MODAL LOGIC AND PROPOSITIONAL DYNAMIC LOGIC

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Abstract. Lecture notes (under construction) for four lectures of the course *Dynamic Logic* at the Faculty of Arts, Charles University in the fall semester 2025/26 (taught together with Igor Sedlár).

- Lecture 1: Modal Logic - Syntax and Kripke Semantics (Oct 22)
- Lecture 2: Modal Logic - Decidability and Completeness (Oct 29)

Further references: For modal logic - [BdRV01]

1. Modal Logic

1.1. **Why modal logic?** Modal logics are among the most prominent *non-classical logics* for a number of reasons, for example:

- ⇝ Simple yet expressive logical systems
- ⇝ Typically decidable (unlike *e.g.*, first-order logic)
- ⇝ Internal/local perspective on relational structures
- ⇝ Rich interplay with other logical/mathematical frameworks
- ⇝ Applications in theoretical computer science, philosophy, linguistics, ...

Modal formulas like $\Diamond\varphi$ and $\Box\varphi$ can model *non truth-functional* concepts like:

- ⇝ $\Diamond\varphi$: *It is possible that* φ —— $\Box\varphi$: *It is necessary that* φ (Alethic)
- ⇝ $F\varphi$: *Sometime in the future* φ —— $G\varphi$: *Always in the future* φ (Temporal)
- ⇝ $P\varphi$: *It is permitted that* φ —— $O\varphi$: *It is obligatory that* φ (Deontic)
- ⇝ $K\varphi$: *The agent knows that* φ (Epistemic)
- **(!)**⇝ $\langle\pi\rangle\varphi$: *Executing program* π *might result in* φ
  $[\pi]\varphi$: *Executing program* π *always results in* φ

1.2. **Syntax.** The set of formulas is defined inductively from a countable collection $\mathsf{Prop} = \{p_1, p_2, p_3, \dots\}$ of *propositional variables* together with *logical connectives*.

Definition 1.1 (Modal Formulas)**.** The collection Form of modal formulas is inductively defined as follows.

$$\mathsf{Form} \ni \varphi ::= p \in \mathsf{Prop} \mid \varphi \wedge \varphi \mid \neg\varphi \mid \Diamond\varphi$$

As usual, we use abbreviations $\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$, $\varphi \to \psi := \neg\varphi \vee \psi$, $\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi)$, $\top := p \vee \neg p$, $\bot = \neg\top$. Furthermore, we use the abbreviation

$$\Box\varphi := \neg\Diamond\neg\varphi,$$

and we refer to $\Diamond$ and $\Box$ as the 'diamond' and 'box' modalities, respectively. Intuitively, the basic modal language is simply the language of classical propositional logic expanded by these two modalities.

Remark 1.2. The definition of modal formulas (Definition 1.1) can be easily generalised to include *more than one modality*. Indeed, later on we introduce PDL which uses the multi-modal language containing a modality $\langle\pi\rangle$ for every 'program'

π. It is also possible (albeit not needed for this course) to include modalities of *higher arities*, for example a binary modality $\triangle(\varphi, \psi)$ takes two formulas as input.

1.3. **Relational Semantics.** We now know what a modal formula is, but we still need a reasonable notion of *interpretation* for these formulas. The notions of *Kripke frames/models* are the key notion here.

Definition 1.3 (Kripke frame)**.** A *(Kripke) frame* is a structure $\mathfrak{F} = (X, R)$ where X is a set and $R \subseteq X^2$ is a binary relation on X.

The members of X are often called 'possible worlds', we use the more neutral term 'states'. The relation R is called the 'accessibility relation'.

Definition 1.4 (Kripke model)**.** A *(Kripke) model* is a structure $\mathfrak{M} = (X, R, \mathsf{Val})$ where (X, R) is a frame and

$$\mathsf{Val}\colon \mathsf{Prop} \to \mathcal{P}(X)$$

is a *propositional valuation.*

Note that every model $\mathfrak{M} = (X, R, \mathsf{Val})$ is *based on* a frame $\mathfrak{F} = (X, R)$. The set $\mathsf{Val}(p) \subseteq X$ corresponds to the set of states where the propositional variable p is true. This notion of *truth* is extended to all modal formulas in the following.

