Algebraic generalizations of completeness and canonicity

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Dedicated to the memory of Bjarni Jónsson

In their landmark 1951 work, Jónsson and Tarski identified defining properties of boolean algebras with operators (BAOs) dual to relational structures (which we know today as *Kripke frames*) and showed that every BAO $\mathfrak{A}$ can be embedded into such a dual suitably constructed from $\mathfrak{A}$, which they called the *perfect* extension and which we know today as the *canonical* extension of $\mathfrak{A}$. Furthermore, they isolated a class of equations preserved by this process, thus pioneering a line of work which much later came to include results such as the Sahlqvist theorem and its generalizations to, e.g., *inductive (in-)equalities*.

Let us briefly recall these defining properties of duals of relational structures, restricting attention to modal algebras (MAs, i.e., BAOs with a single unary $\Diamond$) for simplicity: they are lattice-$\mathcal{C}$omplete, $\mathcal{A}$tomic (thus also atomistic, being boolean algebras) and completely additi$\mathcal{V}$e, i.e., for any set X of elements, if $\bigvee X$ exists, then $\bigvee\{\Diamond x \mid x \in X\}$ exists and

$$\Diamond \bigvee X = \bigvee\{\Diamond x \mid x \in X\}.$$

Hence, it is natural to call such algebras $\mathcal{CAV}$-BAOs and write $\mathcal{CAV}$ to denote this class. It is also natural to use similar conventions for classes of algebras obtained by dropping some of these conditions, e.g., $\mathcal{CA}$ or $\mathcal{CV}$. Some of these classes are dual disguises of more general semantics of modal logic, e.g., $\mathcal{CA}$-BAOs are dually equivalent to neighbourhood frames (Došen 1989), thus also providing an algebraic framework for *coalgebraic* semantics; $\mathcal{CV}$-BAOs, as shown recently by Holliday, allow a dual representation in terms of *possibility semantics*; and $\mathcal{AV}$-BAOs are dual incarnations of *discrete general frames*.

For every variety of BAOs V defined by equations satisfying the conditions of Jónsson and Tarski, or perhaps by in-equalities studied in Jónsson's 1994 work, or by Sahlqvist/inductive (in-)equalities, the following meta-level "equation" holds:

$$V = \mathbb{S}(V \cap \mathcal{CAV}), \tag{1}$$

i.e., every $\mathfrak{A} \in V$ is (an isomorphic copy of) a subalgebra of a $\mathcal{CAV}$-BAO from the same variety. Several authors, like Goldblatt, call this property being *complex*; in our setting, to be more precise, we should speak of being $\mathcal{CAV}$-*complex*. As shown by Wolter in the 1990's, this is a proper generalization of canonicity. In other words, there is a variety whose defining equations are not preserved in general by canonical (perfect) extensions, yet satisfying (1); furthermore, this variety happens to correspond to a very natural tense logic. Wolter has also shown that $\mathcal{CAV}$-complexity is the algebraic counterpart of two distinct notions of modal completeness: strong *global* completeness and strong *local* completeness, corresponding to the two natural notions of modal consequence.

What happens when $\mathcal{CAV}$ in (1) is replaced by a broader class of algebras? First of all, note that there is a natural generalization of canonicity, proposed by Chellas 1980. This notion allowed Surendonk (2001) to prove that some flagship examples of varieties failing (1) are, e.g., $\mathcal{CA}$-complex. But, in general, for many non-canonical varieties even $\mathcal{C}$-complexity (i.e., closure under completions) is too much too ask. Furthermore, while the Wolterian correspondence between $\mathcal{X}$-complexity and strong *global* $\mathcal{X}$-completeness is quite robust (to wit, it survives

whenever $\mathcal{X}$ is closed under products), strong *local* $\mathcal{X}$-completeness can be a weaker property when $\mathcal{X} \not\subseteq \mathcal{AV}$. Results of Shehtman imply strong local $\mathcal{CA}$-completeness of many logics which are not closed under completions.

Finally, an obvious corollary of canonicity or $\mathcal{CAV}$-complexity of V is *weak* Kripke completeness, i.e., completeness for *theoremhood*, i.e., the satisfaction of the following meta-level "equation":

$$V = \mathbb{HSP}(V \cap \mathcal{CAV}). \tag{2}$$

Given that, on the one hand, a) even (1) itself is a weaker property than canonicity and (2) is still a *much* weaker property than (1) and that on the other hand, b) weak completeness can be proved by more constructive methods, which do not involve the Axiom of Choice and yet establish a strong property (fmp) for possibly non-canonical logics (cf., e.g., Fine 1975, Moss 2007 or Bezhanishvili and Ghilardi 2014), there is some irony in the fact that canonicity appears in many presentations of modal logic mostly *en route* to weak completeness. This state of affairs does not seem to do full justice to either notion. Still, weak completeness is quite often *the* notion of completeness of interest from modal logicians' point of view.[1] An obvious question is thus, again, what happens when $\mathcal{CAV}$ is replaced in (2) by other classes of BAOs? Note, for example, that weak $\mathcal{AV}$-completeness, strong $\mathcal{AV}$-completeness and $\mathcal{AV}$-complexity coincide, so while we can expect numerous negative results, there are some unexpected positive ones too.

More than a decade ago, I attempted to clarify the picture during my PhD studies (Litak 2004, 2005, 2008), unifying, expanding, and building on earlier results by Thomason, Fine, Gerson, van Benthem, Blok, Chagrova, Chagrov, Wolter, Zakharyaschev, Venema and other researchers. As it turns out, every possible combination of $\mathcal{C}$, $\mathcal{A}$, $\mathcal{V}$ and related properties allows to produce examples of logics/varieties for which completeness fails in a different way. Moreover, negative results concerning Kripke completeness, such as the Blok Dichotomy (sometimes also called the *Blok Alternative*), generalize to these weaker completeness notions. The only major piece of the puzzle missing was the status of $\mathcal{V}$-completeness—and I only managed to solve this in a recent collaboration with Holliday, using a first-order formulation of complete additivity inspired by his work on possibility semantics (some additional insights on this issue have been obtained by Andréka, Gyenis and Németi and more recently by van Benthem). We were surprised how natural some of our counterexamples turned out to be.

Where do we go from here? Even as far as weak completeness of modal logics is concerned, there are numerous unanswered questions like availability of broader completeness results in smaller lattices of logics (are all extensions of $\mathsf{K4}$ $\mathcal{AV}$-complete, for example?) or the status of the Blok Dichotomy for $\mathcal{A}$-completeness. Our understanding of the hierarchy of notions refining strong completeness and canonicity seems even more sketchy—and further study could yield dividends for coalgebraic semantics and possibility semantics (which, as observed by Holliday, can be used to present a constructive perspective on canonical extensions). But our ignorance in these matters as far as other non-classical logics are concerned is most striking. We have some isolated results: we know, for example, that MV-algebras are not only non-canonical (Gehrke and Priestley 2002), but fail to be closed under completions (Gehrke and Jónsson 2004) and the same applies to many other varieties of GBL-algebras (Kowalski and Litak 2008). Thanks to Shehtman 1977, we also know that there are Kripke-incomplete si-logics, even uncountably many ones (Litak 2002), but this is the border of *hic sunt leones* area: Kuznetsov's earlier question about the existence of topologically incomplete si-logics remains unanswered until today. And for substructural logics in general, not much more seems to be known. Where will the door opened by Jónsson and Tarski in 1951 finally lead us?

[1] Surely enough, things looks different when a logic is taken to be a *consequence relation* rather than a *set of theorems*.