# Wild Algebras in Cartesian Categorical Logic

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Let $k$ denote an algebraically closed field. A "k-algebra" is a ring with compatible $k$-vector space structure. A "module" over a $k$-algebra $A$ is a $k$-vector space $M$ with an action of $A$ that is compatible with the group and vector space structures on $M$. It is known that the classical first-order theory of modules over the free algebra $k\langle X, Y\rangle$ is undecidable. More precisely, the classical first-order theory is undecidable in that there is no Turing machine algorithm that will establish whether a given sentence of the theory is a theorem [4], [1]. A $k$-algebra $S$ is "wild" if its category of modules admits a "representation embedding." This is a finitely generated $(S, k\langle X, Y\rangle)$-bimodule $M$, free over $k\langle X, Y\rangle$, such that an induced functor $M \otimes_{k\langle X,Y\rangle} -\colon k\langle X, Y\rangle\text{-mod} \to S\text{-mod}$, between categories of finite-dimensional modules, preserves and reflects indecomposability and isomorphism (p. 272 of [4]). The conjecture of M. Prest is that any finite-dimensional wild algebra has an undecidable theory of modules (p. 350 of [4]).

There are two goals of our recent work [3]. First is to reformulate the theory of modules over a fixed $k$-algebra within the cartesian fragment of first-order categorical logic as described in D1.2 of [2]; and then to prove that the cartesian theory of $k\langle X, Y\rangle$-modules in particular is undecidable. The second is to reformulate Prest's conjecture in a manner appropriate for cartesian logic and then to prove it. Our hope is that this rephrasing in cartesian categorical logic will shed light on the original problem in classical model theory. However, we make no claim to resolve the original conjecture.

That the cartesian theory of $k\langle X, Y\rangle$-modules is undecidable can be seen by adapting the proof idea of the original source [1]. That is, it can be seen that each element of a distinguished class of sequents of the cartesian theory is provable in the theory if, and only if, the two words of a corresponding pair of words of a fixed monoid with an undecidable word problem are equivalent. Thus, the module theory can be seen to interpret the undecidable word problem of a finitely presented monoid, making it undecidable as well. The affirmative result here meets the first goal stated above.

To attain the second goal, undecidability must somehow be "transmitted" between cartesian theories of modules over $k\langle X, Y\rangle$ and $S$ of the first paragraph. This is accommplished by giving what amounts to a translation between the theories. This is by the device of "syntactic categories." The syntactic category $\mathscr{C}_{\mathbb{T}}$ of a cartesian theory $\mathbb{T}$ is a categorification of the syntax of $\mathbb{T}$, described for example in D1.4 of [2]. A translation of theories $\mathbb{T} \to \mathbb{T}'$ is essentially the same thing as a functor of syntactic categories $\mathscr{C}_{\mathbb{T}} \to \mathscr{C}_{\mathbb{T}'}$. Our main result is that what we call a "representation embedding" of cartesian theories $\mathbb{T}$ and $\mathbb{T}'$ induces such a functor of syntactic categories. The functor amount to a conservative translation of theories, so that if $\mathbb{T}$ is undecidable, then so is $\mathbb{T}'$.

In a bit more detail, a representation embedding between categories of models of cartesian theories can be defined to be a functor between the categories of **Set**-models that preserves indecomposability and projectivity and that reflects epics when restricted to the full subcategory of indecomposable projective models. The main result, then, is that if there is such a representation embedding between cartesian theories say $\mathbb{T}$ and $\mathbb{T}'$, then if $\mathbb{T}$ is undecidable,

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so is $\mathbb{T}'$. Our reformulation of Prest's conjecture in cartesian logic can be seen to provide a representation embedding in this sense between the respective cartesian theories of $k\langle X, Y\rangle$- and $S$-modules. Thus, the main result is applied to obtain an affirmative resolution of the reformulation in cartesian logic of Prest's conjecture as a corollary.

# References


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