Structure Theorem for a Class of Group-like Residuated Chains à la Hahn

Sándor Jenei*

University of Pécs, Pécs, Hungary
jenei@ttk.pte.hu

Hahn's structure theorem [2] states that totally ordered Abelian groups can be embedded in the *lexicographic product* of *real groups*. Residuated lattices [7] are semigroups only, and are algebraic counterparts of substructural logics [1]. Involutive commutative residuated chains (aka. involutive FL_e-chains) form an algebraic counterpart of the logic **IUL** [6]. The focus of our investigation is a subclass of them, called commutative *group-like* residuated chains. Commutative, group-like residuated chains are totally ordered, involutive commutative residuated lattices such that the unit of the monoidal operation coincides with the constant that defines the involution. The latest postulate forces the structure to resemble totally ordered Abelian groups in many ways. Firstly, similar to lattice-ordered Abelian groups, for complete, densely ordered, group-like FL_e-chains the monoidal operation can be recovered from its restriction solely to its positive cone (see [3, Theorem 1]). Secondly, group-like commutative residuated chains can be characterized as generalizations of totally ordered Abelian groups by weakening the strictly-increasing nature of the partial mappings of the group multiplication by nondecreasing behaviour, see Theorem 1. Thirdly, in quest for establishing a structural description for commutative group-like residuated chains à la Hahn, "partial-lexicographic product" constructions will be introduced. Roughly, only a cancellative subalgebra of a commutative group-like residuated chain is used as a first component of a lexicographic product, and the rest of the algebra is left unchanged. This results in group-like FL_e-algebras, see Theorem 2. The main theorem is about the structure of order-dense group-like FL_e-chains with a finite number of idempotents: Each such algebra can be constructed by iteratively using the partial-lexicographic product constructions using totally ordered Abelian groups as building blocks, see Theorem 3. This result extends the famous structural description of totally ordered Abelian groups by Hahn [2], to order-dense group-like commutative residuated chains with finitely many idempotents. The result is quite surprising.

Theorem 1. *For a group-like FL_e-algebra $(X, \wedge, \vee, \circledast, \to_{\circledast}, t, f)$ the following statements are equivalent: $(X, \wedge, \vee, \circledast, t)$ is a lattice-ordered Abelian group if and only if $\circledast$ is cancellative if and only if $x \to_{\circledast} x = t$ for all $x \in X$ if and only if the only idempotent element in the positive cone of X is t.*

Definition 1. *(Partial-lexicographic products)* Let $\mathbf{X} = (X, \wedge_X, \vee_X, *, \to_*, t_X, f_X)$ be a group-like FL_e-algebra and $\mathbf{Y} = (Y, \wedge_Y, \vee_Y, \star, \to_\star, t_Y, f_Y)$ be an involutive FL_e-algebra, with residual complement $'^*$ and $'^\star$, respectively. Add a top element $\top$ to Y, and extend $\star$ by $\top \star y = y \star \top = \top$ for $y \in Y \cup \{\top\}$, then add a bottom element $\bot$ to $Y \cup \{\top\}$, and extend $\star$ by $\bot \star y = y \star \bot = \bot$ for $y \in Y \cup \{\bot, \top\}$. Let $\mathbf{X_1} = (X_1, \wedge_X, \vee_X, *, \to_*, t_X, f_X)$ be any cancellative subalgebra of $\mathbf{X}$ (by Theorem 1, $\mathbf{X_1}$ is a lattice ordered group). We define $\mathbf{X_{\Gamma(X_1, Y^{\bot\top})}} = \left(X_{\Gamma(X_1,Y^{\bot\top})}, \leq, \circledast, \to_{\circledast}, (t_X, t_Y), (f_X, f_Y)\right)$, where $X_{\Gamma(X_1,Y^{\bot\top})} = (X_1 \times (Y \cup \{\bot, \top\})) \cup ((X \setminus X_1) \times \{\bot\})$, $\leq$ is the restriction of the lexicographic order of $\leq_X$ and $\leq_{Y \cup \{\bot,\top\}}$

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to $X_{\Gamma(X_1,Y^{\bot\top})}$, $\circledast$ is defined coordinatewise, and the operation $\to_\circledast$ is given by $(x_1,y_1)\to_\circledast (x_2,y_2) = ((x_1,y_1)\circledast(x_2,y_2)')'$ where

$$(x,y)' = \begin{cases} (x'^*, y'^\star) & \text{if } x \in X_1 \\ (x'^*, \bot) & \text{if } x \notin X_1 \end{cases}.$$

Call $\mathbf{X}_{\mathbf{\Gamma(X_1,Y^{\bot\top})}}$ the *(type-I) partial-lexicographic product* of X, X_1, and Y, respectively.

