Classical and intuitionistic relation algebras

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In their seminal 1951-52 papers [1] on Boolean algebras with operators (BAOs), Jónsson and Tarski showed that many varieties of BAOs, including the variety of relation algebras, are closed under canonical extensions, and that a relation algebra is complete and atomic with all atoms as functional elements if and only if it is the complex algebra of a generalized Brandt groupoid. The results about canonical extensions were extended to distributive lattices with operators by Gehrke and Jónsson in 1994, and to lattices with operators by Gehrke and Harding in 2001. Here we show results about relation algebras can also be generalized to certain distributive residuated lattices and involutive distributive residuated lattices, in some cases expanded by a Heyting implication. These varieties include (generalized) bunched implication algebras and weakening relation algebras, which have applications in computer science and algebraic logic.

A *relation algebra* $(A, \wedge, \vee, ', \top, \bot, \cdot, ^{\smile}, 1)$ is a Boolean algebra $(A, \wedge, \vee, ', \top, \bot)$ and a monoid $(A, \cdot, 1)$ such that $xy \le z \iff x^{\smile} \cdot z' \le y'$. An excellent introduction to relation algebras is in [2] and several results about them were extended to residuated Boolean monoids in [3].

A *cyclic involutive generalized bunched implication algebra* (or CyGBI-algebra for short) $(B, \wedge, \vee, \to, \top, \bot, \cdot, 1, {\sim})$ is a Heyting algebra $(B, \wedge, \vee, \to, \top, \bot)$ and a monoid $(B, \cdot, 1)$ with a linear cyclic negation $\sim$ that satisfies ${\sim}{\sim}x = x$ and $x \le {\sim}y \iff xy \le {\sim}1$. So they are involutive residuated lattices expanded with a Heyting implication, and both relation algebras and CyGBI-algebras can be defined by identities. A relation algebra is a CyGBI-algebra if we define $x \to y = x' \vee y$ and ${\sim}x = x'^{\smile}$. In a CyGBI-algebra define $x^{\smile} = {\sim}\neg x$ where $\neg x = x \to \bot$, then it is a relation algebra if it satisfies the identities $\neg\neg x = x$ and $(xy)^{\smile} = y^{\smile}x^{\smile}$.

We define algebras of binary relations that are cyclic involutive GBI-algebras and generalize representable relation algebras: Let $\mathbf{P} = (P, \sqsubseteq)$ be a partially ordered set, $Q \subseteq P^2$ an equivalence relation that contains $\sqsubseteq$, and define the set of *weakening relations* on $\mathbf{P}$ by $\mathrm{Wk}(\mathbf{P}, Q) = \{\sqsubseteq \circ R \circ \sqsubseteq \, : R \subseteq Q\}$. Note that this set is closed under intersection $\cap$, union $\cup$ and composition $\circ$, but not under complementation $R' = Q - R$ or converse $R^{\smile}$.

Weakening relations are the natural analogue of binary relations when the category **Set** of sets and functions is replaced by the category **Pos** of partially ordered sets and order-preserving functions. Since sets can be considered as discrete posets (i.e. ordered by the identity relation), **Pos** contains **Set** as a full subcategory, which implies that weakening relations are a substantial generalization of binary relations. They have applications in sequent calculi, proximity lattices/spaces, order-enriched categories, cartesian bicategories, bi-intuitionistic modal logic, mathematical morphology and program semantics, e.g. via separation logic.

Theorem 1. *Let $\mathbf{P} = (P, \sqsubseteq)$ be a poset, Q an equivalence relation that contains $\sqsubseteq$, and for $R, S \in \mathrm{Wk}(\mathbf{P}, Q)$ define $\top = Q$, $\bot = \emptyset$, $1 = \sqsubseteq$, ${\sim}R = R^{\smile\prime}$ and $R \to S = (\sqsupseteq \circ (R \cap S') \circ \sqsupseteq)'$ where $S' = Q - S$. Then $\mathbf{Wk}(\mathbf{P}, Q) = (\mathrm{Wk}(\mathbf{P}, Q), \cap, \cup, \to, \top, \bot, \circ, 1, {\sim})$ is a CyGBI-algebra.*

Algebras of the form $\mathbf{Wk}(\mathbf{P}, Q)$ are called representable weakening relation algebras, and if $Q = P \times P$, then we write $\mathbf{Wk}(\mathbf{P})$ and call this algebra the *full weakening relation algebra on* $\mathbf{P}$. If $\mathbf{P}$ is a discrete poset then $\mathbf{Wk}(\mathbf{P})$ is the full representable set relation algebra on the set P, so algebras of weakening relations play a role similar to representable relation algebras. We define the class RwRA of *representable weakening relation algebras* as all algebras that are embedded in a weakening relation algebra $\mathbf{Wk}(\mathbf{P}, Q)$ for some poset $\mathbf{P}$ and equivalence relation Q that contains $\sqsubseteq$. In fact the variety RRA of representable relation algebras is finitely axiomatized over RwRA.

