Epistemic Extensions of Substructural Inquisitive Logics*

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Abstract

In this paper, we study epistemic extensions of distributive substructural inquisitive logics. Substructural inquisitive logics are logics of questions based on substructural logics of declarative sentences. They generalize basic inquisitive logic which is based on the classical logic of declaratives. We show that if the underlying substructural logic is distributive, the generalization can be extended to embrace also the epistemic modalities "knowing whether" and "wondering whether" that are applicable to questions. We construct a semantic framework for a language of propositional substructural logics enriched with a question forming operator (inquisitive disjunction) and epistemic modalities. We show that within this framework one can define a canonical model with suitable properties for any (syntactically defined) epistemic inquisitive logic. This leads to a general approach to completeness proofs for such logics. A deductive system for the weakest epistemic inquisitive logic is described and completeness proved for this special case using the general method.

Keywords: Epistemic logic, Modal logic, Substructural logic, Inquisitive logic, Logic of questions

1 Introduction

The standard inquisitive logic is a logic of questions based on the classical logic of declarative sentences [5, 4, 7]. Inquisitive logic and inquisitive semantics were generalized in [13, 14, 8] where it was shown that one can similarly base a logic of questions on intuitionistic logic of declaratives. A further step was made in [15] where inquisitive semantics was formulated for a large class of substructural logics. It was shown that any logic that is at least as strong as a particular weak substructructural logic (a version of Full Lambek Logic) can be extended with questions in an analogous way, and for these non-classical inquisitive logics a suitable semantic framework was developed.

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The aim of this paper is to show that this generalization of inquisitive semantics and logic can be stretched a little more to be applicable also to the epistemic extension of the standard inquisitive logic [3, 6]. As a result, we obtain a general semantic framework for non-classical epistemic logics, which – in comparison to other semantic frameworks for non-classical epistemic logics like those from [2] and [18] – has the extra merit of allowing agents to be equipped not only with information states but also with issues. This semantics determines a weakest logic for which we provide a sound and complete axiomatization.

A crucial notion in the standard inquisitive semantics is the notion of an information state defined as a set of possible worlds. Defined in this way, information states form a concrete Boolean algebra. The generalization of inquisitive semantics formulated in [15] is based on the observation that we can completely abstract away from possible worlds and formulate the whole semantics on the basis of more abstract structures of information states without loss of the crucial features of inquisitive semantics.

The inquisitive epistemic logic (IEL) is an interesting extension of the standard inquisitive logic. A natural question arises whether it also can be generalized in the same manner. However, this task is not straightforward. A peculiar feature of the semantics of IEL is that it exploits a strong interaction between possible worlds and information states so that the layer of possible worlds cannot be simply ignored in the generalization. This makes our current task (which is to develop a suitable semantic framework for non-classical inquisitive epistemic logics) rather challenging. We will show that the obstacles can be overcome provided that the underlying logic of declarative sentences is distributive.

We put forward a general semantic approach in which possible worlds are replaced with situations in the sense of situation semantics [1]. In this framework, information states are not represented simply as sets of possible worlds but rather as sets of situations. This shift allows us to avoid various disputable features of classical logic while preserving the characteristic principles of inquisitive epistemic logic.

The paper is structured in the following way. First, Section 2 provides a brief introduction to standard inquisitive epistemic logic and its semantics. Sections 3 and 4 present respectively a semantic and a syntactic generalization of IEL. In Section 4, a general method is developed that allows one to prove completeness results for a large class of inquisitive epistemic logics provided that they are distributive and constructive (i.e. inquisitive disjunction has the disjunction property).

2 Inquisitive epistemic logic

Inquisitive semantics is a framework suitable for a uniform formal representation of the semantic content of statements and questions. In its most basic form it is usually formulated for a formal language consisting of formulas built up from atomic formulas and the constant for contradiction $\bot$ by the connectives $\to$ (implication), $\wedge$ (conjunction) and $\vee\!\!\!\vee$ (inquisitive disjunction). Negation, non-inquisitive disjunction, and equivalence are defined in the standard way: $\neg\varphi =_{def} \varphi \to \bot$, $\varphi \vee \psi =_{def} \neg(\neg\varphi \wedge \neg\psi)$, $\varphi \leftrightarrow \psi =_{def} (\varphi \to \psi) \wedge (\psi \to \varphi)$. We can denote this language as $\mathcal{L}_B$. In this paper, our starting point is the language of Inquisitive Epistemic Logic (IEL) introduced in [3, 6], which adds to $\mathcal{L}_B$ two epistemic modalities (for each agent a): K_a (knowledge modality) and E_a (entertaining modality). We will denote this language as $\mathcal{L}_{BE}$.

Inquisitive disjunction is a question-forming operator. The formula $p \vee\!\!\!\vee\, q$ is interpreted as the question *whether p or q*. Moreover, using this connective one can define the following question mark operator that allows us to express yes-no questions: $?\varphi =_{def} \varphi \vee\!\!\!\vee \neg\varphi$. For example, the formula $?p$ expresses the question *whether p*. Inquisitive disjunction can be arbitrarily embedded under other operators, which allows us to express, for example, conditional questions like $p \rightarrow ?q$ (i.e. the question *whether q, if p*), and hybrids of statements and questions, like $p \wedge ?q$ (i.e. *it is the case that p but is it also the case that q?*).

When applied to statements, K_a is meant to be the standard knowledge operator familiar from epistemic logic (see, e.g. [10]). For example, $K_a p$ is interpreted as *the agent a knows that p*. It is one of the crucial insights of inquisitive epistemic logic that it makes a perfect sense to apply this operator also to questions. For example, $K_a ?p$ is interpreted as *the agent a knows whether p*, which is a statement involving a question as its part. In general, for any question φ, the statement $K_a\varphi$ means that the agent a is equipped with a reliable piece of information that resolves the question. The operator E_a is supposed to behave just like K_a when applied to statements but in combination with questions, it behaves differently. It does not have an exact counterpart in natural language but its meaning could be clarified in the following way: For any question φ, $E_a\varphi$ is the statement that if the agent extends her knowledge to a state in which she has enough information to resolve her issue, then she will also have enough information to resolve the question φ.[1] In other words, having resolved φ is for the agent a necessary, though maybe not sufficient, condition for resolving her issue. The entertaining modality is introduced mainly as a tool by means of which one can define a complex wondering modality: $W_a\varphi =_{def} E_a\varphi \wedge \neg K_a\varphi$. For example, $W_a ?p$ represents the claim *the agent a wonders whether p*, since it says that having resolved the question *whether p* is for the agent a necessary condition for resolving her issue but she does not know yet whether p.

Note that even if φ is a question, both $K_a\varphi$ and $E_a\varphi$ always represent statements. The set of declarative $\mathcal{L}_{BE}$-formulas (i.e. formulas that represent statements) is defined as the least set that contains all atomic formulas, the constant $\bot$, is closed under $\wedge$ and $\rightarrow$, and contains $K_a\varphi$ and $E_a\varphi$ for any (not necessarily declarative) $\mathcal{L}_{BE}$-formula φ. So, a declarative $\mathcal{L}_{BE}$-formula may contain the inquisitive disjunction only in the scope of an epistemic modality.

To deal with questions inquisitive semantics replaces the notion of truth with the notion of support. Unlike truth, which is a relation between possible worlds and formulas, support is a relation between information states and formulas.

Possible worlds can be defined simply as functions assigning truth values to atomic formulas. Information states are defined as sets of possible worlds. We will call information states defined in this way *concrete states* to distinguish them from *abstract states* that will be introduced in the next section as primitive entities of our general framework. Assume that $s \subseteq t$, for two concrete states s and t. This means that the state s excludes every possibility excluded by t and in this sense s is informationally at least as strong as t. In that case we say that s is a refinement of t.

In IEL, sets of possible worlds are not only the objects with respect to which formulas are evaluated, they also explicitly encode information states of agents. Moreover,

[1] A more detailed discussion of issues and their representation can be found below.

the peculiar feature of IEL is that besides information states, agents are also equipped with issues. An issue is identified with the set of those information states that resolve the issue. So, while an information state is represented as a set of possible worlds, an issue is modelled as a non-empty downward closed set of information states, i.e. as a set of states that contains the empty set and is closed under subsets.

More formally, given a set of agents $\mathcal{A}$, a concrete inquisitive epistemic model is a triple $\langle W, \Sigma_{\mathcal{A}}, V\rangle$, where W is a non-empty set (of possible worlds), V is a valuation specifying which atoms are true in which worlds, more precisely, it is defined as a function assigning to each atomic formula a subset of W, and $\Sigma_{\mathcal{A}} = \{\Sigma_a \mid a \in \mathcal{A}\}$ is a set of inquisitive state maps. For each agent $a \in \mathcal{A}$ there is a map Σ_a that assigns to each world from W an issue in the model, i.e. a non-empty downward closed set of subsets of W. $\Sigma_a(w)$ represents the issue of the agent a in the world w. The information state of the agent a in the world w is defined as $\sigma_a(w) = \bigcup \Sigma_a(w)$. Inquisitive state maps are required to satisfy the following conditions:

Factivity: for any $w \in W$, $w \in \sigma_a(w)$.

Introspection: for any $w, v \in W$, if $v \in \sigma_a(w)$, then $\Sigma_a(v) = \Sigma_a(w)$.

Intuitively, factivity requires that the actual world is regarded as possible by the state of the agent, and introspection means that the agent is aware of her own inquisitive state, so if the world v is regarded as possible by the agent's state, then the agent's issue in v must be the same as her issue in the actual world.

Given a concrete inquisitive epistemic model, the support relation ($\vDash$) between subsets of W and $\mathcal{L}_{BE}$-formulas of $\mathcal{L}_{BE}$ can be defined by the following recursive semantic clauses:

- $s \vDash p$ iff $s \subseteq V(p)$,
- $s \vDash \bot$ iff $s = \emptyset$,
- $s \vDash \varphi \to \psi$ iff for any $t \in \mathcal{P}(W)$, if $t \vDash \varphi$ then $s \cap t \vDash \psi$,
- $s \vDash \varphi \wedge \psi$ iff $s \vDash \varphi$ and $s \vDash \psi$,
- $s \vDash \varphi \vee\!\!\!\vee \psi$ iff $s \vDash \varphi$ or $s \vDash \psi$.
- $s \vDash K_a\varphi$ iff for any $w \in s$, $\sigma_a(w) \vDash \varphi$.
- $s \vDash E_a\varphi$ iff for any $w \in s$ and for any $t \in \Sigma_a(w)$, $t \vDash \varphi$.

Our formulation of the semantic clause for implication differs from the one that is usually used in inquisitive semantics, e.g. in [3, 4]:

- $s \vDash \varphi \to \psi$ iff for any $u \subseteq s$, if $u \vDash \varphi$ then $u \vDash \psi$.

Nevertheless, the alternative formulation is equivalent to the standard clause and we prefer this particular formulation because, unlike the standard one, it can be smoothly generalized to the substructural setting introduced in the next section. Let us show that the clauses are indeed equivalent. First, assume that for any $t \in \mathcal{P}(W)$, if $t \vDash \varphi$ then $s \cap t \vDash \psi$. Let $u \subseteq s$ such that $u \vDash \varphi$. Then, according to our assumption, $s \cap u \vDash \psi$, i.e. $u \vDash \psi$ (since $s \cap u = u$). Second, assume that for any $u \subseteq s$, if $u \vDash \varphi$

then $u \vDash \psi$. Take any $t \in \mathcal{P}(W)$ such that $t \vDash \varphi$. Now we have to use the persistence property (see Theorem 1-b below) to conclude that also $s \cap t \vDash \varphi$. Since $s \cap t \subseteq s$, our assumption implies $s \cap t \vDash \psi$.

An $\mathcal{L}_{BE}$-formula is IEL-valid if it is supported by every state of every concrete inquisitive epistemic model. Two $\mathcal{L}_{BE}$-formulas, φ and ψ, are IEL-equivalent if $\varphi \leftrightarrow \psi$ is IEL-valid, i.e. if in every concrete inquisitive epistemic model, φ and ψ are supported by the same states.

It follows from the above clauses that the support condition for the defined symbol of negation can be specified as follows:

- $s \vDash \neg\varphi$ iff for any $t \in \mathcal{P}(W)$, if $t \cap s \neq \emptyset$ then $t \nvDash \varphi$.

For a motivation of the semantic clauses for the operators of $\mathcal{L}_B$, especially of the clause for inquisitive disjunction, see [4] or [7]. As regards the epistemic operators, consider the following examples. Intuitively, an information state s supports the information that the agent a *knows that* p (i.e. $s \vDash K_a p$) iff the information in s excludes every possible world in which a's information state does not support p. The information state s supports the information that the agent a *knows whether* p (i.e. $s \vDash K_a?p$) iff from the perspective of s only such worlds are possible in which the information state of a resolves the question whether p, i.e. it either supports p, or $\neg p$. Moreover, the information state s supports the information that the agent *is wondering whether* p (i.e. $s \vDash E_a?p \wedge \neg K_a?p$) iff from the perspective of s only such worlds are possible in which the information state of a does not resolve the question whether p but the question whether p is resolved by any refinement of a's state in which a's issue is resolved.

The difference between the two modalities, K_a and E_a, is illustrated in Figure 1. The picture represents a part of a concrete inquisitive epistemic model and depicts a possible world w in which both p and q are true and two information states, s and t. Assume that the issue of the agent a in w, i.e. $\Sigma_a(w)$, consists of all subsets of the states s and t.

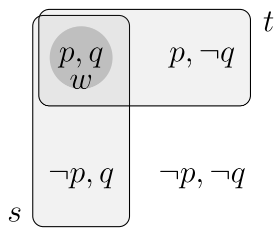

Figure 1: The issue of an agent in a world w

Take the state $u = \{w\}$. According to this state, the agent a entertains the question whether p or q ($u \vDash E_a(p \vee\!\!\!\vee q)$) because every state from her issue supports $p \vee\!\!\!\vee q$. However, she does not know whether p or q, i.e. $u \vDash \neg K_a(p \vee\!\!\!\vee q)$, for her information state, that is the state $s \cup t$, does not support $p \vee\!\!\!\vee q$, since it supports neither p, nor q. As a consequence, the agent wonders whether p or q, i.e. $u \vDash W_a(p \vee\!\!\!\vee q)$. The following theorem states the crucial features of the support relation (see [6]).