Let $\mathbf{X} = (X, \leq_X, *, \to_*, t_X, f_X)$ be a group-like FL$_e$-chain, $\mathbf{Y} = (Y, \leq_Y, \star, \to_\star, t_Y, f_Y)$ be an involutive FL$_e$-algebra, with residual complement $'^*$ and $'^\star$, respectively. Add a top element $\top$ to Y, and extend $\star$ by $\top\star y = y\star\top = \top$ for $y \in Y\cup\{\top\}$. Further, let $\mathbf{X_1} = (X_1, \wedge, \vee, *, \to_*, t_X, f_X)$ be a cancellative, discrete, prime[1] subalgebra of $\mathbf{X}$ (by Theorem 1, $\mathbf{X_1}$ is a discrete lattice ordered group). We define $\mathbf{X}_{\mathbf{\Gamma(X_1,Y^\top)}} = \left(X_{\Gamma(X_1,Y^\top)}, \leq, \circledast, \to_\circledast, (t_X,t_Y), (f_X,f_Y)\right)$, where $X_{\Gamma(X_1,Y^\top)} = (X_1 \times (Y\cup\{\top\})) \cup ((X\setminus X_1)\times\{\top\})$, $\leq$ is the restriction of the lexicographic order of $\leq_X$ and $\leq_{Y\cup\{\top\}}$ to $X_{\Gamma(X_1,Y)}$, $\circledast$ is defined coordinatewise, and the operation $\to_\circledast$ is given by $(x_1,y_1)\to_\circledast (x_2,y_2) = ((x_1,y_1)\circledast(x_2,y_2)')'$ where

$$(x,y)' = \begin{cases} ((x'^*),\top) & \text{if } x \notin X_1 \text{ and } y = \top \\ (x'^*, y'^\star) & \text{if } x \in X_1 \text{ and } y \in Y \\ ((x'^*)_\downarrow, \top) & \text{if } x \in X_1 \text{ and } y = \top \end{cases}.$$

[2] Call $\mathbf{X}_{\mathbf{\Gamma(X_1,Y^\top)}}$ the *(type-II) partial-lexicographic product* of X, X_1, and Y, respectively.

Theorem 2. *$\mathbf{X}_{\mathbf{\Gamma(X_1,Y^{\bot\top})}}$ and $\mathbf{X}_{\mathbf{\Gamma(X_1,Y^\top)}}$ are involutive FL$_e$-algebras. If $\mathbf{Y}$ is group-like then also $\mathbf{X}_{\mathbf{\Gamma(X_1,Y^{\bot\top})}}$ and $\mathbf{X}_{\mathbf{\Gamma(X_1,Y^\top)}}$ are group-like.*

Theorem 3. *Any order-dense group-like FL$_e$-chain which has only a finite number of idempotents can be built by iterating finitely many times the partial-lexicographic product constructions using only totally ordered groups, as building blocks. More formally, let $\mathbf{X}$ be an order-dense group-like FL$_e$-chain which has $n \in \mathbf{N}$ idempotents in its positive cone. Denote $I = \{\bot\top, \top\}$. For $i \in \{1,2,\ldots,n\}$ there exist totally ordered Abelian groups $\mathbf{G}_i$, $\mathbf{H}_1 \leq \mathbf{G}_1$, $\mathbf{H}_i \leq \mathbf{\Gamma}(\mathbf{H}_{i-1}, \mathbf{G}_i)$ $(i \in \{2,\ldots,n-1\})$, and a binary sequence $\iota \in I^{\{2,\ldots,n\}}$ such that $\mathbf{X} \simeq \mathbf{X}_n$, where $\mathbf{X}_1 := \mathbf{G}_1$ and $\mathbf{X}_i := \mathbf{X}_{i-1}{}_{\mathbf{\Gamma}(\mathbf{H}_{i-1},\mathbf{G}_i{}^{\iota_i})}$ $(i \in \{2,\ldots,n\})$.*

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[1] Call a subalgebra $(X_1, \wedge, \vee, \circledast, \to_\circledast, t_X, f_X)$ of an FL$_e$-algebra $(X, \leq_X, \circledast, \to_\circledast, t_X, f_X)$ *prime* if $(X\setminus X_1) * (X\setminus X_1) \subseteq X\setminus X_1$.

[2] $x_\downarrow = \begin{cases} u & \text{if there exists } u < x \text{ such that there is no element in } X \text{ between } u \text{ and } x, \\ x & \text{if for any } u < x \text{ there exists } v \in X \text{ such that } u < v < x \text{ holds.} \end{cases}$