Diagrammatic duality

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Dualities between algebras and representation spaces are a staple topic at the interface of algebra and logic. While dualities are often produced with the aid of dualizing objects as codomains for both algebra and space homomorphisms, other techniques are also useful. As an addition to the palette of alternative techniques, diagrammatic duality is a method for obtaining new dualities founded on existing ones. Whenever algebras of a certain class are equivalent to diagrams in a category of known dualizable algebras, **diagrammatic duality** furnishes representation spaces for the algebras in the class by examining dual diagrams in the category of representation spaces for the known dualizable algebras. We present two examples: Nelson algebras, and algebras from an arbitrary variety.

A *diagram* in a category $\mathbf{C}$ is a graph map $F\colon D \to \mathbf{C}$ from a quiver (or directed graph) D to (the underlying graph of) $\mathbf{C}$. The diagram is *proper* if its domain D is small. For a quiver D and category $\mathbf{C}$, let $\left(\mathbf{C}^D\right)_0$ be the class of diagrams $F\colon D \to \mathbf{C}$. For given diagrams $F\colon D \to \mathbf{C}$ and $G\colon D \to \mathbf{C}$, let $\left(\mathbf{C}^D\right)(F,G)$ be the class of natural transformations $\tau : F \to G$. Then $\mathbf{C}^D$ forms a category. If D is a proper diagram, then categories of the form $\mathbf{C}^D$ are known as *diagram categories*.

Suppose that $\mathfrak{C}$ and $\mathfrak{A}$ are categories of algebras and homomorphisms. Suppose that there is an equivalence $\mathfrak{C} \cong \mathfrak{C}_{\mathfrak{A}}$ between $\mathfrak{C}$ and a subcategory $\mathfrak{C}_{\mathfrak{A}}$ of a diagram category $\mathfrak{A}^V$ with given domain diagram V. Then the objects of $\mathfrak{C}$ are known as *diagrammatic algebras* (relative to $\mathfrak{A}$). In this context it is often convenient to abuse notation and suppress the distinction between $\mathfrak{C}$ and $\mathfrak{C}_{\mathfrak{A}}$, merely stating that a $\mathfrak{C}$-algebra C is equivalent to a diagram $\gamma\colon V \to \mathfrak{A}$.

A *duality* denotes a dual equivalence $D : \mathfrak{A} \rightleftarrows \mathfrak{X} : E$ in which $\mathfrak{A}$ is a category of *algebras* (in the sense of modern universal algebra) and homomorphisms, while $\mathfrak{X}$ is a concrete category of objects known as *spaces*. For an algebra A, the image AD is called the *representation space* of A. For a space X, the image XE is called the *algebra represented* by X. The functor D is called the *dual space functor*. The functor E is called the *represented algebra functor*.

For **Esakia duality** [1], take $\mathfrak{A}$ to be the category **Heyt** of Heyting algebras. Take $\mathfrak{X}$ to be the category **Esakia** of *Esakia spaces*, partially ordered Stone spaces where the downset $C^{\geq}$ of each clopen subset C is clopen. For a Heyting algebra H, the representation space $H^D = \mathbf{Heyt}(H,2)$ carries the induced order and subspace topology from the product 2^H. An Esakia space S represents the algebra $S^E = \mathbf{Esakia}(S,2)$, a Heyting subalgebra of 2^S.

For **Lindenbaum-Tarski duality** [2, p. xiv], [8], take $\mathfrak{A}$ to be the category **Set** of sets (algebras without operations). Take $\mathfrak{X}$ to be the category **CABA** of complete atomic Boolean algebras and homomorphisms preserving all joins and meets. Consider the set $2 = \{0,1\}$, possibly endowed with Boolean algebra structure. For a set A, the representation space A^D is defined to be the set 2^A or $\mathcal{P}(A)$ of (characteristic functions of) subsets of A, with the singletons as atoms. For a complete atomic Boolean algebra B, the represented algebra $B^E := \mathbf{CABA}(B,2)$ is naturally isomorphic to the set of atoms of B.

Nelson algebras

Nelson algebras [6], also known as "N-lattices" [4], provide algebraic semantics for Nelson's *constructive logic with strong negation* [3, 5]. Consider an algebra $(B,\vee,\wedge,\rightarrow,\sim,0,1)$ equipped with three binary operations $\vee,\wedge,\rightarrow$, and with $\sim$ as a unary operation (*strong negation*). Suppose that $(B,\vee,\wedge,0,1)$ is a bounded distributive lattice, with $\leq$ as the lattice ordering. Then the algebra is a *Nelson algebra* if the following conditions are satisfied: the reduct $(B,\vee,\wedge,\sim,0,1)$ is a De Morgan algebra; a reflexive, transitive relation $\preceq$ is defined on B by setting $x \preceq y$ iff $(x \rightarrow y) \rightarrow (x \rightarrow y) = x \rightarrow y$; the lattice order relation $x \leq y$ on B is equivalent to $x \preceq y$ and $\sim x \preceq \sim y$; the equivalence relation χ, defined on B by $x\ \chi\ y$ iff $x \preceq y$ and $y \preceq x$, is a congruence on the reduct $(B,\vee,\wedge,\rightarrow)$ such that the quotient $(B,\vee,\wedge,\rightarrow,0,1)^\chi$ is a Heyting algebra; and for all $x,y \in B$, one has $(x\wedge \sim x, 0) \in \chi$ and $(\sim(x \rightarrow y), x\wedge \sim y) \in \chi$. Now a congruence α on a Heyting algebra H is *Boolean* if the quotient H^α is a Boolean algebra. Then the category of Nelson algebras is equivalent to the category of pairs (H,α), where H is a Heyting algebra and α is a Boolean congruence on H [6, Th. 4.1]. Thus Nelson algebras are diagrammatic relative to (the category **Heyt** of) Heyting algebras: Consider the quiver V given as $a\colon h \rightarrow b$. Then a Nelson algebra B is equivalent to a diagram $\beta\colon V \rightarrow \mathbf{Heyt}$ sending the arrow a to the natural projection of the Boolean congruence α_B from the Heyting algebra B^χ. By this means, Nelson algebras have a diagrammatic duality based on Esakia duality for Heyting algebras.

Classical universal algebras

Consider a *type* $\tau\colon \Omega \rightarrow \mathbb{N}$, with *operator domain* Ω. Then a τ-*algebra* (A,τ) is a set A with an *operation* $\omega\colon A^{\omega\tau} \rightarrow A$ corresponding to each *operator* or element ω of Ω. Let $\underline{\underline{\tau}}$ denote the category of τ-algebras and homomorphisms between them [7, §§IV1.1–2]. The Ω-*cospan* is a quiver Ω_∞ with edge set Ω. Its vertex set is the disjoint union $\Omega + \top$ of Ω with a singleton $\top = \{\infty\}$. The tail map is the identity function on Ω, while the head map is the unique function $\Omega \rightarrow \top$. A τ-algebra A is equivalent to a diagram $\alpha\colon \Omega_\infty \rightarrow \mathbf{Set}$ with edge map $\alpha_1\colon \omega \mapsto (\omega\colon A^{\omega\tau} \rightarrow A)$. Thus the edge map sends each operator to the corresponding operation on the set A. It follows that algebras from any variety are diagrammatic relative to sets, and thus possess a diagrammatic duality based on Lindenbaum-Tarski duality for sets.

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