Filtrations for many-valued modal logic with applications

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Introduction. The methods of filtration and selective filtration are among the oldest and best known techniques for obtaining finite models in modal logic. In the present work we define the filtration construction in the context of many-valued modal logics with arbitrary residuated lattices as truth spaces. We prove an accompanying filtration theorem and show that many-valued filtrations exist by exhibiting the smallest and largest filtrations satisfying the definition. Next, we apply filtrations to show that certain natural many-valued analogues of **T**, **K4** and **S4** have the strong finite model property. A more challenging example perhaps is the many valued analogue of Gödel-Löb logic. We show that this logic is characterized by the class of all finite MV-frames which satisfy a certain many-valued version of transitivity and which contain no infinite non-0 paths. As in the classical case, this latter result requires the use of a selective filtration construction.

Many valued modal logic. In [2, 3] Fitting introduced a family of many-valued modal logics over Heyting algebras where both the valuation and the accessibility relations of the associated Kripke models are many-valued. This can be generalized by replacing Heyting algebras with residuated lattices, as is done in e.g. [1]. We follow this framework. Formulas are given by the following recursion: $\varphi := \bot \mid p \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \varphi \to \varphi \mid \Diamond\varphi \mid \Box\varphi$, with p from a denumerably infinite set PROP of proposition letters. Let $\mathbf{A} = (A, \wedge, \vee, \circ, \to, 1, 0)$ be a residuated lattice. In other words, the reduct $(A, \wedge, \vee, 1, 0)$ is a bounded lattice while the reduct $(A, \circ, 1)$ is a commutative monoid, and moreover $\to$ is the right residual of $\circ$, i.e. $a \circ b \leq c$ iff $a \leq b \to c$ for all $a, b, c \in A$. An $\mathbf{A}$-frame is a triple $\mathfrak{F} = (W, D, B)$ with a non-empty universe W and $\mathbf{A}$-valued accessibility relations $D : W \times W \to \mathbf{A}$ and $B : W \times W \to \mathbf{A}$. An $\mathbf{A}$-model is a pair $\mathfrak{M} = (\mathfrak{F}, V)$ where $\mathfrak{F}$ is an $\mathbf{A}$-frame and $V : \mathsf{PROP} \times W \to \mathbf{A}$ is an $\mathbf{A}$-valued valuation. The valuation can be extended to all formulas. In particular,

$$V(\Diamond\varphi, w) = \bigvee\{D(w, v) \circ V(\varphi, v) \mid v \in W\} \text{ and}$$
$$V(\Box\varphi, w) = \bigwedge\{B(w, v) \to V(\varphi, v) \mid v \in W\}.$$

Let $a \in \mathbf{A}$, then a formula φ is said to be *a-true* in a model at $w \in W$, denoted by $\mathfrak{M}, w \Vdash_a \varphi$, if $V(\varphi, w) \geq a$.

Filtrations. Given a subformula-closed set of formulas Σ, define an equivalence relation $\leftrightsquigarrow_\Sigma$ on an $\mathbf{A}$-model $\mathfrak{M} = (W, D, B, V)$ such that $w \leftrightsquigarrow^{\mathbf{A}}_\Sigma v$ iff $V(\varphi, w) = V(\varphi, v)$ for all $\varphi \in \Sigma$. Let $[w]^{\mathbf{A}}_\Sigma$ denote the equivalence class of $w \in W$ under $\leftrightsquigarrow^{\mathbf{A}}_\Sigma$.

Definition 1. *Let $W_\Sigma = \{[w]_\Sigma \mid w \in W\}$. Let $\mathfrak{M}_f = (W_\Sigma, \mathrm{D}_f, \mathrm{B}_f, V_\Sigma)$ be any model such that:*

(R1) *Let $a \in \mathbf{A}$. If $\mathrm{D}\, wv \geq a$, then $\mathrm{D}_f[w][v] \geq a$.*
(R2) *Let $a, a' \in \mathbf{A}$. If $\mathrm{D}_f[w][v] \geq a$, then for every $\Diamond\varphi \in \Sigma$, if $\mathfrak{M}, v \Vdash_{a'} \varphi$, then*

$\mathfrak{M}, w \Vdash_{a \circ a'} \Diamond\varphi$.
(R3) *Let* $a \in \mathbf{A}$. *If* $\mathrm{B}\,wv \geq a$, *then* $\mathrm{B}_f[w][v] \geq a$.
(R4) *Let* $a, a' \in \mathbf{A}$. *If* $\mathrm{B}_f[w][v] \geq a$, *then for every* $\Box\varphi \in \Sigma$, *if* $\mathfrak{M}, w \Vdash_{a'} \Box\varphi$, *then* $\mathfrak{M}, v \Vdash_{a \circ a'} \varphi$.
(V) $V_\Sigma([w], p) = V(w, p)$ *for all* $p \in \Sigma$.

