Structure Theorem for a Class of Group-like Residuated Chains à la Hahn

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Hahn’s structure theorem [2] states that totally ordered Abelian groups can be embedded in the lexicographic product of real groups. Residuated lattices [7] are semigroups only, and are algebraic counterparts of substructural logics [1]. Involutive commutative residuated chains (aka. involutive FLe-chains) form an algebraic counterpart of the logic IUL [6]. The focus of our investigation is a subclass of them, called commutative group-like residuated chains. Commutative, group-like residuated chains are totally ordered, involutive commutative residuated lattices such that the unit of the monoidal operation coincides with the constant that defines the involution. The latest postulate forces the structure to resemble totally ordered Abelian groups with finitely many idempotents. The result extends the famous structural description of totally ordered Abelian groups by Hahn [2], to order-dense group-like commutative residuated chains with finitely many idempotents. The result is quite surprising.

Theorem 1. For a group-like FLe-algebra \((X, \land, \lor, \ast, \rightarrow, t, f)\) the following statements are equivalent: \((X, \land, \lor, \ast, \rightarrow, t, f)\) is a lattice-ordered Abelian group if and only if \(\ast\) is cancellative if and only if \(x \rightarrow_x x = t\) for all \(x \in X\) if and only if the only idempotent element in the positive cone of \(X\) is \(t\).

Definition 1. (Partial-lexicographic products) Let \(X = (X, \land_X, \lor_X, \ast, \rightarrow, t_X, f_X)\) be a group-like FLe-algebra and \(Y = (Y, \land_Y, \lor_Y, \ast, \rightarrow, t_Y, f_Y)\) be an involutive FLe-algebra, with residual complement \('\) and \(''\), respectively. Add a top element \(T\) to \(Y\), and extend \(\ast\) by \(T \ast y = y \ast T = T\) for \(y \in Y \cup \{T\}\), then add a bottom element \(\bot\) to \(Y \cup \{\bot\}\), and extend \(\ast\) by \(\bot \ast y = y \ast \bot = \bot\) for \(y \in Y \cup \{\bot\}\). Let \(X_1 = (X_1, \land_X, \lor_X, \ast, \rightarrow, t_X, f_X)\) be any cancellative subalgebra of \(X\) (by Theorem 1, \(X_1\) is a lattice ordered group). We define \(X_{\Gamma(X_1, Y, T)} = (X_{\Gamma(X_1, Y, T)}, \leq, \ast, \rightarrow, (t_X, t_Y), (f_X, f_Y))\), where \(X_{\Gamma(X_1, Y, T)} = (X_1 \times (Y \cup \{\bot, T\})) \cup ((X \setminus X_1) \times \{\bot\})\), \(\leq\) is the restriction of the lexicographic order of \(\leq_X\) and \(\leq_{Y \cup \{\bot, T\}}\).

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to $X_{\Gamma(X_1,Y\uparrow)}$, $\ast$ is defined coordinatewise, and the operation $\to_{\ast}$ is given by $(x_1,y_1) \to_{\ast} (x_2,y_2) = ((x_1,y_1) \ast (x_2,y_2))'$ where

$$(x,y)' = \begin{cases} (x',y') & \text{if } x \in X_1, \\ (x',\perp) & \text{if } x \notin X_1. \end{cases}$$

Call $X_{\Gamma(X_1,Y\uparrow)}$ the \textit{(type-I) partial-lexicographic product} of $X, X_1$, and $Y$, respectively.

Let $X = (X, \leq_X, \ast, \to_{\ast}, t_X, f_X)$ be a group-like FL$_e$-chain, $Y = (Y, \leq_Y, \ast, \to_{\ast}, t_Y, f_Y)$ be an involutive FL$_e$-algebra, with residual complement $\r'$ and $\r'$, respectively. Add a top element $\top$ to $Y$, and extend $\ast$ by $\top \ast y = y \ast \top = \top$ for $y \in Y \cup \{\top\}$, Further, let $X_1 = (X_1, \leq, \ast, \to_{\ast}, t_X, f_X)$ be a cancellative, discrete, prime\footnote{Call a subalgebra $(X_1, \leq_X, \ast, \to_{\ast}, t_X, f_X)$ of an FL$_e$-algebra $(X, \leq_X, \ast, \to_{\ast}, t_X, f_X)$ prime if $(X \setminus X_1) \ast (X \setminus X_1) \subseteq X \setminus X_1$.} subalgebra of $X$ (by Theorem 1, $X_1$ is a discrete lattice ordered group). We define $X_{\Gamma(X_1,Y\downarrow)} = (X_{\Gamma(X_1,Y\uparrow)}, \leq, \ast, \to_{\ast}, (t_X, t_Y), (f_X, f_Y))$, where $X_{\Gamma(X_1,Y\uparrow)} = (X_1 \times (Y \cup \{\top\})) \cup ((X \setminus X_1) \times \{\top\})$, $\leq$ is the restriction of the lexicographic order of $\leq_X$ and $\leq_Y \cup \{\top\}$ to $X_{\Gamma(X_1,Y)}$, $\ast$ is defined coordinatewise, and the operation $\to_{\ast}$ is given by $(x_1,y_1) \to_{\ast} (x_2,y_2) = ((x_1,y_1) \ast (x_2,y_2))'$ where

$$(x,y)' = \begin{cases} ((x'), \top) & \text{if } x \notin X_1 \text{ and } y = \top, \\ ((x'), y') & \text{if } x \in X_1 \text{ and } y \in Y, \\ ((x'), \perp) & \text{if } x \in X_1 \text{ and } y = \top. \end{cases}$$

2 Call $X_{\Gamma(X_1,Y\downarrow)}$ the \textit{(type-II) partial-lexicographic product} of $X, X_1$, and $Y$, respectively.

**Theorem 2.** $X_{\Gamma(X_1,Y\uparrow)}$ and $X_{\Gamma(X_1,Y\downarrow)}$ are involutive FL$_e$-algebras. If $Y$ is group-like then also $X_{\Gamma(X_1,Y\uparrow)}$ and $X_{\Gamma(X_1,Y\downarrow)}$ are group-like.

**Theorem 3.** Any order-dense group-like FL$_e$-chain which has only a finite number of idempotents can be built by iterating finitely many times the partial-lexicographic product constructions using only totally ordered groups, as building blocks. More formally, let $X$ be an order-dense group-like FL$_e$-chain which has $n \in \mathbb{N}$ idempotents in its positive cone. Denote $I = \{\bot, \top\}$. For $i \in \{1,2,\ldots,n\}$ there exist totally ordered Abelian groups $G_i, H_i \leq \Gamma(H_{i-1}, G_i)$ ($i \in \{2,\ldots,n-1\}$), and a binary sequence $i \in \{2,\ldots,n\}$ such that $X \simeq X_n$, where $X_1 := G_1$ and $X_i := \bigcap_{i-1}^{i-1} \Gamma(H_{i-1}, G_{i-1})$ ($i \in \{2,\ldots,n\}$).

**References**


