Yes/No Formulae as a Description of Theories of Intuitionistic Kripke Models

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Abstract

The notion of logical equivalence still remains one of the most interesting subjects of investigation. In many logical systems the question that arises is how to describe the theory of a considered structure by means of a single formula.

We transfer this problem to intuitionistic first-order logic and consider Kripke semantics. By a Kripke model for a first-order language \( L \) we define a structure \( K = (K, \leq, \{K_\alpha : \alpha \in K\}, \models_K) \) (for a more general definition see [2]). To any node \( \alpha \in K \) there is assigned a classical first-order structure \( K_\alpha \) for \( L \), and for any two nodes \( \alpha, \alpha' \in K \) we require that

\[
\alpha \leq \alpha' \implies K_\alpha \subseteq K_{\alpha'}.
\]

The forcing relation \( \models_K \) on \( K \) is defined in the standard way, inductively over the construction of a formula (see [1], [2]).

Since quantifiers \( \forall \) and \( \exists \) are not mutually definable, and implication refers to all nodes accessible from a certain node, as a measure of formula’s complexity we consider the characteristic of a formula (see [1]). We say that characteristic of a formula \( \varphi \), \( \text{char}(\varphi) \), equals \((\rightarrow p, \forall q, \exists r)\) whenever there are \( p \) nested implications, \( q \) nested universal quantifiers and \( r \) nested existential quantifiers in \( \varphi \).

Given two Kripke models \( K = (K, \leq, \{K_\alpha : \alpha \in K\}, \models_K) \) and \( M = (M, \leq, \{M_\beta : \beta \in M\}, \models_M) \), we consider a relation of logical equivalence with respect to all formulae of characteristic not greater than \((\rightarrow p, \forall q, \exists r)\). For nodes \( \alpha \in K \), \( \beta \in M \) and any sequences \( \pi \) and \( \delta \) of elements of structures \( K_\alpha \) and \( M_\beta \) respectively, we define a relation \( \equiv_{p,q,r} \) as follows

\[
(\alpha, \pi) \equiv_{p,q,r} (\beta, \delta) : \iff (\alpha \models_K \varphi[\pi] \iff \beta \models_M \varphi[\delta])
\]

for all formulae \( \varphi[\pi] \) with \( \text{char}(\varphi) \leq (\rightarrow p, \forall q, \exists r) \).

Since intuitionistic connectives differ significantly from the classical ones, one might expect a more complex solution of the aforementioned problem. We confine our considerations to a class of strongly finite Kripke models. We say that Kripke model \( K \) is strongly finite if and only if both the frame and first-order structures assigned to the nodes are finite. Moreover, the finite signature of language \( L \) is considered with no function symbols.

Under these assumptions we construct so-called Yes Formulae and No Formulae which describe theory of a node, the former will encode positive information and the latter – negative information of a node. For a strongly finite Kripke model \( K = (K, \leq, \{K_\alpha : \alpha \in K\}, \models_K) \), its node \( \alpha \in K \) and a sequence \( \pi \) of elements of the structure \( K_\alpha \), we introduce a symbol

\[ Y_{p,q,r}^\alpha \pi \]

to denote a formula of characteristic not greater than \((\rightarrow p, \forall q, \exists r)\) that is forced at \( \alpha \) by \( \pi \). Similarly, a formula of characteristic at most \((\rightarrow p, \forall q, \exists r)\) that is refuted at \( \alpha \) by \( \pi \) is denoted by

\[ N_{p,q,r}^\alpha \pi \].
Formulas $Y_{p,q,r}^{α,π}$ and $N_{p,q,r}^{α,π}$ are defined inductively over $p, q, r ≥ 0$ in the following way:

$$Y_{0,0,0}^{α,π}(X) = (∩\{φ: char(φ) = (¬0,¬0,¬0), α ⊨ φ(π)\})(X)$$

$$N_{0,0,0}^{α,π}(X) = (∪\{φ: char(φ) = (¬0,¬0,¬0), α ⊭ φ(π)\})(X)$$

$$Y_{p+1,q,r}^{α,π}(X) = (∪_{α′ ≥ α} (N_{p,q,r}^{α′,π} → Y_{p,q,r}^{α′,π})(X))$$

$$N_{p+1,q,r}^{α,π}(X) = (∩_{α′ ≥ α} (N_{p,q,r}^{α′,π} → N_{p,q,r}^{α′,π})(X))$$

$$Y_{p,q+r+1}^{α,π}(X) = (∩_{a ∈ K_α} (∃_Y Y_{p,q,r}^{α,π}(X, y))$$

$$N_{p,q+r+1}^{α,π}(X) = (∣_{a ∈ K_α} N_{p,q,r}^{α,π}(X, y))$$

For a strongly finite Kripke model $K$, its node $α ∈ K$ and a sequence $π$ of elements of $K_α$, by $Th_{p,q,r}(α, π)$ we denote a set of all formulae of characteristic not greater than $(¬p,¬q,¬r)$ forced at $α$ by $π$, and by $Th_{p,q,r}(α, π)$ we will mean a set of all formulae of characteristic at most $(¬p,¬q,¬r)$ refuted at $α$ by $π$. We claim that

$$Y_{p,q,r}^{α,π} ⊨ Th_{p,q,r}(α, π) \text{ and } N_{p,q,r}^{α,π} ⊨ Th_{p,q,r}(α, π).$$

Using this fact, we can characterise the notion of $(p,q,r)$-equivalence, $≡_{p,q,r}$. Consider strongly finite Kripke models $K$ and $M$, and nodes $α ∈ K$, $β ∈ M$. Let $π$ and $β$ be sequences of elements of worlds $K_α$ and $M_β$ respectively. For $p, q, r ≥ 0$,

$$(α, π) ≡_{p,q,r} (β, β)$$

if and only if

$$β ∩ M Y_{p,q,r}^{α,π}(β) \text{ and } β ∩ M N_{p,q,r}^{α,π}(β).$$

Hence, logical equivalence between strongly finite rooted Kripke models $(K, α)$ and $(M, β)$ can be described as follows:

$$(K, α) ≡_{p,q,r} (M, β)$$

if and only if

$$β ∩ M Y_{p,q,r}^{α,π}(β) \text{ and } β ∩ M N_{p,q,r}^{α,π}(β)$$

for all $p, q, r ≥ 0$ and all sequences $π$ of $K_α$ and $β$ of $M_β$.

References