Then $\mathfrak{M}_f$ *is called an* **A**-valued filtration of $\mathfrak{M}$ through Σ.

Theorem 2 (**A**-valued Filtration Theorem)**.** *Let* $\mathfrak{M}_f = (W_\Sigma, \mathrm{D}_f, \mathrm{B}_f, V_\Sigma)$ *be a filtration of* $\mathfrak{M}$ *through a subformula closed set* Σ *over* **A**. *Then, for all formulas* $\varphi \in \Sigma$, *all states* w *in* $\mathfrak{M}$ *and any truth value* $a \neq 0$ *in* **A**, *we have that* $\mathfrak{M}, w \Vdash_a \varphi \iff \mathfrak{M}_f, [w]^{\mathbf{A}}_{\Sigma} \Vdash_a \varphi$. *Moreover, if* **A** *and* Σ *are both finite, then so is* $\mathfrak{M}_f$.

We show that **A**-valued filtrations exist by exhibiting the smallest and largest filtrations (in the sense of producing, respectively, the smallest and largest relations D_f and B_f in terms of the order of **A**) satisfying the definition. We also define a filtration which preserves a-transitivity of models (see below).

Applications: Many-valued modal logics with the FMP. In this section, for simplicity, we restrict to **A**-frames $\mathfrak{F} = (W, D, B)$ where $D = B$ which we notate as $\mathfrak{F} = (W, R)$. Moreover, we will assume that **A** is a finite Heyting algebra. In [1] Bou et. al. axiomatize the logic $\Lambda(\mathsf{Fr}, \mathbf{A}^c)$ of the class of all **A**-frames. In analogue to the classical case, let $\mathbf{T}(\mathbf{A})$, $\mathbf{K4}(\mathbf{A})$ and $\mathbf{S4}(\mathbf{A})$ be the logics obtained by adding to to the system of Bou et. al. the axioms $\Box p \to p$ and $\Box p \to \Box\Box p$ individually and in combination. Let **gl** be the Löb formula $\Box(\Box p \to p) \to \Box p$ and let $\mathbf{GL}(\mathbf{A})$ be the logic obtained by adding **gl** and **4** to an axiomatization of $\Lambda(\mathsf{Fr}, \mathbf{A}^c)$.

An **A**-frame $\mathfrak{F} = (W, R)$ is a-reflexive if $Rwv \geq a$ for all $w, v \in W$; it is a-transitive if $a \leq (Rwv \wedge Rvu \to Rwu)$ for all $w, v, u \in W$. It follows that $\mathfrak{F}$ is 1 transitive iff $Rwv \wedge Rvu \leq Rwu$ for all $w, v, u \in W$. A non-0 path in $\mathfrak{F}$ is a finite or infinite sequence $w_0, w_1, \ldots$ such that $Rw_iw_{i+1} > 0$ for all $i \geq 0$.

Lemma 3. *The axioms* $\Box p \to p$ *and* $\Box p \to \Box\Box p$ *are canonical for* 1*-reflexivity and* 1*-transitivity, i.e., the canonical models (see [1]) of* $\mathbf{T}(\mathbf{A})$, $\mathbf{K4}(\mathbf{A})$ *are* 1*-reflexive and* 1*-transitive, respectively.*

Now we can obtain the following theorem by judicious application of filtrations.

Theorem 4. $\mathbf{T}(\mathbf{A})$ *and* $\mathbf{K4}(\mathbf{A})$ *are characterized by the classes of all finite* 1*-reflexive and* 1*-transitive* **A***-frames, respectively.* $\mathbf{S4}(\mathbf{A})$ *is by the class of all finite* 1*-reflexive and* 1*-transitive* **A***-frames.*

A more intricate argument, combining transitive and selective filtration, establishes the following analogue of the well-known classical result.

Theorem 5. *Let* **A** *be a finite Heyting chain. Then* $\mathbf{GL}(\mathbf{A})$ *is determined by the class of finite,* 1*-transitive* **A***-frames with no infinite non-*0 *paths.*

References

[1] Félix Bou, Francesc Esteva, Lluis Godo, and Ricardo Oscar Rodríguez. On the minimum many-valued modal logic over a finite residuated lattice. *Journal of Logic and Computation*, 21(5):739–790, 2011.

[2] M.C. Fitting. Many-valued modal logic. *Fundamenta Informaticae*, 15:235–254, 1991.

[3] M.C. Fitting. Many-valued modal logic II. *Fundamenta Informaticae*, 17:55–73, 1992.