Theorem 1. *In every concrete inquisitive epistemic model:*

(a) Empty set property: for every $\mathcal{L}_{BE}$-formula φ, $\emptyset \vDash \varphi$.

(b) Persistence property: for every $\mathcal{L}_{BE}$-formula φ and all states s, t, if $s \vDash \varphi$ and $t \subseteq s$ then $t \vDash \varphi$.

(c) Union closure property: for every declarative $\mathcal{L}_{BE}$-formula α and any set of states X, if $s \vDash \alpha$, for each $s \in X$, then $\bigcup X \vDash \alpha$.

Note that for declarative formulas the persistence property and the union closure property are together equivalent to the following characterisation of support for declarative $\mathcal{L}_{BE}$-formulas:

Truth-conditionality of declaratives: $s \vDash \alpha$ iff $\{w\} \vDash \alpha$ for all $w \in s$.

For the basic propositional connectives $\neg, \wedge, \vee, \rightarrow$ the support conditions at singleton states boil down to the standard truth-conditions. For example, $\{w\} \vDash \neg\varphi$ iff $\{w\} \nvDash \varphi$. Thus, truth-conditionality of declaratives ensures that the set of IEL-valid formulas in the language restricted to these connectives is identical with the set of classical tautologies.

Every $\mathcal{L}_{BE}$-formula φ can be associated with a finite set of declarative formulas $\mathcal{R}(\varphi)$ called resolutions of φ. The function $\mathcal{R}$ is defined by the following recursive clauses:

- $\mathcal{R}(p) = \{p\}$, $\mathcal{R}(\bot) = \{\bot\}$,
- $\mathcal{R}(K_a\varphi) = \{K_a\varphi\}$, $\mathcal{R}(E_a\varphi) = \{E_a\varphi\}$,
- $\mathcal{R}(\varphi \rightarrow \psi) = \{\bigwedge_{\alpha \in \mathcal{R}(\varphi)} \alpha \rightarrow f(\alpha) \mid f : \mathcal{R}(\varphi) \rightarrow \mathcal{R}(\psi)\}$,
- $\mathcal{R}(\varphi \wedge \psi) = \{\alpha \wedge \beta \mid \alpha \in \mathcal{R}(\varphi), \beta \in \mathcal{R}(\psi)\}$,
- $\mathcal{R}(\varphi \mathbin{\vee\!\!\!\vee} \psi) = \mathcal{R}(\varphi) \cup \mathcal{R}(\psi)$.

It holds for any declarative α that $\mathcal{R}(\alpha) = \{\alpha\}$. If $\mathcal{R}(\varphi)$ has more than one element then φ represents a question and the elements of $\mathcal{R}(\varphi)$ can be viewed as possible direct answers to the question φ. The connection between φ and $\mathcal{R}(\varphi)$ is stated in the following theorem together with disjunction property and its strengthened variant, the so-called splitting property (for more details see [3], [6] or [4]).

Theorem 2. *For any $\mathcal{L}_{BE}$-formulas φ, ψ:*

(a) Disjunctive normal form: if $\mathcal{R}(\varphi) = \{\alpha_1, \ldots, \alpha_n\}$ then φ is IEL*-equivalent to $\alpha_1 \mathbin{\vee\!\!\!\vee} \ldots \mathbin{\vee\!\!\!\vee} \alpha_n$.*

(b) Disjunction property: if $\varphi \mathbin{\vee\!\!\!\vee} \psi$ is IEL*-valid then φ is* IEL*-valid or ψ is* IEL*-valid.*

(c) Splitting property: for any declarative $\mathcal{L}_{BE}$-formula α, if $\alpha \rightarrow (\varphi \mathbin{\vee\!\!\!\vee} \psi)$ is IEL*-valid then $\alpha \rightarrow \varphi$ is* IEL*-valid or $\alpha \rightarrow \psi$ is* IEL*-valid.*

Table 1: Axiomatization of inquisitive epistemic logic IEL

Axioms of Basic Inquisitive Logic

INT	Axioms of intuitionistic logic, with $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ in the role of disjunction
Split	$(\alpha \to (\varphi \mathbin{\vee\!\!\!\vee} \psi)) \to ((\alpha \to \varphi) \mathbin{\vee\!\!\!\vee} (\alpha \to \psi))$, for declarative α
DN	$\neg\neg\alpha \to \alpha$, for declarative α

Modal axioms:

E1	$E_a(\varphi \to \psi) \to (E_a\varphi \to E_a\psi)$	E2	$E_a\varphi \to E_aE_a\varphi$
E3	$E_a\alpha \to \alpha$, for declarative α	E4	$\neg E_a\varphi \to E_a\neg E_a\varphi$
K1	$K_a(\varphi \to \psi) \to (K_a\varphi \to K_a\psi)$	KD	$K_a(\varphi \mathbin{\vee\!\!\!\vee} \psi) \leftrightarrow (K_a\varphi \vee K_a\psi)$
KE	$E_a\alpha \leftrightarrow K_a\alpha$, for declarative α		

Rules:

MP	$\varphi, \varphi \to \psi / \psi$
NecE	$\varphi / E_a\varphi$
NecK	$\varphi / K_a\varphi$

The set of IEL-valid $\mathcal{L}_{BE}$-formulas is axiomatized by the Hilbert system presented in Table 1. For a completeness proof, see also [3]. The axioms INT, Split, DN plus the rule MP, when formulated in the language $\mathcal{L}_B$, axiomatize what is known as Basic Inquisitive Logic (InqB). The most distinguishing feature of InqB is the Split axiom that plays a key role in the proof of the syntactic counterpart of the disjunctive normal form theorem. Notice that in the presence of the Split axiom disjunction property is equivalent to splitting property. We will formulate this claim more precisely in Section 4 (Lemma 5).

The axioms E1-E4 plus K1, and the rules NecE, NecK are standard principles of epistemic logic extrapolated to the language with questions. The axioms KD and KE are specific principles of inquisitive epistemic logic. The axiom KD has a very intuitive meaning. For example, the agent *a knows whether p or q*, i.e. $K_a(p \mathbin{\vee\!\!\!\vee} q)$, iff either the agent knows that p, or she knows that q, i.e. $K_ap \vee K_aq$. The axiom KE says that the modalities E_a and K_a can differ only if they are applied to questions.

A peculiar feature of all versions of inquisitive logic proposed in the literature is that they are closed under substitutions of declarative formulas but not under uniform substitution. This holds also for IEL. The fact that IEL is closed under substitutions of declarative $\mathcal{L}_{BE}$-formulas is evident from the formulation of the axiomatic system since every axiom and every rule allows this type of substitution. All the restrictions to declarative formulas are necessary. For example, we have illustrated above that $E_a(p \mathbin{\vee\!\!\!\vee} q)$ and $K_a(p \mathbin{\vee\!\!\!\vee} q)$ are not IEL-equivalent which shows that the axiom schema KE cannot be strengthen to the full language $\mathcal{L}_{BE}$.

3 Non-Classical Inquisitive Epistemic Semantics

In this section, we generalize the semantics of IEL in a way that allows the background logic of declaratives to be non-classical. The reasons for such a generalization are mathematical as well as philosophical. The mathematical reasons are that we want to isolate the general mathematical features of the semantics of IEL that are responsible for the characteristic properties of inquisitive semantics such as those expressed in Theorems 1 and 2. We will show that if we extract such features and define the semantic structures through these features (see Definition 3 below) we will obtain a class of semantic models that is much broader than the class of concrete inquisitive epistemic models, and that nevertheless preserves all the essential mathematical properties of inquisitive semantics expressed in Theorems 1 and 2. On the syntactic side we will see that this class of models corresponds to a logic of declaratives that is much weaker than classical logic. This shows that only a fragment of classical logic is needed for the inquisitive disjunction to behave properly and interact in a suitable way with the other operators, including the epistemic modalities. Such a fragment of classical logic is embodied in our basic logic InqSE that will be introduced in the next section.

The philosophical reasons for this generalization concern the well-known problems of classical logic that have been discussed over more than one hundred years, especially since the work of C.I. Lewis (e.g. [12]). There is a widespread intuition that some features of classical logic are problematic, as for example the validity of the following formulas:

$$(p \wedge \neg p) \rightarrow q,\ p \rightarrow (q \rightarrow p),\ \neg(p \rightarrow q) \rightarrow p,\ (p \rightarrow q) \vee (q \rightarrow r).$$

It seems that, at least in a sense, these forms are invalid and the aim of non-classical logics is to avoid them. If one admits that such forms are problematic, one can notice that these problems project to the inquisitive and epistemic layer of IEL. For a simple example, consider the formula $\neg(p \rightarrow q) \rightarrow p$. In words: *if p does not imply q then p is true.* This is classically valid, and thus $K_a\neg(p \rightarrow q) \rightarrow K_a p$, i.e. *if a knows that p does not imply q, then a knows that p*, is IEL-valid. As a consequence, $K_a\neg(p \rightarrow q) \rightarrow K_a?p$, i.e. *if a knows that p does not imply q, then a knows whether p*, is also IEL-valid.

The aim of this paper is neither to analyse the exact nature of such problems, nor to find an optimal logic that avoids them. Instead, we just acknowledge that there are some well-known questionable features of classical logic that affect also IEL and that in general motivate the investigation of its non-classical variants. Our aim is to propose a general semantic framework that is as flexible as possible, so that it is not dependent on the questionable features of classical logic, but, at the same time, it still preserves the essential properties of IEL, and thus it allows us to analyze questions and their interaction with the epistemic operators in the style of inquisitive semantics. Though we will discuss in the next section a concrete logic InqSE, which is the weakest logic determined by our framework, we do not claim that this is the optimal inquisitive epistemic logic. Our framework is designed for a very large class of logical systems, including the system IEL itself, and InqSE is just a lower bound of these systems.

A similar project, but focused on the language of inquisitive logic without the epistemic modalities, was pursued in [15]. However, it turns out that the semantics proposed in [15] is not suitable for our present purpose. The reason is that the mentioned semantics does not have the means to simulate, on a more general level, the interaction between the layer of worlds and the layer of states that is crucially exploited in IEL in the semantic clauses for the modal operators. To be able to handle the modalities we need to preserve this feature of the basic inquisitive semantics, namely the presence of the two distinct layers. Instead of following the strategy of [15], we will rather generalize the approach of [16], where a semantics for the inquisitive extension of one particular logic, namely the relevant logic R, was developed, and where the two-layered character of the basic framework was preserved. This approach is also akin to Fine's relational-operational semantics for relevant logic [11].

To allow the weakening of classical logic we will have to replace possible worlds with something more general. For this purpose, we will use the key notion of situation semantics [1], namely the notion of a *situation*. A crucial feature of situations is that they are (typically) partial. A situation represents only a part of a bigger reality. In this respect, situations contrast with possible worlds, which are complete in the sense that every sentence of a given language is true or false in them. Unlike worlds, situations may be related one to another in a variety of ways: they can overlap, one situation can be a part of another situation and so on. Hence, a space of situations has much more fine-grained and richer structure than a space of possible worlds. Barwise and Perry demonstrated in [1] that this extra structure can be effectively exploited when one wants to analyse the meaning of various kinds of utterances.

In the semantics of IEL, information states are represented as sets of possible worlds. If $w \notin s$, for a world w and state s, then the world w is not compatible with the information represented by s and thus, in this sense, it is not an open possibility from the perspective of s. The role of information states is to determine which possible worlds are excluded and which are not. In our framework, information states will play the same role with respect to situations. Hence, the role of an information state is to determine, in an analogous way, which situations are excluded and which are not. For technical reasons we will need to require a persistence property that can be vaguely stated in the following way:

P If a situation t is not excluded by the information state s, then no possible extension of t is excluded by s.

This requirement can be accepted if we understand situations in a modal way: A situation t does not determine only what is true and what is false in it but it determines also what are its possible extensions, that is, what are the situations that have t as one of their parts. Then if an information state s is not compatible with a situation u which is, from the perspective of t, a possible extension of t then s is also not compatible with t itself. This is just an equivalent formulation of the persistence property P.

In the semantics of IEL singletons are special information states that characterize completely a single possible world. Algebraically speaking singletons are completely join-irreducible elements in the Boolean algebra of information states, i.e. exactly those states that are different from the bottom element and cannot be expressed as non-empty unions (joins) of other states. In our generalized framework we will allow for algebras of information states that are not Boolean. In the abstract framework we will identify a situation with the information state that characterizes the situation completely. The specific algebraic feature of such a completely characterizing state will be the same as in the semantics of IEL: it is a completely join-irreducible element in the algebra of information states.

Definition 1. *Let $L = \langle S, \sqsubseteq \rangle$ be a complete lattice (of information states) where, for any $X \subseteq S$, $\bigsqcup X$ denotes the join of X w.r.t. $\sqsubseteq$. An element $s \in S$ is called a situation in L iff it is completely join-irreducible, i.e., for any $X \subseteq S$, if $s = \bigsqcup X$ then $s = t$, for some $t \in X$.*

Given a lattice of information states, $s \sqsubseteq t$ can be understood in accordance with the semantics of IEL as meaning that the state s is a refinement of the state t. Moreover, if s is a situation, we regard s as a possibility that is not excluded by the state t. In the semantics that we will propose, it will be assumed that every state s is completely determined by the set of situations that are not excluded by s, in particular, s is determined as the join of this set. If $s \sqsubseteq t$ for two situations, s and t, we may also say that the situation t is a part of the situation s.