Definition 1.5 (Truth)**.** Let $\mathfrak{M} = (X, R, \mathsf{Val})$ be a model and let $x \in X$ be a state. We inductively define the relation

$$\mathfrak{M}, x \Vdash \varphi$$

(φ is true at state x in model $\mathfrak{M}$) for all formulas φ via

$$\begin{array}{lll}
\mathfrak{M}, x \Vdash p & \text{iff} & x \in \mathsf{Val}(p), \\
\mathfrak{M}, x \Vdash \varphi_1 \wedge \varphi_2 & \text{iff} & \mathfrak{M}, x \Vdash \varphi_1 \text{ and } \mathfrak{M}, x \Vdash \varphi_2, \\
\mathfrak{M}, x \Vdash \neg\varphi & \text{iff} & \mathfrak{M}, x \nVdash \varphi, \\
\mathfrak{M}, x \Vdash \Diamond\varphi & \text{iff} & \exists y\colon xRy \text{ and } \mathfrak{M}, y \models \varphi.
\end{array}$$

From this definition we also get the truth-conditions for $\top, \bot, \to, \vee$ as usual in propositional logic. More importantly, we get the truth-condition for the box-modality as

$$\mathfrak{M}, x \Vdash \Box\varphi \qquad \text{iff} \qquad \forall y\colon \text{ if } xRy \text{ then } \mathfrak{M}, y \models \varphi.$$

That is, the modalities $\Diamond, \Box$ can be seen as 'scanning' the related states xRy of the state x, where the $\Diamond$ scans 'existentially' and $\Box$ scans 'universally'.

Example 1.6. Let us consider the Kripke frame $\mathfrak{F} = (X, R)$ with $X = \{x_1, \ldots, x_6\}$ and R being indicated in Figure 1. Let us consider the model $\mathfrak{M}$ which arises from adding the following propositional valuation (we will only consider formulas with two propositional variables p, q, so it is irrelevant how the valuation is defined on the remainder of Prop)

$$\mathsf{Val}(p) = \{x_1, x_2, x_4\} \quad \mathsf{Val}(q) = \{x_1, x_3, x_4, x_5\}.$$

Then, for example, we can check the following:

- $\mathfrak{M}, x_1 \Vdash q \wedge \Diamond\neg q$
- $\mathfrak{M}, x_2 \Vdash \Box\bot$
- $\mathfrak{M}, x_3 \Vdash \Diamond\Diamond q$

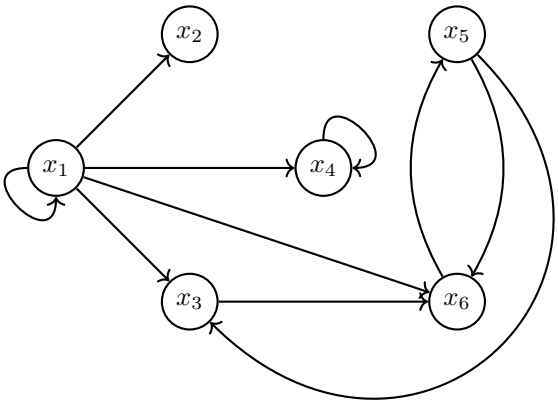

FIGURE 1. Exmple of a Kripke frame

- $\mathfrak{M}, x_4 \Vdash p \to \Diamond p$
- $\mathfrak{M}, x_5 \Vdash \Box \neg p$
- $\mathfrak{M}, x_6 \Vdash \Diamond \Box \neg p$

While the notion of *truth* is satisfied relative to a model, the notion of *validity* happens on the level of frames (by quantifying over all possible models based over this frame, similarly to tautologies in propositional logic).

