Filter pairs: A new way of presenting logics

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In the work we present we introduce the notion of filter pair as a tool for creating and analyzing logics. We sketch the basic idea of this notion:

By *logic* we mean a pair $(\Sigma, \vdash)$ where Σ is a signature, i.e. a collection of connectives with finite arities, and $\vdash$ is a Tarskian consequence relation, i.e. an idempotent, increasing, monotone, finitary and structural relation between subsets and elements of the set of formulas $Fm_\Sigma(X)$ built from Σ and a set X of variables.

It is well-known that every logic gives rise to an algebraic lattice contained in the powerset $\wp(Fm_\Sigma(X))$, namely the lattice of theories. This lattice is closed under arbitrary intersections (since intersections of theories are theories) and suprema of directed subsets.

Conversely an algebraic lattice $L \subseteq \wp(Fm_\Sigma(X))$ that is closed under arbitrary intersections and unions of increasing chains gives rise to a finitary closure operator (assigning to a subset $A \subseteq Fm_\Sigma(X)$ the intersection of all members of L containing A). This closure operator need not be structural — this is an extra requirement.

We observe that the structurality of the logic just defined is equivalent to the *naturality* (in the sense of category theory) of the inclusion of the algebraic lattice into the power set of formulas with respect to endomorphisms of the formula algebra: Structurality means that the preimage under a substitution of a theory is a theory again or, equivalently, that the following diagram commutes for any substitution σ:

$$\begin{tikzcd}
Fm_\Sigma(X) \arrow[d, "\sigma"'] & L \arrow[r, hook, "i"] & \wp(Fm_\Sigma(X)) \\
Fm_\Sigma(X) & L \arrow[u, "\sigma^{-1}|_L"] \arrow[r, hook, "i"] & \wp(Fm_\Sigma(X)) \arrow[u, "\sigma^{-1}"']
\end{tikzcd}$$

Further, it is equivalent to demand this naturality for all Σ-algebras and homomorphisms instead of just the formula algebra. We thus arrive at the definition of *filter pair*:

Definition. **(i)** *A* filter pair *for the signature* Σ *is a contravariant functor* G *from* Σ*-algebras to algebraic lattices together with a natural transformation* $i\colon G \to \wp(-)$ *from* G *to the functor that takes an algebra to the power set of its underlying set, which preserves arbitrary infima and suprema of directed subsets.*

(ii) *The* logic associated to a filter pair (G, i) *is the logic associated (in the above fashion) to the algebraic lattice given by the image* $i(G(Fm_\Sigma(X))) \subseteq \wp(Fm_\Sigma(X))$.

Thus a filter pair can be seen as a *presentation* of a logic, different from the usual style of presentation by axioms and derivation rules. For a given logic L one can take $G := Fi_L$, the functor which associates to a Σ-algebra the lattice of L-filters on it; this shows that every logic admits a presentation by a filter pair.

A more interesting case is when G is the functor associating to a Σ-structure the lattice of congruences relative to some quasivariety K, that is, $G\colon A \mapsto \{\theta \mid A/\theta \in K\}$ – we call these filter pairs *congruence filter pairs*. There is a huge supply of congruence filter pairs by the following result:

Theorem. *Let K be a quasivariety, and $\tau = \langle \epsilon, \delta \rangle$ a set of equations (i.e. pairs of unary formulas in the signature of K). For every Σ-algebra A denote by $Con_K(A) := \{\theta \mid A/\theta \in K\}$ the set of congruences relative to K. Then*

$$(G\colon A \mapsto Con_K(A), \quad i\colon \theta \mapsto \{a \in A \mid \epsilon(a) = \delta(a) \text{ in } A/\theta\})$$

defines a filter pair

It follows from [**BP**, Thm 5.1(ii)] that every algebraizable logic admits a presentation by such a congruence filter pair. But strictly more logics arise in this way, even non-protoalgebraic logics. A presentation by a congruence filter pair can give means of determining the position of a logic in the Leibniz hierarchy; e.g. the logic is algebraizable if the natural transformation i is injective. Similar criteria can be given for being truth-equational or Lindenbaum algebraizable.

Further, we have algebraic tools available for dealing with logics which admit a presentation by a congruence filter pair. For example, (under an additional technical assumption) the logic presented by a congruence filter pair has the Craig interpolation property if the quasivariety has the amalgamation property. Further correspondence principles between algebraic and logical properties are under investigation.

Thus on the one hand filter pairs give a way of analyzing logics, on the other hand they give a new way of attacking the problem of associating a logic to a given quasivariety.

In the talk we will motivate and introduce the notions of filter pair and congruence filter pair, and make the above statements concrete in examples.

We will further offer a point of view on congruence filter pairs as being an approach to algebraizing logic which is dual to the one via the Leibniz operator. The Leibniz operator is a map from the lattice of filters to the lattice of congruences which is "trying to be" a *right* adjoint (and actually is a right adjoint for protoalgebraic logics). The natural transformation i of a congruence filter pair, in contrast, has a *left* adjoint going from the lattice of filters to the lattice of congruences. Thus the congruence filter pair approach to defining and analyzing logics, while being equivalent to the Leibniz operator approach in the algebraizable case, diverges into a different direction in the non-algebraizable case. We will also make this point of view concrete with sample calculations.

References

[**AJMP**] P. Arndt, R. Jansana, H.L. Mariano, D. Pinto, **Filter pairs and their associated logics**, in preparation.

[**BP**] W. J. Blok, D. Pigozzi, **Algebraizable logics**, Memoirs of the AMS **396**, American Mathematical Society, Providence, USA, 1989.