We will also need the notion of an issue. Intuitively, an issue can be viewed as a demand of an agent to possess knowledge concerning some matter, i.e. to reach a sufficiently rich information state. Formally, an issue is represented by the set of those information states that resolve the issue, i.e. those that satisfy the demand. As in the semantics of IEL, we will assume that if a state resolves an issue then any refinement of the state resolves the issue as well. In other words, the set of states resolving the issue is downward closed. Moreover, we will assume that there is an inconsistent state that trivially resolves every issue. This determines the defining conditions of an issue.

Definition 2. *Let $L = \langle S, \sqsubseteq \rangle$ be a complete lattice (of information states). An issue in L is any non-empty subset of S that is downward closed w.r.t. $\sqsubseteq$.*

Now we can define the semantic structures of our general framework.

Definition 3. *An abstract epistemic information model (AEI-model, for short) is a structure $\mathcal{M} = \langle S, \sqsubseteq, C, \cdot, 1, \Sigma_{\mathcal{A}}, V \rangle$ such that*

(a) $\langle S, \sqsubseteq \rangle$ *is a complete lattice (of information states),*

(b) every state from S is identical to the join of a set of situations,

(c) $\cdot$ *is a binary operation on S with respect to which 1 is a left-identity: $1 \cdot s = s$,*

(d) $\sqcap$*, i.e. the meet w.r.t. $\sqsubseteq$, and $\cdot$ distribute over arbitrary joins from both directions, that is, for any $s \in S$, and any $X \subseteq S$:*

$$s \cdot \bigsqcup X = \bigsqcup \{s \cdot t \mid t \in X\} \text{ and } \bigsqcup X \cdot s = \bigsqcup \{t \cdot s \mid t \in X\},$$

$$s \sqcap \bigsqcup X = \bigsqcup \{s \sqcap t \mid t \in X\} \text{ and } \bigsqcup X \sqcap s = \bigsqcup \{t \sqcap s \mid t \in X\},$$

(e) C is a symmetric relation among the states of S,

(f) $sC(\bigsqcup X)$ *iff there is $t \in X$ such that sCt,*

(g) $\Sigma_{\mathcal{A}} = \{\Sigma_a \mid a \in \mathcal{A}\}$*, where each Σ_a is a function assigning issues to situations and satisfying for any situations s, t: if $s \sqsubseteq t$ then $\Sigma_a(s) \subseteq \Sigma_a(t)$,*

(h) $V(p) \in S$.

As in the semantics proposed in [15], C represents a *compatibility relation* among states. Intuitively, compatibility between two states means that the two states do not support mutually incompatible pieces of information. The left-to-right implication of the condition (f), when combined with the condition (b), boils down to the requirement that if a state is compatible with another state, then the former has to be compatible with a situation that is not excluded by the latter. The right-to-left implication of (f) amounts to saying that if a state is compatible with another state then it must be compatible with any weaker state. The operation $\cdot$ represents *fusion* of two states, the state 1 represents the *logical state* with respect to which the "logic" of the model will be defined. However, note that the state 1 does not have to be the top element in the structure. A more detailed discussion of how the compatibility relation, fusion and the logical state may be interpreted can be found in [16].

For each agent a, Σ_a represents an *inquisitive state map* assigning to each situation an issue interpreted as the issue of the agent in that situation. The monotonicity condition expressed in (g) for the inquisitive state maps intuitively says that if a situation s is a refinement of a situation t, that is, if t is a part of s, then any state that resolves the agent's issue in the stronger situation s resolves the agent's issue also in the weaker situation t. A valuation is a function that assigns to every atomic formula an informational content. It is assumed in (h) that this informational content forms a single information state that is present in the lattice of information states.

Given an AEI-model $\mathcal{M} = \langle S, \sqsubseteq, C, \cdot, 1, \Sigma_{\mathcal{A}}, V\rangle$, the set of situations in $\mathcal{M}$ will be denoted as $Sit(\mathcal{M})$. For any $s \in S$, $Sit(s)$ denotes the set of situations that refine s, i.e. $Sit(s) = \{t \in Sit(\mathcal{M}) \mid t \sqsubseteq s\}$. One can easily verify that (b) from Definition 3 is equivalent to $s = \bigsqcup Sit(s)$, for every state s.

Note also that since the lattice is complete, it has the least element that can be denoted as 0. The state 0, which can be conceived of as representing the trivially inconsistent state, is not a situation. Nevertheless, it can be expressed as a join of a set of situations, namely the join of the empty set, for it is the case that $0 = \bigsqcup\emptyset$. Since $\cdot$ distributes over arbitrary joins from both direction, it holds for any state that $s \cdot \bigsqcup\emptyset = \bigsqcup\emptyset = \bigsqcup\emptyset \cdot s$, i.e. $s \cdot 0 = 0 = s \cdot 0$. Distributivity of fusion with respect to joins also implies monotonicity of fusion, i.e.:

- if $s \sqsubseteq t$ and $u \sqsubseteq v$ then $s \cdot u \sqsubseteq t \cdot v$

Moreover, since situations are completely join-irreducible, and the meet distributes over arbitrary joins, the following holds for every situation s and every set of states X:

- if $s \sqsubseteq \bigsqcup X$ then for some $t \in X$, $s \sqsubseteq t$.

AEI-models can serve as semantic structures for a language of substructural logics enriched with inquisitive disjunction and the epistemic modalities. We will denote it as $\mathcal{L}_{SE}$ and define it as follows:[2]

$$\varphi := p \mid \bot \mid \mathtt{t} \mid \neg\varphi \mid \varphi \to \varphi \mid \varphi \wedge \varphi \mid \varphi \otimes \varphi \mid \varphi \vee \varphi \mid \varphi \mathbin{\vee\!\!\!\vee} \varphi \mid K_a\varphi \mid E_a\varphi$$

[2]For the sake of simplicity, we will have only one implication in the language, even though we do not generally require that fusion is commutative. If needed the second implication may be easily added into the language.

In contrast to $\mathcal{L}_{BE}$, negation $\neg$ and the non-inquisitive disjunction $\vee$ are not regarded as defined symbols in $\mathcal{L}_{SE}$. Equivalence $\leftrightarrow$ will be still regarded as defined by means of conjunction $\wedge$ and implication $\rightarrow$. The constant t is a constant for logical truth and $\otimes$ is intensional conjunction known from substructural logics. In general, it is still not completely clear how exactly this connective relates to natural language but it plays an important role for example in fuzzy logic, linear logic, or relevant logic. Its significance for the analysis of reasoning in the context of relevant logic was discussed in great detail, for example, in [17].

We intend to interpret the modal operators similarly to how they are interpreted in IEL. However, in the basic general setting we do not impose such restrictions as factivity and introspection that are assumed in IEL. As a consequence, $K_a\alpha$ can be hardly read as "the agent a knows that α" because on the syntactic side we will not have the characteristic principle of knowledge $K_a\alpha \rightarrow \alpha$. For a declarative $\mathcal{L}_{SE}$-formula α, the formula $K_a\alpha$ should be rather read as "α holds according to the a's information" (without implying that the information is correct). Correspondingly, for a question φ, the formula $K_a\alpha$ should be read as "φ is resolved by a's information".

In accordance with the semantics of IEL we will denote the information state of the agent a in the situation s as $\sigma_a(s)$ and we define:

$$\sigma_a(s) = \bigsqcup \Sigma_a(s).$$

Then the support conditions are defined as follows:

- $s \Vdash p$ iff $s \sqsubseteq V(p)$,
- $s \Vdash \bot$ iff $s = 0$,
- $s \Vdash \mathsf{t}$ iff $s \sqsubseteq 1$,
- $s \Vdash \neg\varphi$ iff for any $t \in S$, if tCs then $t \nVdash \varphi$,
- $s \Vdash \varphi \rightarrow \psi$ iff for any $t \in S$, if $t \Vdash \varphi$, then $s \cdot t \Vdash \psi$,
- $s \Vdash \varphi \wedge \psi$ iff $s \Vdash \varphi$ and $s \Vdash \psi$,
- $s \Vdash \varphi \otimes \psi$ iff for some $t, u \in S$, $s \sqsubseteq t \cdot u$, $t \Vdash \varphi$ and $u \Vdash \psi$,
- $s \Vdash \varphi \vee \psi$ iff for some $t, u \in S$, $s \sqsubseteq t \sqcup u$, $t \Vdash \varphi$ and $u \Vdash \psi$,
- $s \Vdash \varphi \mathbin{\vee\!\!\!\vee} \psi$ iff $s \Vdash \varphi$ or $s \Vdash \psi$,
- $s \Vdash K_a\varphi$ iff for any $t \in Sit(s)$, $\sigma_a(t) \Vdash \varphi$,
- $s \Vdash E_a\varphi$ iff for any $t \in Sit(s)$ and for any $u \in \Sigma_a(t)$, $u \Vdash \varphi$.

We define $Th(\mathcal{M})$ as the set of $\mathcal{L}_{SE}$-formulas supported by the state 1 in $\mathcal{M}$. If $\varphi \in Th(\mathcal{M})$, we say that φ is valid in $\mathcal{M}$. We say that φ is InqSE-valid if it is valid in every AEI-model. φ and ψ are InqSE-equivalent if $\varphi \leftrightarrow \psi$ is InqSE-valid. Moreover, we say that a set of $\mathcal{L}_{SE}$-formulas Γ InqSE-entails an $\mathcal{L}_{SE}$-formula φ if for any AEI-model $\mathcal{M}$ and any state s in $\mathcal{M}$, if s supports every formula from Γ then s supports φ.

Lemma 1. *For any AEI-model $\mathcal{M}$ and for any $\mathcal{L}_{SE}$-formulas $\varphi, \varphi_1, \ldots, \varphi_n, \psi$:*

(a) $\varphi \to \psi$ *is valid in* $\mathcal{M}$ *iff for any state* s *in* $\mathcal{M}$*, if* $s \Vdash \varphi$ *then* $s \Vdash \psi$*,*

(b) $\{\varphi_1, \ldots, \varphi_n\}$ InqSE-*entails* ψ *iff* $(\varphi_1 \wedge \ldots \wedge \varphi_n) \to \psi$ *is* InqSE-*valid.*

Proof. (a) $1 \Vdash \varphi \to \psi$ iff for any state s, if $s \Vdash \varphi$ then $1 \cdot s \Vdash \psi$ iff (since $1 \cdot s = s$) for any state s, if $s \Vdash \varphi$ then $s \Vdash \psi$.

(b) $\{\varphi_1, \ldots, \varphi_n\}$ InqSE-entails ψ iff for any AEI-model $\mathcal{M}$ and any state s in $\mathcal{M}$, if s supports $\varphi_1 \wedge \ldots \wedge \varphi_n$ then s supports φ iff (using (a)) for any AEI-model $\mathcal{M}$, $(\varphi_1 \wedge \ldots \wedge \varphi_n) \to \psi$ is valid in $\mathcal{M}$ iff $(\varphi_1 \wedge \ldots \wedge \varphi_n) \to \psi$ is InqSE-valid. □

Lemma 1-b cannot be extended to the standard semantic version of the deduction theorem. For example $\{p, q\}$ InqSE-entails p but $\{p\}$ does not InqSE-entail $q \to p$.

It is useful to observe that at situations, the support conditions for the modalities K_a and E_a may be significantly simplified.

Lemma 2. *For any situation* s *of any AEI-model, and any* $\mathcal{L}_{SE}$*-formula* φ*:*

(a) $s \Vdash K_a\varphi$ *iff* $\sigma_a(s) \Vdash \varphi$*,*

(b) $s \Vdash E_a\varphi$ *iff for any* $t \in \Sigma_a(s)$*,* $t \Vdash \varphi$*.*

Any concrete inquisitive epistemic model can be viewed as a particular example of an AEI-model where S is the powerset of the set of possible worlds W, $\sqsubseteq$ is the subset relation, sCt is defined as $s \cap t \neq \emptyset$, fusion $\cdot$ coincides with intersection, and 1 is identical to the set W. Situations in these models are singletons. To build an intuition concerning our abstract framework it might be illuminating to view the conditions defining AEI-models in Definition 3 as a collection of some abstract features of concrete inquisitive epistemic models. The support conditions for atomic formulas, and connectives of the language $\mathcal{L}_{BE}$ also directly generalize corresponding semantic clauses of IEL. An important fact that we will now explain in more detail is that the crucial semantic features of IEL are preserved in this generalization.

Definition 4. *The set of declarative formulas of the language* $\mathcal{L}_{SE}$ *is defined as the least set that contains every atomic formula, the constants* $\bot$ *and* t*, is closed under* $\neg$*,* $\to$*,* $\wedge$*,* $\otimes$*,* $\vee$*, and contains* $K_a\varphi$ *and* $E_a\varphi$ *for every* $\mathcal{L}_{SE}$*-formula* φ*.*

We will usually use α, β, γ as variables for declarative $\mathcal{L}_{SE}$-formulas and φ, ψ, χ as variables for arbitrary $\mathcal{L}_{SE}$-formulas. As in $\mathcal{L}_{BE}$, declarative $\mathcal{L}_{SE}$-formulas may involve inquisitive disjunction but only in the scope of the modalities K_a and E_a. The following result is a generalization of Theorem 1. Just instead of the empty set, inclusion and union, we have now respectively the more general 0, $\sqsubseteq$ and $\bigsqcup$.

Theorem 3. *In every AEI-model:*

(a) *Zero state property: for every* $\mathcal{L}_{SE}$*-formula* φ*,* $0 \Vdash \varphi$*.*

(b) *Persistence property: for every* $\mathcal{L}_{SE}$*-formula* φ *and any states* s, t*, if* $s \Vdash \varphi$ *and* $t \sqsubseteq s$ *then* $t \Vdash \varphi$*.*

(c) *Join closure property: for every declarative* $\mathcal{L}_{SE}$*-formula* α *and any set of states* X*, if* $s \Vdash \alpha$*, for each* $s \in X$*, then* $\bigsqcup X \Vdash \alpha$*.*

Proof. Each of these claims can be proved by induction. We will not go through all the cases but we will illustrate every property with several selected examples of the inductive steps.