Definition 1.7 (Validity)**.** Let $\mathfrak{F} = (X, R)$ be a Kripke frame and $x \in \mathfrak{F}$. The formula φ is *valid* at x, written

$$\mathfrak{F}, x \Vdash \varphi$$

if and only if $\mathfrak{M}, x \Vdash \varphi$ for *every* model $\mathfrak{M}$ based on $\mathfrak{F}$. We write

$$\mathfrak{F} \Vdash \varphi$$

if $\mathfrak{F}, x \Vdash \varphi$ for all $x \in \mathfrak{F}$ and say the formula φ is *valid on* $\mathfrak{F}$.

Example 1.8. For *every* frame $\mathfrak{F} = (X, R)$, it holds that

$$\mathfrak{F} \Vdash \Diamond(\varphi \vee \psi) \leftrightarrow (\Diamond\varphi \vee \Diamond\psi).$$

To see this, let $\mathfrak{M}$ be an arbitrary model on $\mathfrak{F}$ and let $x \in X$. We have

$$\begin{aligned} \mathfrak{M}, x \Vdash \Diamond(\varphi \vee \psi) &\Leftrightarrow \exists xRy \colon \mathfrak{M}, y \Vdash \varphi \vee \psi \\ &\Leftrightarrow \exists xRy \colon \mathfrak{M}, y \Vdash \varphi \text{ or } \exists xRy \colon \mathfrak{M}, y \Vdash \psi \\ &\Leftrightarrow \mathfrak{M}, x \models \Diamond\varphi \vee \Diamond\psi \end{aligned}$$

as desired.

Example 1.9. For every Kripke frame $\mathfrak{F} = (X, R)$ we have

$$\mathfrak{F}, x \Vdash p \to \Diamond p \text{ if and only if } xRx,$$

in particular $\mathfrak{F} \Vdash p \to \Diamond p$ if and only if R is *reflexive*.

To see this, first assume xRx and let $\mathfrak{M}$ be an arbitrary model based on $\mathfrak{F}$. If $\mathfrak{M}, x \models p$, then we get $\mathfrak{M}, x \models \Diamond p$ because xRx. Conversely, assume $\neg xRx$ and define a model $\mathfrak{M}$ with $\mathsf{Val}(p) = \{x\}$. Then we have $\mathfrak{M}, x \Vdash p$ but $\mathfrak{M}, x \nVdash \Diamond p$.

1.4. **Bisimulation.** Given two models $(X, , R, \mathsf{Val})$ and (X', R', Val'), how can we express the fact that two states $x \in X$ and $x' \in X'$ have the same 'behaviour'? The key property (besides satisfying the same atomic propositions) is that every 'step' along the accessibility relation in one model can be 'simulated' in the other one.

Definition 1.10 (Bisimulation)**.** Let $\mathfrak{M} = (X, R, \mathsf{Val})$ and $\mathfrak{M}' = (X', R', \mathsf{Val}')$ be models. A (non-empty) binary relation $B \subseteq X \times X'$ is a *bisimulation* between these models if the following hold

(1) If xBx', then $x \in \mathsf{Val}(p) \Leftrightarrow x' \in \mathsf{Val}'(p)$.
(2) If xBx' and xRy, then there is y' with $x'R'y'$ and yBy'.
(3) If xBx' and $x'R'y'$, then there is y with xRy and yBy'.

We call x and x' *bisimilar* if there exists any bisimulation with xBx'.

One reason why this notion is important is the following *logical invariance result.*

Proposition 1.11. *Let $\mathfrak{M} = (X, R, \mathsf{Val})$ and $\mathfrak{M}' = (X', R', \mathsf{Val}')$ be models and let $x \in X$ and $x' \in X'$ be bisimilar. Then x and x' are* logically equivalent *in the sense that*

$$\mathfrak{M}, x \Vdash \varphi \text{ if and only if } \mathfrak{M}', x' \Vdash \varphi$$

for every modal formula φ.

