Proper multi-type display calculi for classical and intuitionistic inquisitive logic*

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Introduction. In this work, we define proper multi-type display calculi for both classical and intuitionsitic inquisitive logic, which enjoy Belnap-style cut-elimination and subformula property.

Inquisitive logic is the logic of inquisitive semantics, a semantic framework developed by Ciardelli, Groenendijk and Roelofsen [5, 1] which captures both assertions and questions in natural language. A distinguishing feature of inquisitive logic is that formulas are evaluated on *information states*, i.e., sets of possible worlds, rather than on single possible worlds. Inquisitive logic defines a *support* relation between states and formulas, the intended understanding of which is that in uttering a sentence, a speaker proposes to enhance the current common ground to one that supports the sentence. This semantics is also known as *team semantics*, which was introduced by Hodges [6, 7] in the context of *dependence logic* [8]. Recent work [2] generalised the original classical logic-based framework of inquisitive logic [1], and introduced inquisitive logic on the basis of intuitionistic propositional logic.

The Hilbert-style presentations of both classical and intuitionistic inquisitive logic are not closed under uniform substitution, and some axioms are sound only for a certain subclass of formulas, called *standard formulas*. This and other features make the quest for analytic calculi for the logics not straightforward. A first step in this direction was taken in [3], where a multi-type sequent calculus was developed for classical inquisitive logic. However, this calculus does not enjoy display property. In this work, we generalise the methodology of [3] and propose a proper multi-type display calculi for both classical and intuitionistic inquisitive logic. We develop a certain algebraic and order-theoretic analysis of the support semantics, which provides the guidelines for the design of a multi-type environment accounting for two domains of interpretation, for standard and for general formulas, as well as for their interaction. This multi-type environment in its turn provides the semantic environment for the multi-type calculi for both classical and intuitionistic inquisitive logic we propose in this work.

Classical and intuitionistic inquisitive logic. The following grammar defines the language of both classical (CInq) and intuitionistic inquisitive logic (IInq) presented as a language of two types:

$$\mathsf{Standard} \ni \alpha ::= p \mid 0 \mid \alpha \sqcap \alpha \mid \alpha \twoheadrightarrow \alpha \qquad \mathsf{General} \ni A ::= {\downarrow}\alpha \mid A \wedge A \mid A \vee A \mid A \to A$$

Standard formulas of CInq and IInq adopt the standard semantics for classical and intuitionistic propositional logic, respectively. General type formulas are evaluated on *information states*, which *sets* of classical valuations for CInq, or *sets* of possible worlds in intuitionistic Kripke models $M = (W, R, V)$ for IInq. The *support* relation $S \models \phi$ of a general type formula ϕ in either logic on a state S is defined as:

$$\begin{array}{rcl@{\qquad}rcl} S \models {\downarrow}\alpha & \text{iff} & v \models \alpha \text{ for all } v \in S & S \models \phi \vee \psi & \text{iff} & S \models \phi \text{ or } S \models \psi \\ S \models \phi \wedge \psi & \text{iff} & S \models \phi \text{ and } S \models \psi & S \models \phi \to \psi & \text{iff} & \text{for any } T \leq S, \text{ if } T \models \phi, \text{ then } T \models \psi \end{array}$$

where the extension relation $\leq$ between information states is defined as $T \leq S$ iff $T \subseteq S$ in the CInq case, and as $T \leq S$ iff $T \subseteq R[S]$ in the IInq case. IInq and CInq are complete with respect to the systems below:

System of IInq: Rule: Modus Ponens for both types

Axioms:
- Axiom schemata of (disjunction-free) intuitionistic logic (**IPC**) for $\mathsf{Standard}$-formulas
- Axiom schemata of **IPC** for $\mathsf{General}$-formulas
- $({\downarrow}\alpha \to (A \vee B)) \to ({\downarrow}\alpha \to A) \vee ({\downarrow}\alpha \to B)$ (Split axiom)

System of CInq: The system of IInq extended with two extra axioms: • $\sim\sim \alpha \twoheadrightarrow \alpha$ • $\neg\neg{\downarrow}\alpha \to {\downarrow}\alpha$