(a) First, we consider the inductive step for $\otimes$. Our inductive assumption is that $0 \Vdash \varphi$ and $0 \Vdash \psi$. Since $0 = 0 \cdot 0$ we immediately obtain $0 \Vdash \varphi \otimes \psi$. Second, we consider the inductive step for $\to$. For any t, it holds that $t \cdot 0 = 0$. So, by the inductive assumption, if $t \Vdash \varphi$, then $t \cdot 0 \Vdash \psi$, i.e. $0 \Vdash \varphi \to \psi$. For K_a and E_a, (a) is simply a consequence of the fact that $Sit(0)$ is empty.

(b) First, we consider the inductive step for $\neg$. Assume that $s \Vdash \neg\varphi$ and $t \sqsubseteq s$. Let uCt. Then, due to the right-to-left implication of (f) in Definition 3, uCs, and so $u \nVdash \varphi$. Hence $t \Vdash \neg\varphi$. Second, we consider the inductive step for $\to$. Assume $s \Vdash \varphi \to \psi$ and $t \sqsubseteq s$. Let $u \Vdash \varphi$. Then $s \cdot u \Vdash \psi$. Since $t \cdot u \sqsubseteq s \cdot u$, the inductive assumption implies $t \cdot u \Vdash \psi$, and thus $t \Vdash \varphi \to \psi$. The inductive step for modalities follows from the fact that $s \sqsubseteq t$ implies $Sit(s) \subseteq Sit(t)$.

(c) First, we consider the inductive step for $\vee$. Assume that $s \Vdash \alpha \vee \beta$, for any $s \in X$. Then for any $s \in X$, there are t_s and u_s such that $t_s \Vdash \alpha$, $u_s \Vdash \beta$ and $s \sqsubseteq t_s \sqcup u_s$. Then, by the induction hypothesis, $\bigsqcup_{s\in X} t_s \Vdash \alpha$, $\bigsqcup_{s\in X} u_s \Vdash \beta$. Since also $\bigsqcup X \sqsubseteq \bigsqcup_{s\in X} t_s \sqcup \bigsqcup_{s\in X} u_s$, we obtain $\bigsqcup X \Vdash \alpha \vee \beta$. Second, we consider the inductive step for implication. Assume that $s \Vdash \alpha \to \beta$, for any $s \in X$. Let $t \in S$ such that $t \Vdash \alpha$. Then, $s \cdot t \Vdash \beta$, for any $s \in X$. By the inductive assumption $\bigsqcup_{s\in X}(s \cdot t) \Vdash \beta$. Then due to distributivity of fusion over join (Definition 3-d) $\bigsqcup X \cdot t \Vdash \beta$. Hence $\bigsqcup X \Vdash \alpha \to \beta$. The case of K_a can be proved as follows: Let X be a set of states such that for any $s \in X$, $s \Vdash K_a\varphi$. Let $t \in Sit(\bigsqcup X)$. Since t is completely join-irreducible, it follows that $t \in Sit(s)$, for some $s \in X$. Hence $\sigma_a(t) \Vdash \varphi$. It follows that $\bigsqcup X \Vdash K_a\varphi$. The case of E_a is analogous. □

Since we assume $s = \bigsqcup Sit(s)$, we obtain also a version of truth-conditionality for declarative $\mathcal{L}_{SE}$-formulas, as a consequence of the persistence property and the join closure property.

Truth-conditionality of declaratives: $s \Vdash \alpha$ iff $t \Vdash \alpha$ for all $t \in Sit(s)$.

The generalization of Theorem 2 also holds but we will leave its proof to the next section.

4 Non-Classical Inquisitive Epistemic Logics

In this section, we will provide a syntactic characterization of the weakest inquisitive epistemic logic and its extensions. An axiomatic system for which we will prove completeness with respect to the class of all AEI-models is presented in Table 2. If φ is provable in this system we say that it is InqSE-provable. The non-inquisitive and non-modal part of this system corresponds to distributive, non-associative, non-commutative Full Lambek Logic with a paraconsistent negation, and with only one implication. To be more precise, the underlying substructural logic corresponds to Došen's basic system E^+ from [9] enriched with the axiom A2 for $\bot$, the distributive axiom D1 and the rules R7, R8, R9 for t and $\neg$. The non-modal part also corresponds to the basic substructural inquisitive logic introduced in [15] extended with the distributive axiom D1. Note that the modal rules present a weak modal basis and the specific principles of inquisitive epistemic logic, i.e. the axioms KD and KE, are preserved in our generalization. The formulation of the system makes it obvious that the set of InqSE-provable formulas is closed under substitution of declarative $\mathcal{L}_{SE}$-formulas but, like IEL, it is not closed under uniform substitution. This is clear form the fact that, for example, the formula $K_a p \leftrightarrow E_a p$ is an axiom of the system (as an admissible instance of the schema KE) but since the system is weaker than IEL not all substitutional instances of this formula are provable in the system.

For the language $\mathcal{L}_{SE}$ we introduce the notion of resolution defined by the following equations, which extend the definition of resolution for the language $\mathcal{L}_{BE}$ in a natural way:

- $\mathcal{R}(p) = \{p\}$, $\mathcal{R}(\bot) = \{\bot\}$, $\mathcal{R}(\mathsf{t}) = \{\mathsf{t}\}$,
- $\mathcal{R}(K_a\varphi) = \{K_a\varphi\}$, $\mathcal{R}(E_a\varphi) = \{E_a\varphi\}$,
- $\mathcal{R}(\neg\varphi) = \{\bigwedge_{\alpha\in\mathcal{R}(\varphi)} \neg\alpha\}$,
- $\mathcal{R}(\varphi \to \psi) = \{\bigwedge_{\alpha\in\mathcal{R}(\varphi)} \alpha \to f(\alpha) \mid f : \mathcal{R}(\varphi) \to \mathcal{R}(\psi)\}$,
- $\mathcal{R}(\varphi \circ \psi) = \{\alpha \circ \beta \mid \alpha \in \mathcal{R}(\varphi), \beta \in \mathcal{R}(\psi)\}$, for any $\circ \in \{\wedge, \otimes, \vee\}$,
- $\mathcal{R}(\varphi \vee\!\!\!\vee \psi) = \mathcal{R}(\varphi) \cup \mathcal{R}(\psi)$.

A crucial feature of the logic InqSE is that it is sufficient for the disjunctive normal form result, i.e. every $\mathcal{L}_{SE}$-formula φ is provably equivalent to the inquisitive disjunction of the declarative $\mathcal{L}_{SE}$-formulas from $\mathcal{R}(\varphi)$.

Lemma 3. *Let φ be an $\mathcal{L}_{SE}$-formula and $\mathcal{R}(\varphi) = \{\alpha_1, \ldots, \alpha_n\}$. Then the formula $\varphi \leftrightarrow (\alpha_1 \vee\!\!\!\vee \ldots \vee\!\!\!\vee \alpha_n)$ is* InqSE*-provable.*

Proof. This can be proved by induction on the complexity of the formula φ. The distributive axioms D2-D6 are used in the proof. The most complex case is the inductive step for implication which is handled by the axiom D6. The inductive steps for $\neg, \to, \wedge, \otimes, \vee, \vee\!\!\!\vee$ can be proved as in [15] where a parallel claim was formulated for a language without the epistemic modalities and logic without the distributive axiom D1. The inductive steps for modalities follow directly from the definition of resolutions. □

The following result expresses soundness of the system.

Lemma 4. *Every axiom in Table 2 is valid in every AEI-model and every rule in Table 2 preserves validity in every AEI-model.*

Proof. We will not go through all the axioms and rules. Instead we will select only several interesting cases as an illustration. Let us fix any AEI-model $\mathcal{M}$ with a set of states S. We will often use Lemma 1 without reference. As examples of non-modal axioms, we will prove validity of A5 and A8.

A5: Assume $s \Vdash \varphi$ in $\mathcal{M}$. Due to the zero state property (Lemma 3-a) $0 \Vdash \psi$, for any $\mathcal{L}_{SE}$-formula ψ. Moreover, $s \sqsubseteq s \sqcup 0$ and thus $s \Vdash \varphi \vee \psi$.

A8: Let $s \Vdash \alpha \vee \alpha$, for a declarative α. So, there are $t, u \in S$ such that $s \sqsubseteq t \sqcup u$, $t \Vdash \alpha$ and $u \Vdash \alpha$. Due to the join closure property (Lemma 3-c), $t \sqcup u \Vdash \alpha$, and consequently, due to the persistence property (Lemma 3-b), $s \Vdash \alpha$.

Table 2: Axiomatization of the weakest substructural inquisitive epistemic logic InqSE

Non-modal axioms:

A1	$\varphi \to \varphi$	A2	$\bot \to \varphi$
A3	$(\varphi \land \psi) \to \varphi$	A4	$(\varphi \land \psi) \to \psi$
A5	$\varphi \to (\varphi \lor \psi)$	A6	$\psi \to (\varphi \lor \psi)$
A7	$(\varphi \lor \psi) \to (\psi \lor \varphi)$	A8	$(\alpha \lor \alpha) \to \alpha$ (for declarative α)
A9	$\varphi \to (\varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \psi)$	A10	$\psi \to (\varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \psi)$

Modal axioms:

KD	$K_a(\varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \psi) \leftrightarrow (K_a\varphi \lor K_a\psi)$
KE	$K_a\alpha \leftrightarrow E_a\alpha$ (for declarative α)

Distributive axioms:

D1	$(\varphi \land (\psi \lor \chi)) \to ((\varphi \land \psi) \lor (\varphi \land \chi))$
D2	$(\varphi \otimes (\psi \lor \chi)) \to ((\varphi \otimes \psi) \lor (\varphi \otimes \chi))$
D3	$(\varphi \land (\psi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \chi)) \to ((\varphi \land \psi) \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} (\varphi \land \chi))$
D4	$(\varphi \otimes (\psi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \chi)) \to ((\varphi \otimes \psi) \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} (\varphi \otimes \chi))$
D5	$(\varphi \lor (\psi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \chi)) \to ((\varphi \lor \psi) \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} (\varphi \lor \chi))$
D6	$(\alpha \to (\psi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \chi)) \to ((\alpha \to \psi) \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} (\alpha \to \chi))$ (for declarative α)

Non-modal rules:

R1	$\varphi, \varphi \to \psi / \psi$	R2	$\varphi \to \psi / (\psi \to \chi) \to (\varphi \to \chi)$
R3	$\chi \to \varphi, \chi \to \psi / \chi \to (\varphi \land \psi)$	R4	$\varphi \to \chi, \psi \to \vartheta / (\varphi \lor \psi) \to (\chi \lor \vartheta)$
R5	$\varphi \to (\psi \to \chi) / (\varphi \otimes \psi) \to \chi$	R6	$(\varphi \otimes \psi) \to \chi / \varphi \to (\psi \to \chi)$
R7	$\mathsf{t} \to \varphi / \varphi$	R8	$\varphi / \mathsf{t} \to \varphi$
R9	$\varphi \to \neg\psi / \psi \to \neg\varphi$	R10	$\varphi \to \chi, \psi \to \chi / (\varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \psi) \to \chi$

Modal rules:

MR1	$\varphi \to \psi / E_a\varphi \to E_a\psi$	MR2	$E_a\varphi \land E_a\psi / E_a(\varphi \land \psi)$
MR3	$\varphi \to \psi / K_a\varphi \to K_a\psi$	MR4	$K_a\varphi \land K_a\psi / K_a(\varphi \land \psi)$

We will prove soundness of the modal axioms. Since $\sigma_a(s) = \bigsqcup \Sigma_a(s)$, soundness of KE holds due to the join closure property and the persistence property. Let us prove KD.

KD($\rightarrow$): First, assume that $s \Vdash K_a(\varphi \vee\!\!\!\vee \psi)$. That means that for any $t \in Sit(s)$, $\sigma_a(t)$ supports φ or $\sigma_a(t)$ supports ψ. Let $Sit_a^\varphi(s) = \{t \in Sit(s) \mid \sigma_a(t) \Vdash \varphi\}$ and $Sit_a^\psi(s) = \{t \in Sit(s) \mid \sigma_a(t) \Vdash \psi\}$. It holds that for every $t \in Sit_a^\varphi(s)$, $t \Vdash K_a\varphi$ and for every $t \in Sit_a^\psi(s)$, $t \Vdash K_a\psi$. Since support of declarative formulas is closed under arbitrary joins, it holds that $\bigsqcup Sit_a^\varphi(s) \Vdash K_a\varphi$ and $\bigsqcup Sit_a^\psi(s) \Vdash K_a\psi$. Moreover, $(\bigsqcup Sit_a^\varphi(s)) \sqcup (\bigsqcup Sit_a^\psi(s)) = \bigsqcup(Sit_a^\varphi(s) \cup Sit_a^\psi(s)) = \bigsqcup Sit(s) = s$. So, $s \Vdash K_a\varphi \vee K_a\psi$.

KD($\leftarrow$): Second, assume that $s \Vdash K_a\varphi \vee K_a\psi$. Then there are states s_1, s_2 such that $s \sqsubseteq s_1 \sqcup s_2$ and $s_1 \Vdash K_a\varphi$ and $s_2 \Vdash K_a\psi$. Take any $t \in Sit(s)$. Then $t \sqsubseteq s_1 \sqcup s_2$ and thus $t \sqsubseteq s_1$ or $t \sqsubseteq s_2$. So, $t \in Sit(s_1)$ or $t \in Sit(s_2)$. It follows that $\sigma_a(t) \Vdash \varphi$ or $\sigma_a(t) \Vdash \psi$. Hence, $\sigma_a(t) \Vdash \varphi \vee\!\!\!\vee \psi$ and consequently (since t is an arbitrary element of $Sit(s)$) $s \Vdash K_a(\varphi \vee\!\!\!\vee \psi)$.