Proof. Exercise (Hint: Use induction on the structure of φ, where (2) and (3) in the definition of bisimulation will be needed for the case of $\Diamond\varphi$). □

The converse of this proposition (*i.e.*, logical equivalence implies bisimilarity) does not necessarily hold. It does, however, hold on *image-finite* models, which are models based on frames (X, R) for which the sets $R[x] := \{y \mid xRy\}$ is finite for every $x \in X$. This result, commonly referred to as *Hennessy-Milner Theorem*, states (if read contrapositively) that modal logic is *expressive enough* to distinguish any two non-bisimilar states in image-finite models by a modal formula.

Theorem 1.12. *Let $\mathfrak{M} = (X, R, \mathsf{Val})$ and $\mathfrak{M}' = (X', R', \mathsf{Val}')$ be image-finite models and let $z \in X$ and $z' \in X'$ be logically equivalent. Then z and z' are bisimilar.*

Proof. The idea is to show that logical equivalence

$$x \leftrightsquigarrow x' :\Leftrightarrow (\mathfrak{M}, x \Vdash \varphi \text{ iff } \mathfrak{M}', x' \Vdash \varphi \text{ for all formulas } \varphi)$$

defines itself a bisimulation. For example, we show that property (2) of Definition 1.10 holds. Let $x \leftrightsquigarrow x'$ and xRy, and towards contradiction assume that there is no y' with $x'R'y'$ and $y \leftrightsquigarrow y'$. That is, if $R'[x'] = \{y'_1, \ldots, y'_n\}$ then for every $i = 1, \ldots, n$ we can find a formula φ_n such that $\mathfrak{M}, y \Vdash \varphi_i$ but $\mathfrak{M}', y'_i \nVdash \varphi_i$. Now we define

$$\varphi = \Diamond(\varphi_1 \wedge \cdots \wedge \varphi_n)$$

and observe that $\mathfrak{M}, x \Vdash \varphi$ (because xRy) but $\mathfrak{M}', x' \nVdash \varphi$ (since no $x'R'y'_i$ satisfies $\varphi_1 \wedge \cdots \wedge \varphi_n$). This contradicts the initial assumption that x and x' are logically equivalent. □

Remark 1.13. The relationship between modal logic and bisimulation is even stronger, as shown by the so-called *van Benthem Characterisation Theorem* (see, *e.g.*, [BdRV01, Section 2.6]). This theorem is based on the *standard translation* of modal logic into first-order logic and identifies the modal formulas precisely with the first-order formulas which are *bisimulation-invariant.*

1.5. **Finite model property via filtration.** Our next goal is to prove the following.

Theorem 1.14 (Finite model property)**.** *If a formula φ is true in some model, then it is also true in some* finite *model (of size at most 2^n, where n is the number of* subformulas *of φ).*

In particular, this yields *decidability* results, since this means that truth (and validity) can be *checked algorithmically.*

In the following, we use sets of formulas $\Gamma \subseteq \mathsf{Form}$ which are *closed under subformulas*, meaning that

- $\varphi \vee \psi \in \Gamma \Rightarrow \varphi, \psi \in \Gamma$,
- $\neg\varphi \in \Gamma \Rightarrow \varphi \in \Gamma$,
- $\Diamond\varphi \in \Gamma \Rightarrow \varphi \in \Gamma$.

For such a set, we now define *filtrations*, which are defined on certain *quotients.*

Definition 1.15. (Filtration) Let $\mathfrak{M} = (X, R, \mathsf{Val})$ be a model and let $\Gamma \subseteq \mathsf{Form}$ be closed under subformulas. We define the relation $\leftrightsquigarrow_\Gamma$ on X via

$$x \leftrightsquigarrow_\Gamma y \text{ if and only if } \forall \varphi \in \Gamma \colon \mathfrak{M}, x \Vdash \varphi \Leftrightarrow \mathfrak{M}, y \Vdash \varphi$$

and note that this defines an *equivalence relation* on X, and we denote the corresponding quotient by X^f_Γ. A *filtration of $\mathfrak{M}$ through* Γ is a model

$$\mathfrak{M}^f_\Gamma = (X^f_\Gamma, R^f, \mathsf{Val}^f)$$

such that

(1) $xRy \Rightarrow |x| R^f |y|$,
(2) If $|x| R^f |y|$, then for all $\Diamond\varphi \in \Gamma \colon \mathfrak{M}, y \Vdash \varphi \Rightarrow \mathfrak{M}, x \Vdash \Diamond\varphi$,
(3) $\mathsf{Val}^f(p) = \{|x| \mid \mathfrak{M}, x \Vdash p\}$ for propositional variables $p \in \Gamma$.