Theorem 2. *1.* RwRA *is a discriminator variety closed under canonical extensions.*

2. RRA *is the subvariety of* RwRA *defined by* $\neg\neg x = x$.

3. The class RwRA *is not finitely axiomatizable relative to the variety of all CyGBI-algebras.*

A *groupoid* is defined as a partial algebra $\mathbf{G} = (G, \circ, ^{-1})$ such that $\circ$ is a partial binary operation and $^{-1}$ is a (total) unary operation on G that satisfy the following axioms:

1. $(x \circ y) \circ z \in G$ or $x \circ (y \circ z) \in G \implies (x \circ y) \circ z = x \circ (y \circ z)$,
2. $x \circ y \in G \iff x^{-1} \circ x = y \circ y^{-1}$,
3. $x \circ x^{-1} \circ x = x$ and $(x^{-1})^{-1} = x$.

Typical examples of groupoids are disjoint unions of groups and the *pair-groupoid* $(X \times X, \circ, ^{\smile})$, where $(x, y)^{\smile} = (y, x)$ and $(x, y) \circ (z, w) = (x, w)$ if $y = z$ (undefined otherwise). A *partially-ordered groupoid* $(G, \leq, \circ, ^{-1})$, or *po-groupoid* for short, is a groupoid $(G, \circ, ^{-1})$ such that $(G, \leq)$ is a poset and

4. $x \leq y$ and $x \circ z, y \circ z \in G \implies x \circ z \leq y \circ z$,
5. $x \leq y \implies y^{-1} \leq x^{-1}$,
6. $x \circ y \leq z \circ z^{-1} \implies x \leq y^{-1}$.

If $\mathbf{P} = (P, \sqsubseteq)$ is a poset with dual poset $\mathbf{P}^\partial = (P, \sqsupseteq)$ then $\mathbf{P} \times \mathbf{P}^\partial = (P \times P, \leq, \circ, ^{\smile})$ is a po-groupoid, called a *po-pair-groupoid*, with $(a, b) \leq (c, d) \iff a \sqsubseteq c$ and $b \sqsupseteq d$. The set of order-ideals of $\mathbf{P}$ is denoted by $\mathcal{O}(\mathbf{P})$.

Theorem 3. *Let* $\mathbf{G} = (G, \leq, \circ, ^{-1})$ *be a po-groupoid. Then* $(\mathcal{O}(\mathbf{G}), \cap, \cup, \to, \top, \emptyset, \cdot, 1, \sim)$ *is a CyGBI-algebra, where* $X \to Y = \mathcal{O}(\mathbf{G}) - {\uparrow}(X - Y)$, $X \cdot Y = {\downarrow}\{x \cdot y : x \in X, y \in Y\}$, $1 = {\downarrow}\{x \circ x^{-1} : x \in G\}$ *and* ${\sim}X = \mathcal{O}(\mathbf{G}) - \{x^{-1} : x \in X\}$.

For example, for a poset $\mathbf{P} = (P, \sqsubseteq)$ the weakening relation algebra $\mathbf{Wk}(\mathbf{P})$ is obtained from the po-pair-groupoid $\mathbf{G} = \mathbf{P} \times \mathbf{P}^\partial$, and for an equivalence relation $Q \subseteq P^2$, $\mathbf{Wk}(\mathbf{P}, Q)$ is obtained from the sub-po-groupoid $(Q, \leq, \circ, ^{\smile})$. If one takes the 2-element chain $\mathbf{P} = \mathbf{C}_2 = (\{0, 1\}, \sqsubseteq)$ with the usual order $0 \sqsubseteq 1$, then $P^2 = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$ and

$$\mathrm{Wk}(\mathbf{C}_2) = \{\emptyset, \{(0, 1)\}, \{(0, 0), (0, 1)\}, \{(0, 1), (1, 1)\}, \{(0, 0), (0, 1), (1, 1)\}, P^2\}.$$

Theorem 4. *For an n-element chain $\mathbf{C}_n$ the algebra $\mathbf{Wk}(\mathbf{C}_n)$ has cardinality $\binom{2n}{n}$.*

References

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