As examples of distributive axioms, we prove D2 and D6. D2: Assume $s \Vdash \varphi \otimes (\psi \vee \chi)$. So, there are $t, u \in S$ such that $t \Vdash \varphi$, $u \Vdash \psi \vee \chi$, and $s \sqsubseteq t \cdot u$. Then, there are $u_1, u_2 \in S$ such that $u_1 \Vdash \psi$, $u_2 \Vdash \chi$ and $u \sqsubseteq u_1 \sqcup u_2$. It follows that $t \cdot u_1 \Vdash \varphi \otimes \psi$ and $t \cdot u_2 \Vdash \varphi \otimes \chi$. Moreover, due to monotonicity of fusion and its distributivity over join (Definition 3-d) $s \sqsubseteq t \cdot u \sqsubseteq t \cdot (u_1 \sqcup u_2) = (t \cdot u_1) \sqcup (t \cdot u_2)$. Thus, $s \Vdash (\varphi \otimes \psi) \vee (\varphi \otimes \chi)$.

D6: Assume $s \nVdash \alpha \rightarrow \psi$ and $s \nVdash \alpha \rightarrow \chi$, where α is declarative. Then there are $t, u \in S$ such that $t \Vdash \alpha$ but $s \cdot t \nVdash \psi$, and $u \Vdash \alpha$ but $s \cdot u \nVdash \chi$. Due to the join closure property, $t \sqcup u \Vdash \alpha$, and due to the persistence property $(s \cdot t) \sqcup (s \cdot u) \nVdash \psi \vee\!\!\!\vee \chi$. Since join distributes over fusion (Definition 3-d), it holds that $s \cdot (t \sqcup u) = (s \cdot t) \sqcup (s \cdot u)$, and we obtain $s \nVdash \alpha \rightarrow (\psi \vee\!\!\!\vee \chi)$.

As an example of a non-modal rule, let us consider $R6$: Assume $1 \Vdash (\varphi \otimes \psi) \rightarrow \chi$. We will prove $1 \Vdash \varphi \rightarrow (\psi \rightarrow \chi)$. Assume $s \Vdash \varphi$. We need to show that $s \Vdash \psi \rightarrow \chi$. Let $t \Vdash \psi$. Then $s \cdot t \Vdash \varphi \otimes \psi$, and hence $s \cdot t \Vdash \chi$, which is what we needed to show.

As an example of a modal rule, we will prove that MR1 preserves validity. Let $1 \Vdash \varphi \rightarrow \psi$. Assume that $s \Vdash E_a\varphi$, i.e. for any $t \in Sit(s)$ and any $u \in \Sigma_a(t)$, $u \Vdash \varphi$. Then for any $t \in Sit(s)$ and any $u \in \Sigma_a(t)$, $u \Vdash \psi$, and thus $s \Vdash E_a\psi$. We have shown that $1 \Vdash E_a\varphi \rightarrow E_a\psi$. □

Lemmas 3 and 4 together directly lead to a generalization of Theorem 2-a, that is, to the semantic version of the disjunctive normal form. We will now describe a general construction of canonical models that can be used to obtain completeness results. This construction can be carried out for any axiomatic extension of InqSE. Such extensions will be called *inquistive epistemic logics.*

Definition 5. *A set of $\mathcal{L}_{SE}$-formulas Λ is called an inquisitive epistemic logic if it satisfies the following three conditions: (a) Λ contains all the instances of the axioms of* InqSE *(from Table 2); (b) Λ is closed under the rules of* InqSE *(from Table 2); (c) Λ is closed under substitutions of declarative formulas (i.e. if $\varphi \in \Lambda$ and ψ is obtained from φ by the substitution of a declarative formula for every occurrence of some atomic formula, then $\psi \in \Lambda$). We say that Λ is a constructive inquisitive epistemic logic, if, in addition, inquisitive disjunction has the disjunction property w.r.t. Λ, i.e. $\varphi \vee\!\!\!\vee \psi \in \Lambda$ only if $\varphi \in \Lambda$ or $\psi \in \Lambda$. We say that Λ has the splitting property if for any declarative $\mathcal{L}_{SE}$-formula α, if $\alpha \rightarrow (\varphi \vee\!\!\!\vee \psi) \in \Lambda$ then $\alpha \rightarrow \varphi \in \Lambda$ or $\alpha \rightarrow \psi \in \Lambda$.*

It turns out that in the presence of D6, disjunction property and splitting property are equivalent.

Lemma 5. *Let Λ be an inquisitive epistemic logic. Λ is constructive iff Λ has the splitting property.*

Proof. First, assume that Λ is constructive. Assume $\alpha \to (\varphi \vee\!\!\!\vee \psi) \in \Lambda$. Then, applying D6, we obtain $(\alpha \to \varphi) \vee\!\!\!\vee (\alpha \to \psi) \in \Lambda$, and disjunction property implies $\alpha \to \varphi \in \Lambda$ or $\alpha \to \psi \in \Lambda$. Second, assume that Λ has the splitting property. Assume that $\varphi \vee\!\!\!\vee \psi \in \Lambda$. Using R8, we obtain $\mathsf{t} \to (\varphi \vee\!\!\!\vee \psi) \in \Lambda$. By splitting property, $\mathsf{t} \to \varphi \in \Lambda$ or $\mathsf{t} \to \psi \in \Lambda$. Applying R7, we obtain $\varphi \in \Lambda$ or $\psi \in \Lambda$. □

For any inquisitive epistemic logic Λ we define the notion of a declarative prime Λ-theory.

Definition 6. *Let Λ be an inquisitive epistemic logic and Δ a set of declarative $\mathcal{L}_{SE}$-formulas. Δ is called a (declarative) Λ-theory if it is non-empty and the following two conditions are satisfied for any declarative $\mathcal{L}_{SE}$-formulas α, β: (a) if $\alpha \in \Delta$ and $\beta \in \Delta$, then $\alpha \wedge \beta \in \Delta$; (b) if $\alpha \in \Delta$ and $\alpha \to \beta \in \Lambda$ then $\beta \in \Delta$. Δ is prime if, in addition, $\alpha \vee \beta \in \Delta$ implies $\alpha \in \Delta$ or $\beta \in \Delta$.*

It is worth mentioning that if Λ contains all the instances of the schema $\alpha \to (\beta \to (\alpha \wedge \beta))$, as for example classical and intuitionistic logic do, the condition (a) in the previous definition becomes redundant. However, this schema is not InqSE-provable so we need to retain the condition in the definition.

Let Λ be an inquisitive epistemic logic and Δ a set of $\mathcal{L}_{SE}$-formulas. We will write $\Delta \vdash_\Lambda \varphi$ if there are $\psi_1, \dots, \psi_n \in \Delta$ such that $(\psi_1 \wedge \dots \wedge \psi_n) \to \varphi \in \Lambda$. In the special case when Δ is a Λ-theory, and so is closed under $\wedge$, the condition simplifies: $\Delta \vdash_\Lambda \varphi$ iff there is $\alpha \in \Delta$ such that $\alpha \to \varphi \in \Lambda$. Note that Λ has the splitting property iff the following condition hodls: For any Λ-theory Δ and any $\mathcal{L}_{SE}$-formulas φ, ψ, if $\Delta \vdash_\Lambda \varphi \vee\!\!\!\vee \psi$ then $\Delta \vdash_\Lambda \varphi$ or $\Delta \vdash_\Lambda \psi$.

We have remarked in the previous section that the deduction theorem for InqSE-entailment does not hold. Correspondingly, the syntactic version of the deduction theorem for $\vdash_\Lambda$ does not generally hold: $\Delta \cup \{\varphi\} \vdash_\Lambda \psi$ is not generally equivalent to $\Delta \vdash_\Lambda \varphi \to \psi$, since $(\chi \wedge \varphi) \to \psi$ is not provably equivalent to $\chi \to (\varphi \to \psi)$ in the basic logic.

Unlike in [15], we assume in this paper that our basic logic InqSE contains the distributive axiom D1. This assumption is crucial for our approach since it allows us to prove the following lemmas that will be needed in the construction of the canonical model.

Lemma 6. *For any set of $\mathcal{L}_{SE}$-formulas Δ, any $\mathcal{L}_{SE}$-formulas φ, ψ and any declarative $\mathcal{L}_{SE}$-formula α:*

$$\Delta \cup \{\varphi \vee \psi\} \vdash_\Lambda \alpha \text{ iff } \Delta \cup \{\varphi\} \vdash_\Lambda \alpha \text{ and } \Delta \cup \{\psi\} \vdash_\Lambda \alpha.$$

Proof. We prove only the right-to-left implication. Assume $\Delta \cup \{\varphi\} \vdash_\Lambda \alpha$ and $\Delta \cup \{\psi\} \vdash_\Lambda \alpha$. So, there are $\eta_1, \dots \eta_n \in \Delta$ such that $(\eta_1 \wedge \dots \wedge \eta_n \wedge \varphi) \to \alpha \in \Lambda$ and there are $\mu_1, \dots \mu_m \in \Delta$ such that $(\mu_1 \wedge \dots \wedge \mu_m \wedge \psi) \to \alpha \in \Lambda$. Let $\xi = \eta_1 \wedge \dots \wedge \eta_n \wedge \mu_1 \wedge \dots \wedge \mu_m$. Then $(\xi \wedge \varphi) \to \alpha \in \Lambda$ and $(\xi \wedge \psi) \to \alpha \in \Lambda$. Due to A8, R4 and transitivity of implication, $((\xi \wedge \varphi) \vee (\xi \wedge \psi)) \to \alpha \in \Lambda$. Due to D1 and transitivity of implication, it follows that $(\xi \wedge (\varphi \vee \psi)) \to \alpha \in \Lambda$, and thus $\Delta \cup \{\varphi \vee \psi\} \vdash_\Lambda \alpha$. □

Lemma 7. *For any inquisitive epistemic logic Λ and any Λ-theory Γ, it holds that $\Gamma = \bigcap\{\Delta$ prime Λ-theory $\mid \Gamma \subseteq \Delta\}$, that is, every Λ-theory Γ is identical to the intersection of the prime Λ-theories that extend Γ.*

Proof. Let $X = \{\Delta \text{ prime } \Lambda\text{-theory} \mid \Gamma \subseteq \Delta\}$. It is obvious that $\Gamma \subseteq \bigcap X$. To prove that also $\bigcap X \subseteq \Gamma$, we need to show that for any declarative $\mathcal{L}_{SE}$-formula $\beta \notin \Gamma$ there is a prime Λ-theory $\Delta \in X$ such that $\beta \notin \Delta$. This amounts to the usual Lindenbaum-style lemma. The construction of Δ is standard. We will recapitulate how it is done to illustrate how the construction depends on Lemma 6 and thus, in turn, on the distributive axiom D1. First, we enumerate the declarative $\mathcal{L}_{SE}$-formulas $\alpha_1, \alpha_2, \ldots$ Next, we define infinite sequence of sets of declarative $\mathcal{L}_{SE}$-formulas $\Delta_0, \Delta_1, \ldots$ where $\Delta_0 = \Gamma$ and

$$\Delta_{n+1} = \begin{cases} \Delta_n \cup \{\alpha_n\} & \text{if } \Delta_n \cup \{\alpha_n\} \nvdash_\Lambda \beta, \\ \Delta_n & \text{otherwise.} \end{cases}$$

Finally, we define $\Delta = \bigcup_{i=0}^{\infty} \Delta_i$. It can be easily shown that Δ is a Λ-theory which does not contain β. To show that Δ is prime, assume $\alpha_i \notin \Delta$ and $\alpha_j \notin \Delta$, i.e. $\Delta \cup \{\alpha_i\} \vdash_\Lambda \beta$ and $\Delta \cup \{\alpha_j\} \vdash_\Lambda \beta$. Now, Lemma 6 implies $\Delta \cup \{\alpha_i \vee \alpha_j\} \vdash_\Lambda \beta$, and so $\alpha_i \vee \alpha_j \notin \Delta$. □

Every inquisitive epistemic logic will be associated with a canonical structure. Prime Λ-theories will be used in the definition of the canonical structure but a specific feature of this construction is that a point in the canonical structure is not a single prime Λ-theory but rather a set of prime Λ-theories. In the definition of the canonical structure, we will use the following notation: For any inquisitive epistemic logic Λ let Λ^d denote the set of all declarative $\mathcal{L}_{SE}$-formulas in Λ. Moreover, for any sets of $\mathcal{L}_{SE}$-formulas Δ, Ω let $\Delta \otimes \Omega = \{\varphi \otimes \psi \mid \varphi \in \Delta, \psi \in \Gamma\}$.

Definition 7. *Let Λ be an inquisitive epistemic logic. The canonical structure of Λ is the structure $\mathcal{M}^\Lambda = \langle S^\Lambda, \sqsubseteq^\Lambda, C^\Lambda, \cdot^\Lambda, 1^\Lambda, \Sigma^\Lambda_{\mathcal{A}}, V^\Lambda \rangle$, where*

- *$X \in S^\Lambda$ iff X is a non-empty upward closed (under set inclusion) set of prime Λ-theories, that is, X is a non-empty set of prime Λ-theories such that for any prime Λ-theories Δ, Γ, if $\Gamma \in X$ and $\Gamma \subseteq \Delta$ then $\Delta \in X$,*
- *$X \sqsubseteq^\Lambda Y$ iff $X \subseteq Y$,*
- *$XC^\Lambda Y$ iff there are $\Gamma \in X$ and $\Delta \in Y$ such that for any declarative $\mathcal{L}_{SE}$-formula α, if $\neg\alpha \in \Gamma$ then $\alpha \notin \Delta$,*
- *$X \cdot^\Lambda Y = \{\Gamma$ prime Λ-theory $\mid \Delta \otimes \Omega \subseteq \Gamma$, for some $\Delta \in X$, $\Omega \in Y\}$,*
- *$1^\Lambda = \{\Gamma$ prime Λ-theory $\mid \Lambda^d \subseteq \Gamma\}$,*
- *$X \in \Sigma^\Lambda_a(Y)$ iff for any $\mathcal{L}_{SE}$-formula φ, if $E_a\varphi \in \bigcap Y$ then there is $\alpha \in \mathcal{R}(\varphi)$ such that $\alpha \in \bigcap X$ (assuming that Y is completely join-irreducible),*
- *$\Gamma \in V^\Lambda(p)$ iff $p \in \Gamma$.*

Note that in every canonical structure there is the least element, namely the state $\{\Xi\}$, where Ξ is the set of all declarative $\mathcal{L}_{SE}$-formulas, which is the strongest prime Λ-theory. Let us prove a few lemmas concerning the canonical structure.

