Fixed-point elimination in the Intuitionistic Propositional Calculus

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We approach Intuitionistic Logic and Heyting algebras from fixed-point theory and μ-calculi [1]. A μ-calculus is a prototypical computational logic, obtained from a base logic or a base algebraic system by addition of distinct forms of iteration, least and greatest fixed-points, so to increase expressivity. We consider therefore $\mathbf{IPC}_\mu$, the Intuitionistic Propositional μ-Calculus, whose formula-terms are generated by the grammar

$$\phi = x \mid \top \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \to \phi \mid \mu_x.\phi \mid \nu_x.\phi\,,$$

where it is required in the last two productions that the variable x occurs positively in ϕ. Formulas are interpreted over complete Heyting algebras, with $\mu_x.\phi$ (resp. $\nu_x.\phi$) denoting the least fixed-point (resp. the greatest fixed-point) of the intepretation of $\phi(x)$, as a monotone function of the variable x. These extremal fixed-points exist, by the Knaster-Tarski theorem.

Ruitenburg [3] proved that for each formula $\phi(x)$ of the **IPC** there exists a number $\rho(\phi)$ such that $\phi^{\rho(\phi)}(x)$—the formula obtained from ϕ by iterating $\rho(\phi)$ times substitution of ϕ for the variable x—and $\phi^{\rho(\phi)+2}(x)$ are equivalent in Intuitionistic Logic. An immediate consequence of this result is that a syntactically monotone intuitionisitc formula $\phi(x)$ converges both to its least fixed-point and to its greatest fixed-point in at most $\rho(\phi)$ steps. In the language of μ-calculi, we have $\mu_x.\phi(x) = \phi^{\rho(\phi)}(\bot)$ and $\nu_x.\phi(x) = \phi^{\rho(\phi)}(\top)$. These identities witness that the $\mathbf{IPC}_\mu$ is degenerated, meaning that every formula from the above grammar is equivalent to a fixed-point free formula. They also witness that nor completeness neither the Knaster-Tarski theorem are needed to interpret the above formulas over Heyting algebras.

Ruitenburg's result is not the end of the story. We aim at computing explicit representations of fixed-point expressions by means of fixed-point free formulas. Such an algorithm would provide an axiomatization of fixed-points in the **IPC** and also a decision procedure for the $\mathbf{IPC}_\mu$. We also aim at computing closure ordinals of intuitionisitc formulas $\phi(x)$, that is, the least number n such that $\mu_x.\phi(x) = \phi^n(\bot)$ and the least number m for which $\nu_x.\phi(x) = \phi^m(\top)$. Notice that bounds on Ruitenberg's numbers $\rho(\phi)$ might be over-approximation of closure ordinals of ϕ, for example, for an arbitrary intuitionistic formula ϕ, $\nu_x.\phi(x) = \phi^k(\top)$ for $k = 1$, while $\rho(\phi)$ might be arbitrarily large. We tackled these problems in a recent work [2]. We achieve there an effective transformation of intuitionisitc μ-formulas into equivalent fixed-point free intuitionisitc formulas. Such a transformation allows to estimate upper bounds of closure ordinals, which are tight in many cases.

We sketch in what follows the ideas by which we devise our effective transformation.

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Lemma. *Every polynomial* $\mathfrak{f} : H \longrightarrow H$ *over a Heyting algebra* H *is* compatible*, meaning that the equation* $\mathfrak{f}(x) \wedge y = \mathfrak{f}(x \wedge y) \wedge y$ *holds.*

A first consequence of the above statement is that, for such a polynomial, $\mathfrak{f}^2(\top) = \mathfrak{f}(\top)$, so $\mathfrak{f}(\top)$ is the greatest fixed-point of $\mathfrak{f}$ when $\mathfrak{f}$ is monotone. This observation is generalized to systems of equations as follows.

Lemma. *If* H *is an Heyting algebra and, for* $i = 1, \ldots, n$, $\mathfrak{f}_i : H^n \longrightarrow H$ *is a monotone polynomial, then* $\langle \mathfrak{f}_1, \ldots, \mathfrak{f}_n \rangle^n(\top)$ *is the greatest fixed-point of* $\langle \mathfrak{f}_1, \ldots, \mathfrak{f}_n \rangle : H^n \longrightarrow H^n$.

Fact. If $f : P \longrightarrow Q$ and $g : Q \longrightarrow P$ are monotone functions such that the least fixed-point $\mu.(g \circ f)$ of $g \circ f$ exists, then $f(\mu.(g \circ f))$ is the least fixed-point of $f \circ g$.

These statements allow us to give an explicit representation of $\mu_x.\phi(x)$ when all the occurrences of the variable x are under the left side of an implication. Namely, if we write $\phi(x) = \psi_0[\psi_1(x)/y_1, \ldots, \psi_n(x)/y_n]$ with y_i under the left side of just one implication, then

$$\begin{aligned}\mu_x.\phi(x) &= \psi_0(\, \nu_{y_1,\ldots,y_n}.\langle \psi_1(\psi_0(y_1, \ldots, y_n)), \ldots, \psi_n(\psi_0(y_1, \ldots, y_n))\rangle \,) \\ &= \psi_0(\, \langle \psi_1(\psi_0(y_1, \ldots, y_n)), \ldots, \psi_n(\psi_0(y_1, \ldots, y_n))\rangle^n(\top) \,)\,.\end{aligned}$$

Other two important consequences of compatibility of polynomials are the following distribution laws of least fixed-points w.r.t. the residuated structure:

$$\mu.(\bigwedge_{i\in I} \mathfrak{f}_j) = \bigwedge_{i\in I} \mu.\mathfrak{f}_i\,, \qquad\qquad \mu.(\alpha \to \mathfrak{f}) = \alpha \to \mu.\mathfrak{f}\,, \tag{1}$$

which holds when $\mathfrak{f}$ and $\mathfrak{f}_i$ are monotone polynomials and α is a constant.

Fact. The least fixed-point of a monotone function $f(x, x)$ can be computed by firstly computing the least fixed-point of $f(x, y)$ in the variable y, parametrizing in the variable x, and then by computing the least fixed-point of the resulting monotone function in the variable x.

This observation allows us to split the search of an explicit representation of the least fixed-point of a formula into two steps: first we can assume that every occurrence of the variable x is under the left side of an implication; then we can assume that there are no occurrences of the variable x under the left side of an implication. A formula with the latter property is then equivalent to a conjunction of disjunctive formulas, that is, formulas generated by the grammar below on the left:

$$\phi = x \mid \beta \vee \phi \mid \phi \vee \beta \mid \alpha \to \phi \mid \phi \vee \phi\,, \qquad\qquad \mu_x.\phi = (\bigwedge_{\alpha\in\mathsf{Head}(\phi)} \alpha) \to (\bigvee_{\beta\in\mathsf{Side}(\phi)} \beta)\,, \tag{2}$$

where α and β are formulas with no occurrence of the variable x. The first of the relations (1) reduces the computation of the least fixed-point of a formula to the computation of the least fixed-point of a disjunctive formula ϕ. For such a formula, call α a head formula and β a side formula; let $\mathsf{Head}(\phi)$ denote the set of head formulas in a parse tree of ϕ and, similarly, let $\mathsf{Side}(\phi)$ be the set of side formulas in the same parse tree of ϕ. Using the second of the relations (1) and the fact that disjunctive formulas give rise to monotone inflationary functions, an expression for the least fixed-point of a disjunctive formula appears on the right of (2).

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