Exponentiability in Stone Spaces and Priestley Spaces*

Evgeny Kuznetsov[1,2]

[1] I. Javakhishvili Tbilisi State University,
Tbilisi, Georgia.
[2] Free University of Tbilisi,
Tbilisi, Georgia.
e.kuznetsov@freeuni.edu.ge

Introduction. Exponentiability of objects and morphisms is one of the important good properties for a category. The problem of exponentiability is studied in many contexts and starts its history since 1940's. Among the past works on the subject the most useful for the author were: the great note [4]; the articles [1],[7], [8]; and the books [3], [5],[6].

Due to the existence of important dualities between Stone spaces and Boolean algebras, as well as between Priestley spaces and distributive lattices, our aim is to characterize exponentiable objects and exponentiable morphisms in the categories of Stone spaces and Priestley spaces. The presented work is a part of the more extensive program, which aims to study local homeomorphisms of the so called logical spaces e.g. Stone spaces, Priestley spaces, Spectral spaces, Esakia spaces. This is motivated by the importance of local homeomorhphisms not only in topology, but in algebraic geometry and other areas of mathematics due to their attractive properties.

Given objects X, Y in the small category $\mathbf{C}$ with finite limits, the object Y^X (if it exists in $\mathbf{C}$) is said to be an *exponential* of Y by X, if for any object A in $\mathbf{C}$ there is a natural bijection between the set of all morphisms from $A \times X$ to Y and the set of all morphisms from A to Y^X, i.e. $\mathbf{C}(A \times X, Y) \cong \mathbf{C}(A, Y^X)$. An object X of a category $\mathbf{C}$ is said to be *exponentiable* if the exponent Y^X exists in $\mathbf{C}$ for any object Y. Given object X in $\mathbf{C}$, consider the family $\mathbf{C}/X$ of morphisms $f : Y \to X$ with codomain X in $\mathbf{C}$. Let morphisms between members of the mentioned family be obvious commutative triangles. It is easy to check that the family together with the defined morphisms between them is a category. It is the case that if $\mathbf{C}$ has all finite limits, then $\mathbf{C}/X$ also does so. Let us note that the product of two objects of $\mathbf{C}/X$ is a pullback in $\mathbf{C}$ with the obvious projection to X. As in the case of $\mathbf{C}$, given two morphisms $f : Y \to X$ and $g : Z \to X$ the object g^f (if it exists in $\mathbf{C}/X$) is said to be an exponential of g by f, if for any object $h : W \to X$ in $\mathbf{C}/X$ there is a natural bijection between the set of morphisms from $h \times_X f$ to g and the set of morphisms from h to g^f, i.e. $\mathbf{C}/X(h \times_X f, g) \cong \mathbf{C}/X(h, g^f)$. A morphism f of a category $\mathbf{C}$ is said to be *exponentiable morphism* if the exponent g^f exists in $\mathbf{C}/X$ for any morphism g of $\mathbf{C}$.

Note that for categories of structured sets, the problem of exponentiability reduces to finding appropriate corresponding structure on the set of structure-preserving maps between structured sets. In the following subsections we state the main result already obtained regarding exponentiable objects and morphisms in the categories of Stone spaces and Priestley spaces. For brevity, the supporting lemmas and propositions are omitted.

Exponentiability in Stone spaces. A compact, Hausdorff, and zero-dimensional topological space is called a *Stone space*. The first category we are interested in is the category of Stone spaces and continuous maps. Let us denote the mentioned category by **Stone**. Investigation of exponentiability of object in **Stone** showed that only the finite spaces are exponentiable (unlike the case of all topological spaces where only so-called core-compacts are exponentiable, which can be infinite [2],[4]). Thus we obtain the following result:

Proposition 1. *A Stone space X is exponentiable in* **Stone** *if and only if X is finite.*

*Other people who contributed to this work include David Gabelaia (Razmadze Mathematical Institute) and Mamuka Jibladze (Razmadze Mathematical Institute).

After that we are able to prove the full characterization of exponentiable maps of Stone spaces. That is the following result holds:

Proposition 2. *The map $f : X \to B$ between Stone spaces is exponentiable in* **Stone**$/B$ *if and only if f is a local homeomorphism.*

Exponentiability in Priestley spaces. A partially ordered topological space $(X, \leq)$ is called a Priestley space, if X is compact Hausdorff space and for any pair $x, y \in X$ with $x \nleq y$, there exists a clopen up-set U of X such that $x \in U$ and $y \notin U$. It turns out that the topology on a Priestley space is compact Hausdorff and zero-dimensional, i.e. is a Stone topology. The second category we are interested in is the category of Priestley spaces and continuous order-preserving maps. Let us denote this category by **PS** (**P**riestley **S**paces). Investigation of exponentiability of objects in **PS** showed that, similarly to the case of Stone spaces, only finite spaces are exponentiable in **PS**. Hence the following:

Proposition 3. *A Priestley space X is exponentiable in* **PS** *if and only if X is finite.*

Due to this fact, given a Priestley space B we get the following corollary about exponentiability of $\pi_2 : X \times B \to B$ in $\mathbf{PS}/B$:

Corollary 3.1. *$\pi_2 : X \times B \to B$ is exponentiable in* $\mathbf{PS}/B$ *if and only if X is finite.*

Moreover, we were able to prove a necessary condition for exponentiability of a map between Priestley spaces. An order preserving map $f : X \to B$ is called an interpolation-lifting map if given $x \leq y$ in X and $f(x) \leq b \leq f(y)$, there exists $x \leq z \leq y$ such that $f(z) = b$.

Proposition 4. *If $f : X \to B$ is exponentiable in* $\mathbf{PS}/B$ *then f is interpolation-lifting.*

We are still unable to find a necessary and sufficient condition for exponentiability of Priestley maps. Already obtained results draw quite interesting picture of considered categories. Only the smallest part of the considered categories (only finite objects) have such strong property as exponentiability. Further work is in progress, namely we are investigating whether exponentiable morphisms in **PS** are precisely the local homeomorphisms that are also interpolation-lifting maps.

References

[1] F. Cagliari and S. Mantovani. Local homeomorphisms as the exponentiable morphisms in compact hausdorff spaces. *Topology Appl.*, 41:263–272, 1991.

[2] B.J. Day and G.M. Kelly. On topological quotient maps by pullback and products. *Proc. Cambridge Phil. Soc.*, 67:553–558, 1970.

[3] R. Engelking. *General Topology.* Polish Scientific Publishers, Warszawa, 1977.

[4] M. Escardo and R. Heckmann. Topologies on spaces of continuous functions. *Topology Proceedings*, 26(2):545–564, 2001-2002.

[5] P. Johnstone. *Stone spaces.* Cambridge University Press, 1986.

[6] P. Johnstone. *Sketches of an Elephant: A Topos Theory Compendium.* Clarendon Press, 2003.

[7] S.B. Niefield. Cartesianess: topological spaces, uniform spaces and affine schemes. *J. Pure Appl.Algebra*, 23:147–165, 1982.

[8] S.B. Niefield. Exponentiable morphisms: posets, spaces, locales, and grothendieck toposes. *Theory Appl. Categ.*, 8(2):16–32, 2001.