Bunched Hypersequent Calculi for Distributive Substructural Logics and Extensions of Bunched Implication Logic

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The algebraic semantics for distributive Full Lambek logic DFL is the class of algebras $\mathbf{A} = (A, \wedge, \vee, \otimes, /, \backslash, \mathbf{1}, \mathbf{0})$ such that $(A, \wedge, \vee)$ is a distributive lattice (i.e. $x \wedge (y \vee z) \leq (x \wedge y) \vee (x \wedge z)$), and $(A, \otimes, \mathbf{1})$ is a monoid, and $\mathbf{0}$ is an arbitrary element of A, and $x \otimes y \leq z$ iff $x \leq z/y$ iff $y \leq x \backslash z$ for all $x, y, z \in A$. Meanwhile the algebraic semantics for the logic of Bunched Implication BI is the class of Heyting algebras equipped with an additional monoidal operation $\otimes$ and associated implications $/$ and $\backslash$ satisfying $x \otimes y \leq z$ iff $x \leq z/y$ iff $y \leq x \backslash z$. Thus BI has the intuitionistic implication $\rightarrow$ *and* the multiplicative left $/$ and right $\backslash$ implications. Here we propose a new proof calculus formalism called *bunched hypersequents* which can be used to study those subclasses of these algebras that satisfy suitable inequalities. In particular, we construct *analytic proof calculi* such that the inequalities that hold on the class of algebras are precisely those that can be proved *in* the proof calculus in a finite number of steps. Here the term *analytic* means that the proofs in the proof calculus need only contain subterms of the inequality to be proved. In the language of proof-theory, such proof calculi are said to have the *subformula property*. The subformula property (and the ensuing restriction on the space of possible proofs) is crucial for using the calculus for investigating various logical properties such as decidability, complexity, interpolation, conservativity, standard completeness [10], and for developing automated deduction procedures.

Gentzen [7] presented the first analytic calculi, for classical and intuitionistic logic, in his *sequent calculus* formalism. For example, his sequent calculus for intuitionistic logic consists of a small number of unary and binary rules (functions) on sequents; a *sequent* has the form $X \Rightarrow A$ where X is a ;-separated list of formulas and A is a formula. By repeated application of the rules, complicated sequents can be proved (derived) starting from initial sequents of the form $p \Rightarrow p$ such that $B_1; \ldots; B_n \Rightarrow A$ is derivable iff $B_1 \wedge \ldots \wedge B_n \rightarrow A$ is a theorem of intuitionistic logic (i.e. the corresponding inequality is valid on the class of Heyting algebras). This calculus can be used to give direct proofs of e.g. consistency (there is no derivation of $\Rightarrow \bot$) and optimal complexity bounds for the derivability relation.

Unfortunately there are many logics which do not support an analytic treatment in the sequent calculus formalism. The reason is that the form of the proof rules in that formalism are too restrictive. In the last three decades this has led to the introduction of many other formalisms of varying expressivity; prominent examples include the hypersequent [14, 1], display calculus [2] (viewed from a more algebraic perspective as residuated frames [6]), labelled calculus [16, 12] and bunched sequent calculus [5, 11]. The reason for the numerous different formalisms is the tradeoff that exists between an expressive formalism which yields an analytic treatment of many different logics and the difficulty in using such a formalism to prove metalogical properties. As a slogan: typically, the formalism most amenable for proof-theoretic investigation of a logic is the simplest formalism which supports its analyticity.

For distributive substructural logics (including relevant logics)—the logics that are of interest here—bunched sequent calculi, also known as Dunn-Mints systems [5, 11], have been proposed as a means of developing an analytic formulation. In this formalism, sequents have the form $X \Rightarrow A$ where A is a formula and X is a list of formulas with *two* list separators: (";") is the list separator corresponding to the logical connective $\wedge$ and (",") is the list separator corresponding to $\otimes$. Bunched calculi have also been employed to define analytic calculi for the logic of Bunched Implication BI [15]. This logic has been used to reason about dynamic data structures [13] and is a propositional fragment of (intuitionistic) separation logic. Note that although these logics can be formalised using the more powerful formalism of display calculi, the advantage of using a simpler formalism is evident, e.g., when searching for proofs of decidability and complexity of the logic (see [8, 3, 9]).

In this paper we introduce a new proof theoretic framework called bunched hypersequents. Bunched hypersequents extend the bunched sequents by adding a hypersequent structure. In analogy with its extension of traditional sequents, we consider a non-empty set of bunched sequents rather than just a single bunched sequent. This structure allows the definition of new rules which apply to several bunched sequents simultaneously thus increasing the expressive power of the bunched sequent framework. Although a bunched hypersequent is a more complex data structure than a bunched sequent, it is nevertheless a simple and natural extension, retaining many of the useful properties of the sequent calculus (recall the slogan).

The expressive power of the new formalism is demonstrated by introducing analytic bunched hypersequent calculi for a large class of extensions of distributive Full Lambek calculus DFL. The extensions are obtained by suitably extending the procedure in [4] for transforming Hilbert axioms into structural rules. We then consider the case of extensions of the logic of bunched implication BI. Extensions of BI by a certain class of axioms including restricted weakening and restricted contraction are obtained.

Our attempt to extend the BI calculus to obtain a simple analytic calculus for BBI (boolean BI; known to be undecidable) met with a surprising obstacle. While a hypersequent structure extending the bunched calculus for BI *can* be defined (and hence also logics extending BI via the exploitation of the hypersequent structure), there are technical difficulties associated with the interpretation of hypersequent structure at intermediate points of the derivation. In response, we turn the investigation on its head and formulate an analytic hypersequent calculus for a consistent extension of BI which derives a limited boolean principle. The properties of this logic, including its decidability problem, invite further investigation.

References

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