# Yes/No Formulae as a Description of Theories of Intuitionistic Kripke Models

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**Abstract**

The notion of logical equivalence still remains one of the most interesting subjects of investigation. In many logical systems the question that arises is how to describe the theory of a considered structure by means of a single formula.

We transfer this problem to intuitionistic first-order logic and consider Kripke semantics. By a Kripke model for a first-order language $L$ we define a structure $\mathcal{K} = (K, \leq, \{K_\alpha \colon \alpha \in K\}, \Vdash_{\mathcal{K}})$ (for a more general definition see [2]). To any node $\alpha \in K$ there is assigned a classical first-order structure $K_\alpha$ for $L$, and for any two nodes $\alpha, \alpha' \in K$ we require that

$$\alpha \leq \alpha' \Rightarrow K_\alpha \subseteq K_{\alpha'}.$$

The forcing relation $\Vdash_{\mathcal{K}}$ on $\mathcal{K}$ is defined in the standard way, inductively over the construction of a formula (see [1], [2]).

Since quantifiers $\forall$ and $\exists$ are not mutually definable, and implication refers to all nodes accessible from a certain node, as a measure of formula's complexity we consider the *characteristic* of a formula (see [1]). We say that characteristic of a formula $\varphi$, $char(\varphi)$, equals $({}^{\rightarrow}p, {}^{\forall}q, {}^{\exists}r)$ whenever there are $p$ nested implications, $q$ nested universal quantifiers and $r$ nested existential quantifiers in $\varphi$.

Given two Kripke models $\mathcal{K} = (K, \leq, \{K_\alpha \colon \alpha \in K\}, \Vdash_{\mathcal{K}})$ and $\mathcal{M} = (M, \leq, \{M_\beta \colon \beta \in M\}, \Vdash_{\mathcal{M}})$, we consider a relation of logical equivalence with respect to all formulae of characteristic not greater than $({}^{\rightarrow}p, {}^{\forall}q, {}^{\exists}r)$. For nodes $\alpha \in K$, $\beta \in M$ and any sequences $\overline{a}$ and $\overline{b}$ of elements of structures $K_\alpha$ and $M_\beta$ respectively, we define a relation $\equiv_{p,q,r}$ as follows

$$(\alpha, \overline{a}) \equiv_{p,q,r} (\beta, \overline{b}) :\Longleftrightarrow (\alpha \Vdash_{\mathcal{K}} \varphi[\overline{a}] \Leftrightarrow \beta \Vdash_{\mathcal{M}} \varphi[\overline{b}])$$

for all formulae $\varphi(\overline{x})$ with $char(\varphi) \leq ({}^{\rightarrow}p, {}^{\forall}q, {}^{\exists}r)$.

Since intuitionistic connectives differ significantly from the classical ones, one might expect a more complex solution of the aforementioned problem. We confine our considerations to a class of strongly finite Kripke models. We say that Kripke model $\mathcal{K}$ is *strongly finite* if and only if both the frame and first-order structures assigned to the nodes are finite. Moreover, the finite signature of language $L$ is considered with no function symbols.

Under these assumptions we construct so-called *Yes Formulae* and *No Formulae* which describe theory of a node, the former will encode positive information and the latter – negative information of a node. For a strongly finite Kripke model $\mathcal{K} = (K, \leq, \{K_\alpha \colon \alpha \in K\}, \Vdash_{\mathcal{K}})$, its node $\alpha \in K$ and a sequence $\overline{a}$ of elements of the structure $K_\alpha$, we introduce a symbol

$$Y^{\alpha, \overline{a}}_{p,q,r}$$

to denote a formula of characteristic not greater than $({}^{\rightarrow}p, {}^{\forall}q, {}^{\exists}r)$ that is forced at $\alpha$ by $\overline{a}$. Similarly, a formula of characteristic at most $({}^{\rightarrow}p, {}^{\forall}q, {}^{\exists}r)$ that is refuted at $\alpha$ by $\overline{a}$ is denoted by

$$N^{\alpha, \overline{a}}_{p,q,r}.$$

Formulas $Y^{\alpha,\overline{a}}_{p,q,r}$ and $N^{\alpha,\overline{a}}_{p,q,r}$ are defined inductively over $p, q, r \geq 0$ in the following way:

$$Y^{\alpha,\overline{a}}_{0,0,0}(\overline{x}) = \big(\bigwedge\{\varphi\colon \mathit{char}(\varphi) = (^{\to}0,^{\forall}0,^{\exists}0), \alpha \Vdash \varphi(\overline{a})\}\big)(\overline{x})$$