Lemma 8. $\Lambda^d = \bigcap 1^\Lambda$, *for any inquisitive epistemic logic* Λ.

Proof. First, observe that Λ^d is a Λ-theory. Λ^d is closed under modus ponens due to R1. To see that it is also closed under conjunction, assume that $\alpha, \beta \in \Lambda^d$. Then, due to R8, $\mathsf{t} \to \alpha, \mathsf{t} \to \beta \in \Lambda^d$. Using R3, we obtain $\mathsf{t} \to (\alpha \wedge \beta) \in \Lambda^d$, and hence, applying R7, $\alpha \wedge \beta \in \Lambda^d$.

Lemma 7 says that every Λ-theory is the intersection of the prime Λ-theories that extend it. In particular, $\Lambda^d = \bigcap 1^\Lambda$. □

Given any partially ordered set S, it holds for its lattice of upward closed sets ordered by inclusion that the completely join irreducible elements in the lattice are exactly the so-called principal upward closed sets, i.e. sets that are generated by a single point of S. As a consequence, situations in a canonical structure have a simple form. They are exactly those upward closed sets of prime Λ-theories that are generated by a single prime Λ-theory.

Lemma 9. *Situations in a canonical structure* $\mathcal{M}^\Lambda$ *are exactly the sets of prime* Λ-*theories of the form* $\Gamma^\uparrow = \{\Delta \text{ prime } \Lambda\text{-theory} \mid \Gamma \subseteq \Delta\}$, *where* Γ *is a prime* Λ-*theory.*

Note that $\bigcap \Gamma^\uparrow = \Gamma$. So, in the light of the previous lemma, the definition of the inquisitive state maps in the canonical structure can be simplified as follows:

$$X \in \Sigma_a^\Lambda(\Gamma^\uparrow) \text{ iff } E_a\varphi \in \Gamma \text{ implies that there is } \alpha \in \mathcal{R}(\varphi) \text{ such that } \alpha \in \bigcap X.$$

We will verify that every canonical structure satisfies the conditions (a)-(h) from Definition 3.

Lemma 10. *For any inquisitive epistemic logic* Λ, *its canonical structure* $\mathcal{M}^\Lambda$ *is an AEI-model.*

Proof. (a) $\langle S^\Lambda, \subseteq \rangle$ forms a complete lattice. Since there is the largest prime Λ-theory, namely the set of all $\mathcal{L}_{SE}$-formulas Ξ, the intersection of any set M of non-empty upward closed sets of prime Λ-theories is again a non-empty upward closed set, the greatest lower bound of M in the canonical structure.

(b) Since $X = \bigcup\{\Gamma^\uparrow \mid \Gamma \in X\}$, for any $X \in S^\Lambda$, any state from the canonical structure is identical to the join (i.e. union) of a set of situations.

(c) We need to show that $1^\Lambda \cdot^\Lambda X = X$, for any $X \in S^\Lambda$. First, we will prove that $1^\Lambda \cdot^\Lambda X \subseteq X$. Assume that $\Gamma \in 1^\Lambda \cdot^\Lambda X$. So, there are $\Delta \in 1^\Lambda$ (i.e. $\Lambda^d \subseteq \Delta$), and $\Omega \in X$ such that $\Delta \otimes \Omega \subseteq \Gamma$. Let us observe that $\mathsf{t} \in \Lambda^d$, due to A1 and R7. Thus, $\mathsf{t} \in \Delta$, and it follows that $\mathsf{t} \otimes \alpha \in \Gamma$, for any $\alpha \in \Omega$. It holds, due to A1, R7, and R5, that $(\mathsf{t} \otimes \varphi) \to \varphi \in \Lambda$, for any $\mathcal{L}_{SE}$-formula φ. It follows that $\Omega \subseteq \Gamma$, and thus $\Gamma \in X$.

We will prove that also $X \subseteq 1^\Lambda \cdot^\Lambda X$. Assume that $\Gamma \in X$. We need to show that $\Gamma \in 1^\Lambda \cdot^\Lambda X$. It will be sufficient if we prove that there is $\Delta \in 1^\Lambda$ such that $\Delta \otimes \Gamma \subseteq \Gamma$. As in the proof of Lemma 7, we enumerate the declarative $\mathcal{L}_{SE}$-formulas $\alpha_1, \alpha_2, \ldots$ Next, we define infinite sequence of sets of declarative $\mathcal{L}_{SE}$-formulas $\Delta_0, \Delta_1, \ldots$ where $\Delta_0 = \Lambda$ and

$$\Delta_{n+1} = \begin{cases} \Delta_n \cup \{\alpha_n\} & \text{if there are no declarative } \mathcal{L}_{SE}\text{-formulas } \gamma \in \Gamma \text{ and} \\ & \delta \notin \Gamma \text{ such that } \Delta_n \cup \{\alpha_n\} \vdash_\Lambda \gamma \to \delta, \\ \Delta_n & \text{otherwise.} \end{cases}$$

Finally, we define $\Delta = \bigcup_{i=0}^{\infty} \Delta_i$. There are no declarative $\mathcal{L}_{SE}$-formulas $\gamma \in \Gamma$ and $\delta \notin \Gamma$ such that $\Delta \vdash_\Lambda \gamma \to \delta$, for this property holds for Λ and is preserved in the steps from Δ_n to Δ_{n+1} and by the union. Moreover, Δ is a prime Λ-theory. For instance, let us verify that $\alpha \vee \beta \in \Delta$ implies $\alpha \in \Delta$ or $\beta \in \Delta$. Assume $\alpha \notin \Delta$ and $\beta \notin \Delta$. Then there are declarative $\mathcal{L}_{SE}$-formulas $\gamma_1^+, \gamma_2^+ \in \Gamma$ and $\gamma_1^-, \gamma_2^- \notin \Gamma$ such that $\Delta \cup \{\alpha\} \vdash_\Lambda \gamma_1^+ \to \gamma_1^-$, and $\Delta \cup \{\beta\} \vdash_\Lambda \gamma_2^+ \to \gamma_2^-$. Let $\gamma^+ = \gamma_1^+ \wedge \gamma_2^+$ and $\gamma^- = \gamma_1^- \vee \gamma_2^-$. Then $\Delta \cup \{\alpha\} \vdash_\Lambda \gamma^+ \to \gamma^-$ and $\Delta \cup \{\beta\} \vdash_\Lambda \gamma^+ \to \gamma^-$. Due to Lemma 6, $\Delta \cup \{\alpha \vee \beta\} \vdash_\Lambda \gamma^+ \to \gamma^-$. Since $\gamma^+ \in \Gamma$ and $\gamma^- \notin \Gamma$, it follows that $\alpha \vee \beta \notin \Delta$. Now, for any $\gamma \in \Gamma$ and $\delta \in \Delta$ it holds that $\Delta \vdash_\Lambda \gamma \to (\delta \otimes \gamma)$, and hence $\delta \otimes \gamma \in \Gamma$. Thus, we have shown that $\Delta \otimes \Gamma \subseteq \Gamma$ as required.

(d) Since meet and join coincide respectively with intersection and union in the canonical structure, meet distributes over arbitrary joins from both directions. We will show that also fusion distributes over arbitrary joins. Let M be a set of states and X a state in the canonical structure. We will only show the case $X \cdot^\Lambda \bigcup M = \bigcup\{X \cdot^\Lambda Y \mid Y \in M\}$. The following equivalences hold: $\Gamma \in X \cdot^\Lambda \bigcup M$ iff there are $\Delta \in X$, $\Omega \in \bigcup M$ s.t. $\Delta \otimes \Omega \subseteq \Gamma$ iff there is $Y \in M$ and there are $\Delta \in X$, $\Omega \in Y$ s.t. $\Delta \otimes \Omega \subseteq \Gamma$ iff $\Gamma \in \bigcup\{X \cdot^\Lambda Y \mid Y \in M\}$.

(e) Symmetry of C^Λ is a consequence of the following observation. Due to A1 and R9, $\varphi \to \neg\neg\varphi \in \Delta$, for any $\mathcal{L}_{SE}$-formula φ. Hence, for any Λ-theories Γ, Δ the following two conditions are equivalent:

- for any declarative $\mathcal{L}_{SE}$-formula α, if $\neg\alpha \in \Gamma$ then $\alpha \notin \Delta$,
- for any declarative $\mathcal{L}_{SE}$-formula α, if $\neg\alpha \in \Delta$ then $\alpha \notin \Gamma$.

The conditions (f) and (h) are straightforward. We will finish the proof with (g), i.e. we will show that for all Λ-theories Γ, Δ, if $\Gamma^\uparrow \subseteq \Delta^\uparrow$ then $\Sigma_a(\Gamma^\uparrow) \subseteq \Sigma_a(\Delta^\uparrow)$. Let $\Gamma^\uparrow \subseteq \Delta^\uparrow$, and hence $\Delta \subseteq \Gamma$. Assume $X \in \Sigma_a(\Gamma^\uparrow)$. We will verify that $X \in \Sigma_a(\Delta^\uparrow)$. Let $E_a\varphi \in \Delta$. Then $E_a\varphi \in \Gamma$ and hence there is $\alpha \in \mathcal{R}(\varphi)$ such that $\alpha \in \bigcap X$. This finishes the proof. □

The following lemma states a crucial feature of the inquisitive state maps in the canonical structures.

Lemma 11. *Let Λ be an inquisitive epistemic logic and Γ a prime Λ-theory. Assume that $E_a\psi \notin \Gamma$. Then there is $X \in \Sigma_a^\Lambda(\Gamma^\uparrow)$ such that for every $\alpha \in \mathcal{R}(\psi)$, $\alpha \notin \bigcap X$.*

Proof. Assume that $E_a\psi \notin \Gamma$. Let us define $\Omega = \{\chi \mid E_a\chi \in \Gamma\}$. It holds that $\Omega \nvdash_\Lambda \psi$. (Assume the contrary: $\Omega \vdash_\Lambda \psi$, i.e. there are $\chi_1, \ldots, \chi_n \in \Omega$ such that $(\chi_1 \wedge \ldots \wedge \chi_n) \to \psi \in \Lambda$. Then due to the rules MR1 and MR2 and transitivity of implication available already in InqSE, $(E_a\chi_1 \wedge \ldots \wedge E_a\chi_n) \to E_a\psi \in \Lambda$. But $E_a\chi_1, \ldots, E_a\chi_n \in \Gamma$, and so $\Gamma \vdash_\Lambda E_a\psi$, i.e. $E_a\psi \in \Gamma$, which is a contradiction.)

Let $\chi_1, \chi_2, \chi_3, \ldots$ be a list of all formulas from Ω. Let $\Omega^{\geq n} = \{\chi_n, \chi_{n+1}, \chi_{n+2}, \ldots\}$. Note that, due to A9, A10 and R10, for any $\mathcal{L}_{SE}$-formulas η, μ, ζ and any set of $\mathcal{L}_{SE}$-formulas Ψ

$$\{\eta \mathbin{\vee\!\!\!\vee} \mu\} \cup \Psi \vdash_\Lambda \zeta \text{ iff } \{\eta\} \cup \Psi \vdash_\Lambda \zeta \text{ and } \{\mu\} \cup \Psi \vdash_\Lambda \zeta.$$

Thus, since $\{\chi_1\} \cup \Omega^{\geq 2} \nvdash \psi$, Lemma 3 implies that there is $\alpha_1 \in \mathcal{R}(\chi_1)$ such that $\{\alpha_1\} \cup \Omega^{\geq 2} \nvdash \psi$. We set $f(\chi_1) = \alpha_1$. Now assume that $f(\chi_1), \ldots, f(\chi_{n-1})$ are already defined and it holds $\{f(\chi_1),\ldots,f(\chi_{n-1}),\chi_n\}\cup\Omega^{\geq n+1}\nvdash_\Lambda\psi$. Then there must be $\alpha_n\in\mathcal{R}(\chi_n)$ such that $\{f(\chi_1),\ldots,f(\chi_{n-1}),\alpha_n\}\cup\Omega^{\geq n+1}\nvdash_\Lambda\psi$. We set $f(\chi_n)=\alpha_n$. In this stepwise manner we can define the value $f(\chi)$ for every $\chi\in\Omega$. Let $\Omega^f=\{f(\chi)\mid\chi\in\Omega\}$. It must be the case that $\Omega^f\nvdash_\Lambda\psi$. Otherwise there would be n such that $\{f(\chi_1),\ldots,f(\chi_n)\}\vdash_\Lambda\psi$ which would be a contradiction. Now let us define:

$$X^f=\{\Delta\text{ prime }\Lambda\text{-theory}\mid\Omega^f\subseteq\Delta\},$$

$$Cl(\Omega^f)=\{\alpha\text{ declarative }\mathcal{L}_{SE}\text{-formula}\mid\Omega^f\vdash_\Lambda\alpha\}.$$

$Cl(\Omega^f)$ is the smallest Λ-theory extending Ω^f and X^f is the set of all prime Λ-theories extending $Cl(\Omega^f)$, so, due to Lemma 7, it holds that

$$\bigcap X^f=Cl(\Omega^f).$$

Now we will prove that (a) $X^f\in\Sigma_a^\Lambda(\Gamma^\uparrow)$ and (b) for every $\alpha\in\mathcal{R}(\psi)$, $\alpha\notin\bigcap X^f$. To prove the first part, assume $E_a\chi\in\bigcap\Gamma^\uparrow$, i.e. $E_a\chi\in\Gamma$. So, $\chi\in\Omega$ and $f(\chi)\in\Omega^f$. It follows that there is $\alpha\in\mathcal{R}(\chi)$, namely $\alpha=f(\chi)$, such that $\alpha\in\bigcap X^f$. Hence, indeed, $X^f\in\Sigma_a^\Lambda(\Gamma^\uparrow)$.

