## Modular proof theory for axiomatic extensions and expansions of lattice logic

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Lattice logic is the  $\{\wedge, \vee, \top, \bot\}$ -fragment of classical propositional logic without distributivity. Lattice logic is captured by a basic Gentzen-style sequent calculus (cf. e.g. [18]), which we refer to as L0. Such a calculus has the usual rules of Identity (restricted to atomic formulas with empty contexts on both sides of the sequent), Cut (with empty contexts on both sides of the sequents) and the standard introduction rules for the logical connectives in additive form.<sup>1</sup> L0 is perfectly adequate as a proof calculus for lattice logic, when this logic is regarded in isolation. However, the main interest of lattice logic lays in it serving as base for a variety of logics, which are either its axiomatic extensions (e.g. the logics of modular and distributive bounded lattices and their variations [16]), or its proper language-expansions (e.g. the full Lambek calculus [17, 8], bilattice logic [2], orthologic [9], linear logic [15]). Hence, it is sensible to require of an adequate proof theory of lattice logic to be able to account in a modular way for these logics as well. A source of nonmodularity arises from the fact that L0 lacks structural rules. Indeed, the additive formulation of the introduction rules of L0 encodes the information which is stored in standard structural rules such as weakening, contraction, associativity, and exchange. Hence, one cannot use L0 as a base to capture logics aimed at 'negotiating' these rules, such as the Lambek calculus [17] and other substructural logics [8].

To remedy this, in [10] the first and the fourth author introduced two sequent calculi, which we refer here as L1 and L2. L1 is a sequent calculus that adopts the *visibility*<sup>2</sup> principle isolated by Sambin, Battilotti and Faggian in [19] to formulate a general strategy for cut elimination. L2 is a sequent calculus which enjoys the *display*<sup>3</sup> principle isolated by Belnap in [1]. *Properness* (i.e. closure under uniform substitution of all parametric parts in rules, see [20]) is the main interest and added value of L2 and allows for the smoothest Belnap-style proof of cut-elimination. The second attempt is motivated by and embeds in a more general theory—that of the so-called *proper multi-type calculi*, introduced in [13, 5, 6, 4] and further developed in [7, 3, 14, 11]—which creates a proof-theoretic environment designed on the basis of algebraic and order-theoretic insights (see [12]), which aims at encompassing in a uniform and modular way a very wide range of non-classical logics, spanning from dynamic epistemic logic, PDL, and inquisitive logic to lattice-based substructural (modal) logics. Proper multi-type calculi are a natural generalization of Belnap's display calculi [1] (later refined by Wansing's notion of proper display calculi [20]), the salient features of which they inherit. L1 and L2 have a structural language and the introduction rules for the logical connectives are formulated in *multiplicative* form.<sup>4</sup> This more general formulation of the introduction rules implies that

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 $<sup>^{1}</sup>$ A logical rule is in additive form if each occurrence of non-active formulas in the conclusion occurs in each premise and conversely (in the literature such rules are also called context-sharing rules). Moreover, in the unary introduction rules for conjunction and disjunction only one immediate subformula of the principal formula appears as active formula in the premise. An introduction rule for the logical connectives is in multiplicative form if each occurrence of non-active formulas in the conclusion occurs in exactly one premise and conversely (in the literature such rules are also called context-splitting rules). Moreover, in the unary introduction rules for conjunction and disjunction both immediate subformulas of the principal formula appear as active formulas in the premise.

 $<sup>^{2}</sup>$ A sequent calculus verifies the *visibility* property if both the auxiliary formulas and the principal formula of the introduction rules for the logical connectives occur in an *empty* context.

 $<sup>{}^{3}</sup>A$  sequent calculus verifies the *display* property if each substructure can be isolated on exactly one side of the turnstile by means of structural rules. Notice that display property implies visibility, but not vice versa.

<sup>&</sup>lt;sup>4</sup>The multiplicative form of the introduction rules is the most important aspect in which L1 departs from the calculus of [19]. Indeed, the introduction rules for conjunction and disjunction in [19] are in additive form.

the structural rules of weakening, exchange, associativity, and contraction are not anymore subsumed by the introduction rules. L1 and L2 are more uniform and modular compared to L0 in a precise sense. All these calculi block the derivation of the distributivity axiom, as well as of any other weaker form of distributivity. However, in the literature there are no instances of analytic sequent calculi in which axiomatic extension of lattice logic which are weaker than distributive lattice logic are captured using structural rules.

In this talk I will expand on an ongoing work on modular proof theory for axiomatic extensions and expansions of lattice logic. In particular, I will present a sequent calculus enjoying a weaker form of visibility that derives the modularity axiom but still blocks distributivity, thanks to a generalized form of the binary logical rules for conjunction and disjunction.

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