolem, interpretability, stability, etc.) from first-order model theory.

3 Cologic

Whenever $B$ is a category with finite limits, the category $pro-B$ (the formal completion of $B$ under codirected limits) is co-LFP, i.e. $(pro-B)^{op}$ is LFP. Then the logic $FO((pro-B)^{op}, L)$ expresses properties of “cotuples” from an object $M$, i.e. maps $M \to x$, where $x \in B$. For example, a cotuple from a Stone space $S$ (an object of $Stone = pro-FinSet$) is a continuous map from $S$ to a finite discrete space, or equivalently a partition of $M$ into clopen sets. And a cotuple from a profinite group $G$ (an object of $pro-FinGrp$) is a group homomorphism from $G$ to a finite group.

These logics provide a unified framework for the model theory of profinite structures, with connections to several independent bodies of work. I will mention a few:

1. Projective (or Dual) Fraïssé theory, as developed by Irwin and Solecki [3] and recently reformulated in terms of corelations by Panagiotopoulos [5]. The dual ultrahomogeneity exhibited by projective Fraïssé limits can be expressed by $\forall \exists$ sentences in the logic $FO(Stone, L)$.

2. The “cologic” of profinite groups (e.g. Galois groups), developed by Cherlin, van den Dries, and Macintyre [2] and by Chatzidakis, which plays an important role in the model theory of PAC fields. This logic is presented in a multi-sorted first-order framework, which is essentially equivalent to the first-order translation of Theorem 1, applied to $FO(pro-FinGrp, \emptyset)$.

3. The theory of coalgebraic logic, in the special case of cofinitary functors on Stone spaces (see, e.g. [4]), is exactly the theory of equationally defined classes in $FO(Stone, L)$, since $L$-structures are coalgebras for cofinitary functors. This theory has connections to modal logic; for example, when the functor $F$ is the Vietoris functor, $FO(Stone, L)$ embeds modal logics on descriptive general frames.

References