To prove the second part, assume, for the sake of contradiction, that for some $\alpha\in\mathcal{R}(\psi)$, $\alpha\in\bigcap X^f$. Then $\Omega^f\vdash_\Lambda\alpha$. It follows that $\Omega^f\vdash_\Lambda\psi$, which is a contradiction. This finishes the proof. □

We will also need the following facts about prime Λ-theories that can be proved using standard Lindenbaum-style constructions similar to those used in the proofs of Lemmas 7 and 10.

Lemma 12. *For any prime Λ-theory Γ, and all declarative $\mathcal{L}_{SE}$-formulas α,β:*

(a) if $\neg\alpha\notin\Gamma$ then there is a prime Λ-theory Δ such that $\alpha\in\Delta$ and for any declarative $\mathcal{L}_{SE}$-formula γ, if $\neg\gamma\in\Gamma$ then $\gamma\notin\Delta$,

(b) if $\alpha\to\beta\notin\Gamma$ then there are prime Λ-theories Δ,Ω such that $\Gamma\otimes\Delta\subseteq\Omega$, $\alpha\in\Delta$, and $\beta\notin\Omega$,

(c) if $\alpha\otimes\beta\in\Gamma$ then there are prime Λ-theories Δ,Ω such that $\Delta\otimes\Omega\subseteq\Gamma$, $\alpha\in\Delta$, and $\beta\in\Omega$.

To prove the following version of a truth lemma, we will have to use double induction: on the complexity of formula and on modal depth. The modal depth of an $\mathcal{L}_{SE}$-formula φ is denoted as $d(\varphi)$ and it is defined in the following way:

$$d(p)=d(\bot)=d(\mathsf{t})=0,$$

$$d(\neg\varphi)=d(\varphi),$$

$$d(\varphi\circ\psi)=\max\{d(\varphi),d(\psi)\},\text{ for any }\circ\in\{\to,\wedge,\otimes,\vee,\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}\},$$

$$d(K_a\varphi)=d(E_a\varphi)=d(\varphi)+1.$$

One can observe that for any $\mathcal{L}_{SE}$-formula φ, if $\alpha\in\mathcal{R}(\varphi)$ then $d(\alpha)\leq d(\varphi)$.

Lemma 13. *Let Λ be an inquisitive epistemic logic, X an upward closed set of prime Λ-theories, and α a declarative $\mathcal{L}_{SE}$-formula. Then $X\Vdash\alpha$ in $\mathcal{M}^\Lambda$ iff $\alpha\in\bigcap X$.*

Proof. One can proceed by double induction: The primary induction is on modal depth, the secondary induction is on the complexity of declarative $\mathcal{L}_{SE}$-formulas of a fixed modal depth. More specifically, in the first step, one can prove the lemma for every declarative $\mathcal{L}_{SE}$-formula with modal depth 0. The induction hypothesis is that our claim holds for every declarative $\mathcal{L}_{SE}$-formula with modal depth strictly smaller than k (for a positive number k). In the inductive step, one proves that then the claim also holds for any declarative $\mathcal{L}_{SE}$-formula with modal depth k, which is proved by induction on the complexity of formulas of modal depth k. The inductive steps for the operators on the level k are the same as the corresponding steps on the lower levels with the exception that the steps for K_a, E_a appear for the first time on the level 1 of the primary induction. Let us go through the particular cases.

Atomic formulas: $X \Vdash p$ iff $X \subseteq V(p)$ iff $p \in \Gamma$, for any $\Gamma \in X$ iff $p \in \bigcap X$.

The contradiction constant: $X \Vdash \bot$ iff $X = \{\Xi\}$, where Ξ is the set of all declarative $\mathcal{L}_{SE}$-formulas iff $\bot \in \bigcap X$ (due to A2).

The logical truth constant: $X \Vdash \mathsf{t}$ iff $X \subseteq 1^\Lambda$ iff $\Lambda^d \subseteq \Gamma$, for any $\Gamma \in X$ iff $\Lambda^d \subseteq \bigcap X$ iff $t \in \bigcap X$ (due to A1, R7, R8).

The inductive steps for $\neg$, $\rightarrow$, $\wedge$, $\otimes$, and $\vee$ can be proved by rather straightforward arguments with the help of Lemma 12. In particular, (a) of Lemma 12 is used in the left-to-right implication of the inductive step for $\neg$, (b) is needed in the left-to-right implication of the inductive step for $\rightarrow$, and (c) is used in the right-to-left implication of the inductive step for $\otimes$. We will omit the details of these inductive steps and focus instead on the modal cases. They generalize the corresponding inductive steps in [3].

Assume that our claim holds for every declarative $\mathcal{L}_{SE}$-formula of the modal depth strictly smaller than k and ψ is an $\mathcal{L}_{SE}$-formula (not necessarily declarative) such that $d(\psi) = k - 1$. We will show the inductive steps from ψ to $E_a\psi$ and $K_a\psi$.

(a) *The inductive step for E_a:* First, assume that $E_a\psi \in \bigcap X$. Let $\Gamma^\uparrow \in Sit(X)$ in $\mathcal{M}^\Lambda$. This means that $\Gamma \in X$ and thus $E_a\psi \in \Gamma$. Assume $Y \in \Sigma_a^\Lambda(\Gamma^\uparrow)$. Since $E_a\psi \in \Gamma$, it follows from the definition of Σ_a^Λ that there is $\beta \in \mathcal{R}(\psi)$ such that $\beta \in \bigcap Y$. Since $d(\beta)$ is strictly smaller than k, we can use the inductive hypothesis and conclude that $Y \Vdash \beta$ in $\mathcal{M}^\Lambda$. But then also $Y \Vdash \psi$ in $\mathcal{M}^\Lambda$. It follows that $X \Vdash E_a\psi$ in $\mathcal{M}^\Lambda$.

Second, assume that $E_a\psi \notin \bigcap X$. So, there is a prime Λ-theory $\Gamma \in X$ such that $E_a\psi \notin \Gamma$. Since $\Gamma \in X$, $\Gamma^\uparrow \in Sit(X)$. Due to Lemma 11 there is $Y \in \Sigma_a^\Lambda(\Gamma^\uparrow)$ such that for each $\beta \in \mathcal{R}(\psi)$, $\beta \notin \bigcap Y$. Using the inductive assumption, we can conclude that for each $\beta \in \mathcal{R}(\psi)$, $Y \nVdash \beta$, and thus $Y \nVdash \psi$. It follows that $X \nVdash E_a\psi$.

(b) *The inductive step for K_a:* First, assume that $K_a\psi \in \bigcap X$. Then $K_a\psi \in \Gamma$, for every $\Gamma^\uparrow \in Sit(X)$ (i.e. for every $\Gamma \in X$). Let $\mathcal{R}(\psi) = \{\beta_1, \ldots, \beta_n\}$. Due to the axioms KD and KE, $K_a\psi \rightarrow (E_a\beta_1 \vee \ldots \vee E_a\beta_n) \in \Lambda$. So, $E_a\beta_1 \vee \ldots \vee E_a\beta_n \in \Gamma$. Since Γ is prime, for some i, $E_a\beta_i \in \Gamma$. Thus, for each $Y \in \Sigma_a(\Gamma^\uparrow)$, $\beta_i \in \bigcap Y$. By the induction hypothesis, $Y \Vdash \beta_i$, for every $Y \in \Sigma_a(\Gamma^\uparrow)$. It follows that $\bigcup \Sigma_a(\Gamma^\uparrow)$, i.e. $\sigma_a(\Gamma^\uparrow)$ supports β_i, and hence also ψ. Therefore, $X \Vdash K_a\psi$.

Second, assume $K_a\psi \notin \bigcap X$. So, for some $\Gamma^\uparrow \in Sit(X)$ (i.e. for some $\Gamma \in X$), $K_a\psi \notin \Gamma$. Due to the axioms KD and KE, $(E_a\beta_1 \vee \ldots \vee E_a\beta_n) \rightarrow K_a\psi \in \Lambda$. It follows that $E_a\beta_1 \vee \ldots \vee E_a\beta_n \notin \Gamma$. Then for all i, $E_a\beta_i \notin \Gamma$ and it follows from Lemma 11 that for each i there is $Y \in \Sigma_a(\Gamma^\uparrow)$ such that $\beta_i \notin \bigcap Y$. By the induction hypothesis, $Y \nVdash \beta_i$, and hence $\bigcup \Sigma_a(\Gamma^\uparrow)$, i.e. $\sigma_a(\Gamma^\uparrow)$, does not support β_i. Therefore, $\sigma_a(\Gamma^\uparrow) \nVdash \psi$ and so $X \nVdash K_a\psi$. $\square$

Theorem 4. *$Th(\mathcal{M}^\Lambda) \subseteq \Lambda$, for any inquisitive epistemic logic Λ.*

Proof. Assume $\varphi \in Th(\mathcal{M}^\Lambda)$, i.e. $1^\Lambda \Vdash \varphi$ in $\mathcal{M}^\Lambda$. Then $1^\Lambda \Vdash \alpha$, for some $\alpha \in \mathcal{R}(\varphi)$. Due to Lemma 13, $\alpha \in \bigcap 1^\Lambda$, and hence it follows from Lemma 8 that $\alpha \in \Lambda$. But then also $\varphi \in \Lambda$. □

Theorem 5. *For any inquisitive epistemic logic Λ, the following conditions are equivalent:*

(a) $\Lambda \subseteq Th(\mathcal{M}^\Lambda)$,

(b) Λ *is constructive,*

Proof. (a) implies (b): Assume $\Lambda \subseteq Th(\mathcal{M}^\Lambda)$. Let $\varphi \vee\!\!\!\vee \psi \in \Lambda$. Then $1^\Lambda \Vdash \varphi \vee\!\!\!\vee \psi$ in $\mathcal{M}^\Lambda$, and thus $1^\Lambda \Vdash \varphi$ or $1^\Lambda \Vdash \psi$ in $\mathcal{M}^\Lambda$. Hence, due to Theorem 4, $\varphi \in \Lambda$ or $\psi \in \Lambda$.

(b) implies (a): Assume that Λ is constructive. Take any $\varphi \in \Lambda$. So, for some $\alpha \in \mathcal{R}(\varphi)$, $\alpha \in \Lambda$, and thus $\alpha \in \bigcap 1^\Lambda$. Then, due to Lemma 13, $1^\Lambda \Vdash \alpha$, and thus $1^\Lambda \Vdash \varphi$ in $\mathcal{M}^\Lambda$. So, $\varphi \in Th(\mathcal{M}^\Lambda)$.

□

Theorem 6. *Let Λ be a constructive inquisitive epistemic logic. Then for any upward closed set of prime Λ-theories X and any $\mathcal{L}_{SE}$-formula φ, $X \Vdash \varphi$ in $\mathcal{M}^\Lambda$ iff $\bigcap X \vdash_\Lambda \varphi$.*

Proof. First, assume $X \Vdash \varphi$ in $\mathcal{M}^\Lambda$. Then there is $\alpha \in \mathcal{R}(\varphi)$ such that $X \Vdash \alpha$. So, due to Lemma 13, $\alpha \in \bigcap X$. It follows that $\bigcap X \vdash_\Lambda \varphi$. Second, assume $\bigcap X \vdash_\Lambda \varphi$. Since $\bigcap X$ is closed under $\wedge$, there is $\beta \in \bigcap X$ such that $\beta \to \varphi \in \Lambda$. So, applying Lemma 13 and Theorem 5, we obtain $X \Vdash \beta$ and $1^\Lambda \Vdash \beta \to \varphi$. Hence $X \Vdash \varphi$ in $\mathcal{M}^\Lambda$. □

Definition 8. *Let Λ be an inquisitive epistemic logic and $\mathcal{C}$ a class of AEI-models. We say that Λ is sound w.r.t. $\mathcal{C}$ if every $\mathcal{L}_{SE}$-formula from Λ is valid in every model from $\mathcal{C}$. We say that Λ is complete w.r.t. $\mathcal{C}$ if Λ contains every $\mathcal{L}_{SE}$-formula that is valid in every model from $\mathcal{C}$. Let $\mathcal{M}$ be an AEI-model. We say that Λ is sound (complete) w.r.t. $\mathcal{M}$ if it is sound (complete) w.r.t. $\{\mathcal{M}\}$.*

Theorem 4 says that any inquisitive epistemic logic is complete with respect to its canonical model but Theorem 5 adds to that that only constructive logics are sound. Soundness w.r.t. the canonical model and constructivity coincide.

Corollary 1. *If an inquisitive epistemic logic Λ is sound w.r.t. a class of AEI-models $\mathcal{C}$ and $\mathcal{M}^\Lambda \in \mathcal{C}$ then Λ is constructive and complete w.r.t. $\mathcal{C}$.*

We can immediately apply this Corollary to the special case of the logic InqSE. Since it is sound with respect to the class of all AEI-models and its canonical model is in this class, we obtain a completeness result for this logic.

Corollary 2. *The set of* InqSE*-provable $\mathcal{L}_{SE}$-formulas is sound and complete w.r.t. the class of all AEI-models.*

We also obtain analogues of (b) and (c) of Theorem 2 for the logic InqSE.

