Bimodal Bilattice Logic

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Many-valued modal logics provide a natural formalisation of reasoning with modal notions such as knowledge or action in contexts where the two-valued classical picture is not sufficient. Such contexts typically involve reasoning with incomplete, inconsistent or graded information.

A prominent example of a (non-modal) many-valued logic designed to deal with incomplete and incosistent information is is the Dunn–Belnap four-valued logic [4, 2, 3]. Ginsberg [7] generalized the Dunn–Belnap four-valued matrix *FOUR* by introducing the notion of a *bilattice* and shows that bilattices emerge naturally in many computer science applications; see also [5, 6].

Formally, bilattices are sets equipped with two partial orders $\leq_t$ (the "truth order") and $\leq_i$ (the "information order") that both satisfy the lattice properties (plus other assumptions that need not be discussed now). Intuitively, $\leq_t$ orders members of a bilattice with respect to how *truthful* they are; $\leq_i$ orders them with respect to *how much information* they represent. For instance, in Belnap's four-valued matrix the value "true" is above the value "both" with respect to $\leq_t$ but below it with respect to $\leq_i$.

Arieli and Avron [1] study a (non-modal) logic based on bilattices using the full language $\{\wedge, \vee, t, f, \otimes, \oplus, \bot, \top, \neg, -, \supset\}$ containing constants for maximal $(\top, t)$ / minimal $(\bot, f)$ elements and suprema $(\vee, \oplus)$ / infima $(\wedge, \otimes)$ operators for both of the orderings, with two negations $(\neg, -)$ and an implication connective $(\supset)$.

Several modal extensions of Dunn–Belnap and Arieli–Avron have been studied recently [9, 8, 10]. These modal extensions add a modal operator $\Box$ to either the full Arieli–Avron language [8, 10] or to its fragment $\{\wedge, \vee, \neg, f, \supset\}$ [9]. The operator $\Box$ is interpreted in terms of the truth-order infimum (simplifying a bit, the value of $\Box\phi$ in world w of a Kripke model is the truth-order infimum of the values of ϕ in worlds w' accessible from w.)

However, a modal operator $\Box_i$ corresponding to the information-order infimum is a natural addition to consider. If worlds in a Kripke model are seen as "sources" of information, then the value of $\Box_i\phi$ at w is the *minimal information* about ϕ on which all the sources agree. If accessible worlds are seen as possible outcomes of some information-modifying operation (such as adding or removing information), then the value of $\Box_i\phi$ at w is the minimal information about ϕ that is guaranteed to be preserved by the operation. (This extension is briefly considered but not pursued in [8, 10]).

The present paper studies the bimodal bilattice logic arising from such an extension. It is well known that $\Box_i$ is expressible in any language extending $\{\wedge, \vee, \neg, \bot, \Box\}$; define $\Box_i\phi := (\bot \wedge \neg\Box\neg\phi) \vee \Box\phi$. We focus here on the case where $\bot$ is not available and extend the modal language used in [9] with $\Box_i$. For the sake of simplicity, we use Belnap's *FOUR* as our bilattice of truth values (the non-modal logic of arbitrary bilattices is identical to the the non-modal logic of *FOUR*, [1]).

Our main technical result is a sound and complete axiomatization. The axiomatization reflects the fact that $\Box_i\phi$ has a designated value (i.e. one of $\top, t$) iff $\Box\phi$ has a designated value; but $\Box_i$ is distinctive in the context of negation. More specifically, we add the following axioms to the non-modal base: $\Box\phi \equiv \Box_i\phi$, $\Box\neg\phi \equiv \neg\Box_i\phi$, $(\neg\Box\phi \supset f) \equiv \Box(\neg\phi \supset f)$, $\Box t$,

$(\Box\phi \wedge \Box\psi) \supset \Box(\phi \wedge \psi)$, together with the inference rule $\dfrac{\phi \supset \psi}{\Box\phi \supset \Box\psi}$.

Potential applications of the logic in knowledge representation and expressiveness of the language are discussed as well. The work done in this paper is preliminary – a version of the framework with many-valued accessibility is a topic for future research.

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