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Order-theoretic analysis. In the setting of CInq, the base logic, namely classical propositional logic, gives rise to a Boolean algebra $\mathbb{B} = (\mathcal{P}(2^V), \cap, \cup, (\cdot)^c, \varnothing, 2^V)$. The set $\mathcal{P}^{\downarrow}(\mathcal{P}(2^V))$ of downward closed collections of states forms a perfect Heyting algebra $\mathbb{A} := (\mathcal{P}^{\downarrow}(\mathcal{P}(2^V)), \cap, \cup, \Rightarrow, \varnothing, \mathcal{P}(2^V))$ as the complex algebra of the relational structure $(\mathcal{P}(2^V), \subseteq)$. The following mappings between the two algebras

$$\mathrm{f}^* : \mathbb{B} \to \mathbb{A}\,; S \mapsto \{\{v\} \mid v \in S\} \cup \{\varnothing\} \qquad \mathrm{f} : \mathbb{A} \to \mathbb{B};\ S \mapsto \{T \mid T \subseteq S\} \qquad \downarrow : \mathbb{B} \to \mathbb{A};\ S \mapsto \{T \mid T \subseteq S\},$$

turn out to be adjoints to one another: $\mathrm{f}^* \dashv \mathrm{f} \dashv \downarrow$, since $\mathrm{f}\mathcal{S} \subseteq S$ iff $\mathcal{S} \subseteq \downarrow S$ and $\mathrm{f}^* S \subseteq \mathcal{S}$ iff $S \subseteq \mathrm{f}\mathcal{S}$. Similar observations can be made for IInq, and similar mappings can be found between a Heyting algebra for the base logic, intuitionistic logic with single-world semantics, and a Heyting algebra on the higher level.

Proper multi-type display calculi for CInq and IInq. Building on the order-theoretic analysis, we introduce the corresponding structural operators F^*, F and $\Downarrow$ for the mappings f^*, f and $\downarrow$. The structural languages for the standard type and general type and their interpretations are presented as follows:

$$\textsf{Standard} \quad \Gamma ::= \alpha \mid \Phi \mid \Gamma\,,\Gamma \mid \Gamma \sqsupset \Gamma \mid \mathrm{F}X \qquad \textsf{General} \quad X ::= A \mid \Downarrow\Gamma \mid \mathrm{F}^*\Gamma \mid X\,;X \mid X > X$$

Structural symbols	Φ		,		$\sqsupset$		;		$>$		F^*		F		$\Downarrow$	
Operational symbols	(1)	0	$\sqcap$	($\sqcup$)	($\mapsto$)	$\rightarrowtriangle$	$\wedge$	$\vee$	($\rightarrowtail$)	$\to$	(f^*)		(f)	(f)		$\downarrow$

Our calculi for CInq and IInq are built on the basis of the one introduced [3], but there are major differences in the following structural rules that characterise the interaction between the two types:

$$\frac{\mathrm{F}^*\Gamma \vdash \Delta}{\Gamma \vdash \mathrm{F}\Delta}\ \text{f adj} \qquad \frac{\mathrm{F}X \vdash \Gamma}{X \vdash \Downarrow\Gamma}\ \text{d adj} \qquad \frac{X \vdash \Downarrow\mathrm{F}Y}{X \vdash Y}\ \text{d-f elim} \qquad \frac{\Gamma \vdash \Delta}{\mathrm{F}^*\Gamma \vdash \Downarrow\Delta}\ \text{bal} \qquad \frac{X \vdash Y}{\mathrm{F}X \vdash \mathrm{F}Y}\ \text{f mon}$$

$$\frac{X \vdash \Downarrow(\Gamma \sqsupset \Delta)}{X \vdash \mathrm{F}^*\Gamma > \Downarrow\Delta}\ \text{d dis} \qquad \frac{\mathrm{F}X\,,\mathrm{F}Y \vdash Z}{\mathrm{F}(X\,;Y) \vdash Z}\ \text{f dis} \qquad \frac{X \vdash \mathrm{F}^*\Gamma > (Y\,;Z) \qquad X \vdash \mathrm{F}^*\Gamma > (Y\,;Z)}{X \vdash (\mathrm{F}^*\Gamma > Y)\,;(\mathrm{F}^*\Gamma > Z)}\ \text{Split}$$

We adopt a standard display calculus for standard formulas of IInq, and we add the following classical Grishin rule for standard formulas of CInq:

$$\frac{\Pi \vdash \Gamma \sqsupset (\Delta\,,\Sigma)}{\Pi \vdash (\Gamma \sqsupset \Delta)\,,\Sigma}\ \text{CG}$$

The completeness of the calculi is proved by deriving the axioms and rules of the Hilbert systems. In particular, the split axiom in both logics is derived by applying the Split rule, and the double negation law for CInq is derived by applying the Grishin rule for classical standard formulas. The proposed calculi are *proper* multi-type display calculi, a strict and particularly well-behaved subclass of multi-type sequent calculi, therefore cut-elimination and subformula property follow from the general result in [4].

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