Unification in first order logics: superintuitionistic and modal.

Wojciech Dzik[1] and Piotr Wojtylak[2]

[1] University of Silesia, Katowice, Poland
wojciech.dzik@us.edu.pl
[2] University of Opole, Opole, Poland
piotr.wojtylak@math.uni.opole.pl

1. Introduction. We introduce and apply unification in predicate logics that extend intuitionistic predicate logic Q-INT and modal predicate logic Q-S4 (or Q-K4). S. Ghilardi succesfully applied unification in propositional logic [5], [6], [7]. We show that unification in $L \supseteq$ Q-INT is projective iff $L \supseteq$ P.Q–LC, Gödel-Dummett's predicate logic plus Plato's Law (in modal case: $L \supseteq$ mP.Q-S4.3); hence, such L is almost structurally complete: each admissible rule is either derivable or passive and unification in L is unitary. We provide an explicit basis for all passive rules in Q-INT (Q-S4). We show that every unifiable Harrop's formula is projective and we extend the classical results of Kleene (on disjunction and existence quantifier under implication) to projective formulas and to all extensions of $\mathsf{Q} - \mathsf{INT}$. Rules that are admissible in all extensions of Q-INT are given. We prove that L has filtering unification iff L extends Q-KC: = Q-INT + $(\neg A \vee \neg\neg A)$ (Q-K4.2$^+$), and that unification in Q-LC, Q-KC (Q-S4.3, Q-S4.2) is nullary and in Q-INT (Q-S4) it is not finitary, contrary to the propositional cases.

Q-L denotes the least predicate logic extending a propositional logic L, e.g. Q-CL, Q-INT, Q-S4. We follow the axioms and notation of [2], [3]. We consider a standard *first-order* (or *predicate*) language $\{\to, \wedge, \vee, \bot, \forall, \exists\}$ (plus modal $\Box, \Diamond$) with *free individual variables*: $\{a_1, a_2, a_3, \dots\}$, *bound individual variables*: $\{x_1, x_2, x_3, \dots\}$, *predicate variables*: $Pr = \{P_1, P_2, P_3, \dots\}$; no function symbols or $=$. *Formulas* (Fm) are q-formulas (q-Fm) in which no bound variable occurs free. A 2^{nd}-order *substitution for predicate variables* is used.

2. Unifiability. A basis for passive rules. A *unifier* for A in a logic L is a substitution (for predicate variables) τ making A a theorem of L, i.e. $\tau(A) \in L$. A formula A is *unifiable* in L (L-unifiable) if it has a unifier in L. A unifier $v\colon Pr \to \{\bot, \top\}$ is called *ground*. Note: (i) A is L-unifiable iff (ii) there is a ground unifier for A in L iff (iii) A is valid in a classical model with 1-element universe. Hence unifiability is *absolute*. Note: Unifiable $\neq$ Consistent. A rule A/B is *passive*, if A is not unifiable. Consider the following (schematic) rules:

$$(P\forall): \quad \frac{\neg\forall_{\overline{z}} C(\overline{z}) \wedge \neg\forall_{\overline{z}} \neg C(\overline{z})}{\bot} \qquad \Big((\Box P\forall): \quad \frac{\Diamond\exists_{\overline{x}} A(\overline{x}) \wedge \Diamond\exists_{\overline{x}} \neg A(\overline{x})}{\bot}\Big)$$

Theorem 1. *$P\forall$ ($\Box P\forall$) form a basis for all passive rules over* Q–INT *(* Q–K4D*)*

3. Projective unification and Harrop formulas. A unifier ε for a formula A in a logic L is *projective* if $\vdash_L (\Box)A \to \forall_{x_1,\dots,x_n}(\varepsilon(P_i(x_1,\dots,x_n)) \leftrightarrow P_i(x_1,\dots,x_n))$, for each predicate variable P_i. A logic L enjoys *projective unification* if each L-unifiable formula has a projective unifier. P.Q-LC (mP.Q-S4.3) denotes the Gödel-Dummett (S4.3 modal) predicate logic extended with the following formula called *(modal) Plato's Law*
(P): $\exists_x(\exists_x B(x) \to B(x))$, *(mP)*: $\exists_x \Box(\exists_x \Box B(x) \to B(x))$.

Theorem 2. *A superintuitionistic predicate logic L enjoys projective unification if and only if* P.Q-LC $\subseteq L$ *. If a modal logic L enjoys projective unification, then* mP.Q-S4.3 $\subseteq L$.

Corollary 3. *Every logic containing* $\mathsf{P.Q\text{-}LC}$ *is almost structurally complete i.e. every admissible rule is either derivable or passive.*

Corollary 4. $\mathsf{P.Q\text{-}LC}$ *is the least logic* $L \supseteq \mathsf{Q\text{-}INT}$ *in which* $\vee$ *and* $\exists$ *is definable by* $\wedge, \rightarrow, \forall$.