In order to prove Theorem 1.14 we will now prove that (i) for formulas $\varphi \in \Gamma$, truth in $\mathfrak{M}$ and truth in $\mathfrak{M}^f_\Gamma$ coincide, (ii) filtrations through finite Γ are finite and (iii) such filtrations actually exist.

For (i), we essentially check that the way we defined filtrations is 'appropriate'.

Theorem 1.16. *Let $\mathfrak{M}^f_\Gamma$ be a filtration of $\mathfrak{M}$ through Γ. Then*

$$\mathfrak{M}^f_\Gamma, |x| \Vdash \varphi \text{ if and only if } \mathfrak{M}, x \Vdash \varphi$$

for all formulas $\varphi \in \Gamma$.

Proof. By induction on the structure of φ. The case of propositional variables p is covered by Definition 1.15(3) and the cases of Boolean connectives $\neg, \wedge$ hold because Γ is closed under subformulas. The 'interesting' case $\Diamond\varphi$ uses conditions Definition 1.15(1) and (2). □

Next we show (ii) filtrations through finite sets are finite.

Lemma 1.17. *Let Γ be a finite set of formulas closed under subformulas. Then any filtration $\mathfrak{M}^f_\Gamma$ is finite and contains at most 2^n states where n is the size of Γ.*

Proof. We define a map $f \colon X^f_\Gamma \to \mathcal{P}(\Gamma)$ by $f(|x|) = \{\varphi \in \Gamma \mid \mathfrak{M}, x \Vdash \varphi\}$. Note that this is well-defined due to the definition of $\leftrightsquigarrow_\Gamma$. Furthermore, it is injective since $|x| \neq |y|$ implies that there is a formula $\varphi \in \Gamma$ such that $\varphi \in f(|x|)$ but $\varphi \notin f(|y|)$ or vice versa. □

Lastly, we show that (iii) filtrations actually *exist*.

Lemma 1.18. *Let Γ be closed under subformulas and let $\mathfrak{M}$ be a model. The relation on X_Γ^f defined by*

$$|x|R_\Gamma^f|y| \text{ if and only if } \exists x' \in |x|, y' \in |y| \colon x'Ry'$$

defines a filtration of $\mathfrak{M}$ through Γ.

Proof. We need to show that (1) and (2) of Definition 1.15 hold. Note that (1) is immediate by definition. For (2), assume $|x|R_\Gamma^f|y|$ and $\Diamond\varphi \in \Gamma$ such that $\mathfrak{M}, y \Vdash \varphi$. We know that there are $x' \in |x|$ and $y' \in |y|$ with $x'Ry'$. Since $y' \leftrightsquigarrow_\Gamma y$ and $\mathfrak{M}, y \Vdash \varphi$, we have $\mathfrak{M}, y' \Vdash \varphi$. Thus $\mathfrak{M}, x' \Vdash \Diamond\varphi$ since $x'Ry'$. Since $x' \leftrightsquigarrow_\Gamma x$, this finally yields $\mathfrak{M}, x \Vdash \Diamond\varphi$ as desired. □

Thus, we obtain the finite model property of Theorem 1.14 by using for Γ the collection of all subformulas of φ.

1.6. **Normal modal logics.** We now present a syntactic calculus for modal logic which is sound (*i.e.*, all axioms and rules are valid) and complete (*i.e.*, it is able to derive all valid formulas).

