On two concepts of ultrafilter extensions of first-order models and their generalizations

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There exist two known concepts of ultrafilter extensions of first-order models, both in a certain sense canonical. One of them [1] comes from universal algebra where it goes back to a seminal paper by Jónsson and Tarski [2] and also modal logic [3, 4]. Another one [5, 6] has its sources in iterated ultrapowers in model theory [7, 8, 9] and especially algebra of ultrafilters, with ultrafilter extensions of semigroups [10] as its main precursor. By a classical fact of general topology, the space of ultrafilters over a discrete space is its largest compactification. The main result of [5, 6], which confirms a canonicity of the extension introduced there, generalizes this fact to discrete spaces endowed with arbitrary first-order structure. An analogous result for the former type of ultrafilter extensions was obtained in [11].

Here we offer a uniform approach to both types of extensions. It is based on the idea to extend the extension procedure itself. We propose a generalization of the standard concept of first-order models in which functional and relational symbols are interpreted rather by ultrafilters over sets of functions and relations than by functions and relations themselves. We provide two specific operations which turn generalized models into ordinary ones, characterize the resulting ordinary models in topological terms, and establish necessary and sufficient conditions under which the latter are the two canonical ultrafilter extensions of some models. For details, we refer the reader to the forthcoming [12].

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