Theorem 5. *For an infinite rooted Kripke frame* $\mathcal{F} =< W, \leqslant, \mathcal{D} >$, $\mathsf{(m)P}$ *is valid in* $\mathcal{F}$ *iff* $\mathcal{F}$ *has constant domain* $\mathcal{D}$ *and* W *is well (quasi-)ordered.* $\mathsf{IP.Q\text{-}LC}$ *(*$\mathsf{mP.Q\text{-}S4.3}$*) is Kripke incomplete.*

Harrop q-formulas $q\text{-}Fm_H$ (or Harrop formulas Fm_H) are defined by the clauses:
1. all elementary q-formulas (including $\bot$) are Harrop; 2. if $A, B \in q\text{-}Fm_H$, then $A \wedge B \in q\text{-}Fm_H$; 3. if $B \in q\text{-}Fm_H$, then $A \rightarrow B \in q\text{-}Fm_H$; 4. if $B \in q\text{-}Fm_H$, then $\forall_{x_j} B \in q\text{-}Fm_H$.

Theorem 6. *Any unifiable Harrop's formula is projective in* $\mathsf{Q\text{–}INT}$.

Theorem 7. *For any L-projective sentence* A *and any formulas* $B_1, B_2, \exists_x C(x)$, *we have*
(i) *if* $\vdash_L A \rightarrow B_1 \vee B_2$, *then* $\vdash_L (A \rightarrow B_1) \vee (A \rightarrow B_2)$,
(i)' *if* $\vdash_L A \rightarrow \Box B_1 \vee \Box B_2$, *then* $\vdash_L \Box(\Box A \rightarrow B_1) \vee \Box(\Box A \rightarrow B_2)$, *(in the modal case),*
(ii) *if* $\vdash_L A \rightarrow \exists_x C(x)$, *then* $\vdash_L \exists_x (A \rightarrow C(x))$,
(ii)' *if* $\vdash_L A \rightarrow \exists_x \Box C(x)$, *then* $\vdash_L \exists_x \Box(\Box A \rightarrow C(x))$, *(in the modal case).*

Example: The following non-passive rule is admissible in every predicate logic $L \supseteq \mathsf{Q\text{-}INT}$:
$\neg(\exists_x P(x) \wedge \exists_x \neg P(x)) \rightarrow \exists_y Q(y) \;/\; \exists_y [\neg(\exists_x P(x) \wedge \exists_x \neg P(x)) \rightarrow Q(y)]$.

4. Filtering unification and unification types. Recall: σ is *more general than* τ, if $\vdash_L \tau(x) \leftrightarrow \theta(\sigma(x))$, for some substitution θ (σ, τ are defined on finite sets of variables). A *most general unifier, mgu*, for a formula A is a unifier that is more general than any unifier for A. Unification in L is *unitary*, 1, if every L-unifiable formula has a mgu. The other unification types: *finitary, infinitary* and *nullary*, 0, depend on the number of maximal unifiers see [1]. [7] characterized modal logics in which unification is *filtering*, that is, for every two unifiers for a formula there is another unifier that is more general than both of them, (type 1 or 0).

Theorem 8. *Let* L *be a superintuitionistic predicate logic (modal logic extending* $\mathsf{Q\text{-}K4}$*). Unification in* L *is filtering iff the Stone law* $\neg\neg A \vee \neg A$ *(*$2^+ : \Diamond^+\Box^+ A \rightarrow \Box^+\Diamond^+ A$*) is in* L.

Corollary 9. *For every superintuitionistic (modal) predicate logic* L *(containing* $\mathsf{Q\text{-}S4}$*)*
(i) if $\mathsf{Q\text{-}KC} \subseteq L$ *(*$\mathsf{Q\text{-}S4.2} \subseteq L$*), then unification in* L *is either unitary or nullary;*
(ii) if L *enjoys unitary unification, then* $\mathsf{Q\text{-}KC} \subseteq L$ *(*$\mathsf{Q\text{-}S4.2} \subseteq L$*).*

Corollary 10. *Unification in* $\mathsf{Q\text{-}LC}$, $\mathsf{Q\text{-}KC}$ *(*$\mathsf{Q\text{-}S4.3}$, $\mathsf{Q\text{-}S4.2}$*) is nullary and in* $\mathsf{Q\text{-}INT}$ *(*$\mathsf{Q\text{-}S4}$*) it is either infinitary or nullary, contrary to the corresponding propositional cases.*

References

[1] Baader F., Ghilardi S., Unification in Modal and Description Logics, *Logic Journal of the IGPL*, 19(5)(2011), 705-730.

[2] Braüner T., Ghilardi S., First-order Modal Logic, in Blackburn P., (ed.) *Handbook of Modal Logic* 2007 Elsevier, 549–620.

[3] Corsi G., Ghilardi S., Semantical aspects of quantified modal logic, in *Knowledge, Belief, and Strategic Interaction* C Bicchieri, M. L. Dalla Chiara (eds.), (1992), Cambridge Univ. P. 167–195.

[4] Dzik W., Wojtylak P., Unification in Superintuitionistic Predicate Logics, *submitted in 2016.*

[5] Ghilardi S., Unification in Intuitionistic Logic, *Journal of Symbolic Logic* 64(2) (1999), 859–880.

[6] Ghilardi S., Best Solving Modal Equations, *Annals of Pure and Applied Logic* 102 (2000), 183–198.

[7] Ghilardi S., Sacchetti, L., Filtering unification and most general unifiers in modal logic, *The Journal of Symbolic Logic* 69(3) (2004), 879–906.