Definition 1.19 (Normal modal logic)**.** A *normal modal logic* is a set of formulas $\Lambda \subseteq \mathsf{Form}$ which contains all instances of the following *axioms*

- Propositional tautologies (or an axiomatic base thereof)
- (K): $\Box(p \to q) \to (\Box p \to \Box q)$

and is closed under the following *rules*:

- Modus ponens: From $\varphi \to \psi$ and φ infer ψ
- Uniform substitution: From $\phi(p_1, \dots, p_k)$ infer $\phi(\psi_1, \dots, \psi_k)$
- Necessitation (Nec): From φ infer $\Box\varphi$

The *smallest normal modal logic* is called $\mathbf{K}$.

If F is a class of Kripke frames, then the set

$$\Lambda_\mathsf{F} = \{\varphi \mid \forall \mathfrak{F} \in \mathsf{F} \colon \mathfrak{F} \Vdash \varphi\}$$

is a normal modal logic. In what follows, we will show that $\mathbf{K}$ coincides with Λ_F if F is the class of *all* frames.

1.7. **Completeness via canonical model.** The so-called *canonical model* is based on the set of all *maximally consistent sets* defined as follows.

Definition 1.20 ((Maximally) consistent sets)**.** A set of formulas $t \subseteq \mathsf{Form}$ is *consistent* (*w.r.t.* the logic $\mathbf{K}$) if there are *no* $\varphi_1, \dots, \varphi_k \in t$ with $\varphi_1 \wedge \dots \wedge \varphi_k \to \bot \in \mathbf{K}$. It is called *maximally consistent* if it is consistent and every $t \subsetneq t'$ is not consistent.

Lemma 1.21 (Lindenbaum's Lemma)**.** *Every consistent theory is contained in some maximally consistent theory.*

The core idea of the *canonical model construction* is to turn these maximally consistent sets themselves into the states of a model.

Definition 1.22 (Canonical model)**.** The *canonical model* is $\mathfrak{M}^c = (X^c, R^c, \mathsf{Val}^c)$ where

- X^c is the set of all maximally consistent sets,

- $tR^c t'$ if and only if $\psi \in t' \Rightarrow \Diamond\psi \in t$,
- $\mathsf{Val}^c(p) = \{t \in X^c \mid p \in t\}$.

The most important property of this model is often subsumed by the slogan *truth is membership*, as expressed by the following.

Lemma 1.23 (Truth Lemma)**.** *For every formula φ and maximally consistent theory t it holds that*

$$\mathfrak{M}^c, t \Vdash \varphi \text{ if and only if } \varphi \in t.$$

Proof. By induction on the structure of the formula, where only the case $\Diamond\varphi$ is not immediate. The direction '$\Rightarrow$' follows immediately from the definition of R^c. For the other direction '$\Leftarrow$', assuming that $\Diamond\varphi \in t$ we want to construct a maximally consistent set t' such that $\varphi \in t'$ and $tR^c t'$. The set $\{\varphi\} \cup \{\psi \mid \Box\psi \in t\}$ is consistent, thus it is contained in a maximally consistent set t'. By design, we have $\varphi \in t'$ and $\Box\psi \in t \Rightarrow \psi \in t'$, which implies $tR^c t'$ as desired. □

With this, we can now prove the following *completeness result.*

Theorem 1.24 (Completeness)**.** *A formula φ is in $\boldsymbol{K}$ if and only if it is valid on all frames.*

Proof. Soundness '$\Rightarrow$' is an exercise. For the completeness part '$\Leftarrow$', proceed by contrapositive and assume that φ is not in $\mathbf{K}$. Then $\{\neg\varphi\}$ is consistent, thus contained in some maximally consistent theory t. By the above lemma, since $\varphi \notin t$ we have $\mathfrak{M}^c, t \nVdash \varphi$ in the canonical model. Therefore, φ is not valid on all frames. □

References

[BdRV01] P. Blackburn, M. de Rijke, and Y. Venema. *Modal Logic*, volume 53 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, 2001. doi:10.1017/CBO9781107050884.