Corollary 3. *The logic* InqSE *is constructive, i.e., it has disjunction property and splitting property.*

The logic InqSE is very weak. For particular purposes, stronger inquisitive epistemic logics might be more suitable. These logics can be obtained syntactically by adding further axioms or rules, and semantically by imposing further semantic restrictions on AIE-models. To show some examples of such extensions we introduce the following terminology. An AEI-model without the valuation will be called an AEI-frame. An $\mathcal{L}_{SE}$-formula is valid in an AEI-frame if it is valid in every AEI-model on that frame. We have been using two sorts of variables for formulas: $\varphi, \psi, \chi, \ldots$ as variables for arbitrary $\mathcal{L}_{SE}$-formulas and $\alpha, \beta, \gamma, \ldots$ as variables for declarative $\mathcal{L}_{SE}$-formulas. If we take an $\mathcal{L}_{SE}$-formula and replace all atomic formulas with such variables we obtain a "schema". The concrete formulas of the given form will be called instances of the schema. For example, $K_a\alpha \to \alpha$ is a schema but since it involves the variable α only declarative $\mathcal{L}_{SE}$-formulas are allowed to be substituted for this variable. Hence, e.g., $K_a(p \wedge q) \to (p \wedge q)$ is an instance of the schema but $K_a(p \vee\!\!\!\vee q) \to (p \vee\!\!\!\vee q)$ is not. On the other hand, $K_a(p \vee\!\!\!\vee q) \to (p \vee\!\!\!\vee q)$ is an instance of the schema $K_a\varphi \to \varphi$. In the same sense, we will talk about schematic inference rules.

Definition 9. *Let $\mathcal{C}$ be a class of AEI-frames, S a schema and R a schematic inference rule. We say that S strongly characterizes $\mathcal{C}$ if (1) for every AEI-frame $\mathcal{F}$, $\mathcal{F} \in \mathcal{C}$ iff every instance of S is valid in $\mathcal{F}$; and (2) if Λ is an inquisitive epistemic logic that contains every instance of S, then the frame of $\mathcal{M}^\Lambda$ is in $\mathcal{C}$. We say that R strongly characterizes $\mathcal{C}$ if (1) for every AEI-frame $\mathcal{F}$, $\mathcal{F} \in \mathcal{C}$ iff R preserves validity in every model on $\mathcal{F}$; and (2) if Λ is an inquisitive epistemic logic that is closed under the rule R, then the frame of $\mathcal{M}^\Lambda$ is in $\mathcal{C}$.*

Assume that an inquisitive epistemic logic Λ is determined by a system that consists of the axioms and rules of InqSE plus a (possibly infinite) set of additional schematic axioms $\{A_1, A_2, \ldots\}$ and rules $\{R_1, R_2, \ldots\}$. Further assume that the schamata $A_1, A_2, \ldots$ respectively strongly characterize the classes of AEI-frames $\mathcal{C}_1, \mathcal{C}_2, \ldots$ and the rules $R_1, R_2, \ldots$ respectively strongly characterize the classes of AEI-frames $\mathcal{D}_1, \mathcal{D}_2, \ldots$ Then it follows from Corollary 1 that Λ is constructive and it is sound and complete w.r.t. the class of AEI-models based on frames from $\bigcap \mathcal{C}_i \cap \bigcap \mathcal{D}_j$. We will illustrate this application of Corollary 1 with two simple examples.

Theorem 7. *(a) For any agent a, the schematic rules $\varphi/E_a\varphi$ and $\varphi/K_a\varphi$, where φ ranges over arbitrary $\mathcal{L}_{SE}$-formulas, both strongly characterize the class of AEI-frames satisfying the condition: for any state $s \in Sit(1)$, $\sigma_a(s) \sqsubseteq 1$.*

(b) For any agent a, the schemata $E_a\alpha \to \alpha$ and $K_a\alpha \to \alpha$, where α ranges over declarative $\mathcal{L}_{SE}$-formulas, both strongly characterize the class of AEI-frames satisfying the condition: for every situation s, $s \sqsubseteq \sigma_a(s)$.

Proof. (a) Let $\mathcal{C}_1$ be the class of all AEI-frames satisfying the condition: for every state s, if $s \in Sit(1)$ then $\sigma_a(s) \sqsubseteq 1$. (1) First, assume that $\mathcal{F} \in \mathcal{C}_1$. Then, due to the persistence property (Theorem 3-(b)), for any AEI-model on $\mathcal{F}$ and any $\mathcal{L}_{SE}$-formula φ, $1 \Vdash \varphi$ implies both $1 \Vdash E_a\varphi$ and $1 \Vdash K_a\varphi$. Second, assume that $\mathcal{F} \notin \mathcal{C}_1$. Let us define the valuation V so that $V(p) = 1$. Then $1 \Vdash p$ but $1 \nVdash E_ap$ and $1 \nVdash K_ap$. (2) Assume that Λ is an inquisitive epistemic logic closed under the rule $\varphi/E_a\varphi$ or under the rule $\varphi/K_a\varphi$. We show that the frame of the canonical model is in $\mathcal{C}_1$. Assume $\Gamma^\uparrow \in Sit(1^\Lambda)$, i.e. $\Lambda^d \subseteq \Gamma$. We have to show that $\bigcup \Sigma_a(\Gamma^\uparrow) \subseteq 1^\Lambda$. Take any $X \in \Sigma_a(\Gamma^\uparrow)$ and $\Delta \in X$. We want to show that $\Delta \in 1^\Lambda$, i.e. $\Lambda^d \subseteq \Delta$. Let $\alpha \in \Lambda^d$. We assume that

Λ is closed under $\varphi/E_a\varphi$ or $\varphi/K_a\varphi$. Since $E_a\alpha$ is provably equivalent to $K_a\alpha$ (due to KE) we obtain in both cases $E_a\alpha \in \Lambda^d$. Then also $E_a\alpha \in \Gamma$. Since $X \in \Sigma_a(\Gamma^\uparrow)$, $\alpha \in \bigcap X$. It follows that $\alpha \in \Delta$.

(b) First note that, due to KE, $E_a\alpha \to \alpha$ and $K_a\alpha \to \alpha$ are equivalent already in InqSE, so it is sufficient to focus only on one of them. We will consider the case of $E_a\alpha \to \alpha$. The case of $K_a p \to p$ is proved in the same way. Let $\mathcal{C}_2$ be the class of all AEI-frames satisfying the condition: for every situation s, $s \sqsubseteq \sigma_a(s)$. (1) First, assume that $\mathcal{F} \in \mathcal{C}_2$. Take an arbitrary AEI-model on $\mathcal{F}$. To show that $E_a\alpha \to \alpha$ is valid in the model we need to show that for any state s, if $s \Vdash E_a\alpha$ then $s \Vdash \alpha$ (see Lemma 1). Let $s \Vdash E_a\alpha$. Then, due to KE, also $s \Vdash K_a\alpha$ and for any $t \in Sit(s)$, $\sigma_a(t) \Vdash \alpha$. Applying the join closure property (Theorem 3-(c)), we obtain $\bigsqcup\{\sigma_a(t) \mid t \in Sit(s)\} \Vdash \alpha$. Since for any $t \in Sit(s)$, $t \sqsubseteq \sigma_a(t)$ and $s = \bigsqcup Sit(s)$, we have $s \sqsubseteq \bigsqcup\{\sigma_a(t) \mid t \in Sit(s)\}$, so, by persistence property, $s \Vdash \alpha$. Second, assume that $\mathcal{F} \notin \mathcal{C}_2$. Then there is a situation s that is not below $\sigma_a(s)$. Take a valuation V such that $V(p) = \sigma_a(s)$. Then $s \Vdash E_a p$ but $s \nVdash p$. Hence, by Lemma 1, $E_a p \to p$ is not valid in the frame. (2) Assume that Λ is an inquisitive epistemic logic that contains the formula $E_a\alpha \to \alpha$, for every declarative $\mathcal{L}_{SE}$-formula α. We have to show that the frame of the canonical model is in $\mathcal{C}_2$. Thus we are proving that for every prime Λ-theory Γ, $\Gamma^\uparrow \subseteq \bigcup \Sigma_a(\Gamma^\uparrow)$, which is equivalent to $\Gamma^\uparrow \in \Sigma_a(\Gamma^\uparrow)$. This, in turn, is equivalent to the following:

for any $\mathcal{L}_{SE}$-formula φ, if $E_a\varphi \in \Gamma$ then for some $\alpha \in \mathcal{R}(\varphi)$, $\alpha \in \Gamma$.

Assume $E_a\varphi \in \Gamma$ and let $\mathcal{R}(\varphi) = \{\alpha_1, \ldots, \alpha_n\}$. Then $E_a(\alpha_1 \vee\!\!\!\vee \ldots \vee\!\!\!\vee \alpha_n) \in \Gamma$. The system for InqSE allows this derivation:

$\alpha_i \to (\alpha_1 \vee \ldots \vee \alpha_n)$, for each $1 \leq i \leq n$, by A5, A6

$(\alpha_1 \vee\!\!\!\vee \ldots \vee\!\!\!\vee \alpha_n) \to (\alpha_1 \vee \ldots \vee \alpha_n)$, by R10

$E_a(\alpha_1 \vee\!\!\!\vee \ldots \vee\!\!\!\vee \alpha_n) \to E_a(\alpha_1 \vee \ldots \vee \alpha_n)$, by MR1

Since $E_a(\alpha_1 \vee\!\!\!\vee \ldots \vee\!\!\!\vee \alpha_n) \in \Gamma$, we have also $E_a(\alpha_1 \vee \ldots \vee \alpha_n) \in \Gamma$. Since $\alpha_1 \vee \ldots \vee \alpha_n$ is declarative, we obtain $\alpha_1 \vee \ldots \vee \alpha_n \in \Gamma$. Since Γ is prime, $\alpha \in \Gamma$, for some $\alpha \in \mathcal{R}(\varphi)$, which is what we wanted to prove.

□

It follows from Corollary 1 and Theorem 7 that any combination of the rules and axioms considered in the theorem gives us a constructive inquisitive epistemic logic that is sound and complete with respect to the class of AEI-models obtained by adding the respective restricting conditions to the definition of an AEI-model.

We will finish this section with some remarks regarding two possible concerns related to our semantics. The first one is motivational while the second one is rather technical.

The first concern can be formulated as follows. With respect to the interpretation according to which situations are just fragments of bigger reality, it might seem unnatural to require that every agent must have an information state and an issue in every situation.

This problem can be fixed by a slight technical adjustment. We can assume that the inquisitive state maps are partial functions, so that agents have information states and issues only in *some* situations.[3]

If the agent a has an information state and an issue in a situation s, it will mean that the value of $\Sigma_a(s)$ is defined and we will denote this as $s \downarrow a$. Then the condition (g) from the definition of an AEI-model (Definition 3) has to be modified in the following way:

(g)' if s, t are situations such that $t \downarrow a$ and $s \sqsubseteq t$ then $s \downarrow a$ and $\Sigma_a(s) \subseteq \Sigma_a(t)$.

The semantic clauses for modalities also have to be adjusted:

- $s \Vdash K_a\varphi$ iff for any $t \in Sit(s)$, $t \downarrow a$ and $\sigma_a(t) \Vdash \varphi$,
- $s \Vdash E_a\varphi$ iff for any $t \in Sit(s)$, $t \downarrow a$ and for any $u \in \Sigma_a(t)$: $u \Vdash \varphi$.

This modification leads to a broader class of models but one can verify that InqSE is sound even with respect to this more general semantics. Since every AEI-model of the original semantics is also a model of this modified semantics, completeness is preserved. Thus, the adjustments do not have any impact on the resulting logic.

The second concern: In [15], a different framework for substructural inquisitive logics (without the epistemic modalities) was proposed. In that framework the construction of canonical models seems to be significantly simpler. The states in the canonical models are just theories, not sets of prime theories. Would it be possible to simplify the construction so that it would be similar to that of [15]? Since we restrict ourselves to distributive logics for which it holds that every theory is the intersection of prime theories that extend it, it seems that prime theories could play the role of situations. In that perspective, intersection would play the role of join in the canonical model so that it would hold that every state in the model is join of a set of situations.

This strategy leads to the following problem: prime theories are not completely join-irreducible elements in the lattice of theories. They are only finite join-irreducible.[4] So, we would have to modify the definition of a situation in the general formulation of the semantics (Definition 1). Given a complete lattice, a situation in the lattice would have to be defined as a finite join-irreducible element. But this modification has a significant impact. Most importantly, if the defining conditions of an AEI-model are left unaltered, the crucial modal axiom KE ceases to be sound.

It is possible to overcome this problem by several changes in the formulation of the semantics so that it is possible to build the canonical structures directly out of theories. However, while the completeness proof is simpler, the resulting framework is significantly more complicated than the one we have presented, and we will not pursue these modifications in this paper.

[3]Note that already in the semantics for IEL it can happen that the agent has, *in some sense*, no issue in a given world. This happens if $\sigma_a(w) \in \Sigma_a(w)$. Technically, the agent's issue is defined in such a case but it is trivial because it is already resolved by the agent's knowledge. However, this is different from what we are discussing here. We are considering the case when the agent's knowledge and issue are not defined at all for a given situation.

[4]An element s of a given lattice is finite join-irreducible if the following holds: $s = t \sqcup u$ only if $s = t$ or $s = u$. If join is intersection, the prime theories are exactly the finite join-irreducible elements in the lattice of theories.

5 Conclusion

To sum up, in this paper we developed a semantic framework for substructural inquisitive epistemic logics, which generalize the standard semantics for inquisitive epistemic logic. A peculiar feature of the framework is that agents are equipped not only with information states (which is common in epistemic logic) but also with issues. The semantics allows us to evaluate not only statements but also questions and statements with embedded questions like *the agent knows whether* A or *the agent is wondering whether* A. Moreover, the general semantics allows us to base the inquisitive epistemic logic on non-classical logics of declarative sentences so it is immune to many questionable features of classical logic. On the technical side, our main result is a general construction of canonical models for inquisitive epistemic logics that can be used as a crucial tool in completeness proofs. We applied this construction to the case of the weakest inquisitive epistemic logic InqSE and showed with a few examples how it can be used for its extensions. In future work we plan to extend the framework with further group-epistemic and dynamic modalities. In particular, we intend to study distributed and common knowledge, and a public announcement modality within this framework.

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