$$N^{\alpha,\overline{a}}_{0,0,0}(\overline{x}) = \big(\bigvee\{\varphi\colon \mathit{char}(\varphi) = (^{\to}0,^{\forall}0,^{\exists}0), \alpha \nVdash \varphi(\overline{a})\}\big)(\overline{x})$$

$$Y^{\alpha,\overline{a}}_{p+1,q,r}(\overline{x}) = \bigwedge_{\alpha' \geq \alpha} (N^{\alpha',\overline{a}}_{p,q,r} \to Y^{\alpha',\overline{a}}_{p,q,r})(\overline{x})$$

$$N^{\alpha,\overline{a}}_{p+1,q,r}(\overline{x}) = \bigvee_{\alpha' \geq \alpha} (Y^{\alpha',\overline{a}}_{p,q,r} \to N^{\alpha',\overline{a}}_{p,q,r})(\overline{x})$$

$$Y^{\alpha,\overline{a}}_{p,q+1,r}(\overline{x}) = \forall_y \bigvee_{\alpha' \geq \alpha} \bigvee_{a \in K_{\alpha'}} Y^{\alpha',\overline{a}a}_{p,q,r}(\overline{x}, y)$$

$$N^{\alpha,\overline{a}}_{p,q+1,r}(\overline{x}) = \bigvee_{\alpha' \geq \alpha} \bigvee_{a \in K_{\alpha'}} \forall_y\ N^{\alpha',\overline{a}a}_{p,q,r}(\overline{x}, y)$$

$$Y^{\alpha,\overline{a}}_{p,q,r+1}(\overline{x}) = \bigwedge_{a \in K_\alpha} \exists_y\ Y^{\alpha,\overline{a}a}_{p,q,r}(\overline{x}, y)$$

$$N^{\alpha,\overline{a}}_{p,q,r+1}(\overline{x}) = \exists_y \bigwedge_{a \in K_\alpha} N^{\alpha,\overline{a}a}_{p,q,r}(\overline{x}, y)$$

For a strongly finite Kripke model $\mathcal{K}$, its node $\alpha \in K$ and a sequence $\overline{a}$ of elements of $K_\alpha$, by $Th_{p,q,r}(\alpha, \overline{a})$ we denote a set of all formulae of characteristic not greater than $(^{\to}p,^{\forall}q,^{\exists}r)$ forced at $\alpha$ by $\overline{a}$, and by $\widetilde{Th}_{p,q,r}(\alpha, \overline{a})$ we will mean a set of all formulae of characteristic at most $(^{\to}p,^{\forall}q,^{\exists}r)$ refuted at $\alpha$ by $\overline{a}$. We claim that

$$Y^{\alpha,\overline{a}}_{p,q,r} \vdash Th_{p,q,r}(\alpha, \overline{a}) \quad \text{and} \quad N^{\alpha,\overline{a}}_{p,q,r} \vdash \widetilde{Th}_{p,q,r}(\alpha, \overline{a}).$$

Using this fact, we can characterise the notion of $(p, q, r)$-equivalence, $\equiv_{p,q,r}$. Consider strongly finite Kripke models $\mathcal{K}$ and $\mathcal{M}$, and nodes $\alpha \in K$, $\beta \in M$. Let $\overline{a}$ and $\overline{b}$ be sequences of elements of worlds $K_\alpha$ and $M_\beta$ respectively. For $p, q, r \geq 0$,

$$(\alpha, \overline{a}) \equiv_{p,q,r} (\beta, \overline{b})$$

if and only if

$$\beta \Vdash_{\mathcal{M}} Y^{\alpha,\overline{a}}_{p,q,r}(\overline{b}) \quad \text{and} \quad \beta \nVdash_{\mathcal{M}} N^{\alpha,\overline{a}}_{p,q,r}(\overline{b}).$$

Hence, logical equivalence between strongly finite rooted Kripke models $(\mathcal{K}, \alpha)$ and $(\mathcal{M}, \beta)$ can be described as follows:

$$(\mathcal{K}, \alpha) \equiv_{p,q,r} (\mathcal{M}, \beta)$$

if and only if

$$\beta \Vdash_{\mathcal{M}} Y^{\alpha,\overline{a}}_{p,q,r}(\overline{b}) \quad \text{and} \quad \beta \nVdash_{\mathcal{M}} N^{\alpha,\overline{a}}_{p,q,r}(\overline{b})$$

for all $p, q, r \geq 0$ and all sequences $\overline{a}$ of $K_\alpha$ and $\overline{b}$ of $M_\beta$.

# References


[1] Tomasz Połacik. Back and forth between first-order Kripke models. *Logic Journal of the IGPL*, 16(4):335–355, 2008.

[2] Albert Visser. Submodels of Kripke models. Logic Group Preprint Series 189, Utrecht University, 1998.