Relevant Epistemic Logic with Public Announcements and Common Knowledge

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Abstract. Building on our previous work in non-classical dynamic epistemic logic, we add common knowledge operators to a version of public announcement logic based on the relevant logic R. We prove a completeness result with respect to a relational semantics, and we show that an alternative semantics based on information states is dual to the relational one. We add a question-forming inquisitive disjunction operator to the language and prove a completeness result with respect to the information semantics. It is argued that relevant public announcements are particularly suitable for modelling public argumentation.

1 Introduction

Various problematic closure properties of epistemic operators in classical epistemic logic led to the exploration of a number of alternatives to the classical framework; see [9]. One approach to avoiding closure under classical consequence is to represent epistemic states of agents as sets of *abstract situations*, roughly in the sense of Barwise and Perry [2], instead of representing them as sets of possible worlds. This approach leads naturally to epistemic logics based on paraconsistent and substructural propositional logics; see [3,10,19,21,22] for example. Combining these logics with an account of information dynamics is a topic of recent interest: [1,11] explore versions of intuitionistic public announcement logic, [18,20] study paraconsistent bilattice public announcement logic, and [4] outlines a fuzzy version of public announcement logic. Authors of the present paper explored versions of public announcement logic based on relevant and substructural logics in [14] and [25]. The latter paper focuses on modelling epistemic updates in the standard relational semantics for relevant modal logic and the former paper used a more general information-state semantics. All papers mentioned so far use languages without *common knowledge*, a concept linked to information dynamics in a number of important ways.

The aim of this paper is to (i) extend the frameworks presented in [14] and [25] with common knowledge and provide a complete axiomatization, and (ii) explore the relationship between these frameworks. In Section 2 we add common knowledge to the relational semantics introduced in [25]. We argue that the notion of update embodied in the semantics has a natural link to the notion of *public argumentation.* In Section 3 we prove a completeness result for the relational semantics. Then, in Section 4, we introduce an informational semantics following the approach of [14]. The main result here is that informational and relational models are dual, implying that the sound and complete axiomatization of Section 3 is sound and complete with respect to information models as well. In Section 5 we add questions to our object language in the style of inquisitive semantics [7] and show that the completeness proof of Section 3 can be extended to completeness of this enriched language with respect to informational semantics.

2 Relational semantics

Definition 1. *Fix a countable set of propositional variables Pr and a finite set of agent indices G. The language $\mathcal{L}$ contains operators* t *(zero-ary),* $\wedge, \rightarrow, \otimes, [\,]$ *(binary),* $\neg$ *and* B_a, C_A *for all* $a \in G$ *and non-empty* $A \subseteq G$ *(unary). The set of formulas of $\mathcal{L}$, denoted as $Fm_{\mathcal{L}}$, is generated by Pr using the operators of $\mathcal{L}$ in the usual way.*

We define $\varphi \leftrightarrow \psi := (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$, $\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$, $B_A\varphi := \bigwedge_{a\in A} B_a\varphi$, $B^+_A\varphi := B_A C_A\varphi$, and $K_a\varphi := C_{\{a\}}\varphi$. Operators $\mathrm{t}, \neg, \wedge, \otimes, \rightarrow$ are *propositional*, all B_a and C_A are *epistemic operators* and $[\,]$ is the *dynamic operator*. We use r to range over propositional operators, r_i over i-ary propositional operators and $r_{>0}$ over propositional operators of non-zero arity.

We read "$B_a\varphi$" as "Agent a *believes* that φ", although what we mean is, more generally, that a has information that supports (or allows a to conclude that) φ. $B_A\varphi$ means that all agents in the group $A \subseteq G$ believe that φ. We read "$C_A\varphi$" as "φ is *common knowledge* in the group of agents A". $K_a\varphi$ is read "Agent a *knows* that φ" and $B^+_A\varphi$ as "it is *common belief* in group A that φ". Our choice of primitive epistemic operators may seem a bit odd, but it can be shown that belief, knowledge, common belief and common knowledge interact in expected ways. In particular, $B^+_A\varphi$ holds in s iff $B_A\varphi$ holds in s, $B_A B_A\varphi$ holds in s, and so on, which corresponds to the usual semantics of common belief. K_a is an "S4-type" knowledge operator, that is, $K_a\varphi \rightarrow \varphi$ and $K_a\varphi \rightarrow K_a K_a\varphi$ are valid, as we will see below.

The dynamic operator $[\,]$ expresses effects of *public announcement* of a piece of information: we read $[\varphi]\psi$ as "ψ is the case after φ is publicly announced to all agents". Our notion of public announcement differs somewhat from the notion embodied in Public Announcement Logic [12,26]: we do not assume that the announced piece of information is *truthful*, nor is it implied that the information is *accepted* by the agents upon the announcement; we also allow the possibility that the announcement may cause some agents to drop some of the previously accepted information.[1]

[1] We note that while "announcement" seem to us to best express the notion we have in mind, we have hesitated because of its technical connotations. Another term that may be used is "reception" – upon an announcement agents receive a piece of information, but nothing is implied about the nature of the information nor about what the agents make of it.

A natural informal interpretation of such a general notion of public announcement is in terms of *public argumentation*. Imagine a group of agents, engaged in a public discussion. Public announcements can be seen as acts of putting forward arguments for or against claims in the discussion. These arguments do not have to be truthful and they do not have to be persuasive, meaning that the agents do not always accept the arguments. Moreover, arguments can cause agents to reject some of the information they accepted before. All of these features are characteristic aspects of our general rendering of public announcement. We will return to these aspects after introducing the semantics.

We note that our semantics extends the semantic framework of relevant logic introduced by Routley and Meyer in the 1970s. Our axiomatization extends the axiomatization of the relevant logic R, introduced by Anderson and Belnap in the 1960s. We do not have the space to review relevant logic in detail. The reader is referred to [8] or [17], for example.

Definition 2. *A* relevant epistemic model for G *is* $\mathfrak{M} = (S, \sqsubseteq, L, R, C, E, V)$ *where* $(S, \sqsubseteq, L, R, C, V)$ *is a Routley–Meyer model for the relevant logic* R*, i.e.*

- $(S, \sqsubseteq)$ *is a partially ordered set;*
- L *is an up-set in* $(S, \sqsubseteq)$
- R *is a ternary relation on* $(S, \sqsubseteq)$ *that is anti-monotonic in the first two coordinates and monotonic in the third coordinate, and satisfies the following frame conditions (we use the standard notation* $Rstuw := \exists x(Rstx \ \& \ Rxuw)$ *and* $Rs(tu)w := \exists x(Rtux \ \& \ Rsxw)$*)*

$$Rstuw \implies Rs(tu)w \quad (1) \qquad Rstu \implies Rsttu \quad (3)$$
$$Rstuw \implies Rt(su)w \quad (2) \qquad Rstu \implies Rtsu \quad (4)$$

- $s \sqsubseteq t$ *iff there is* $x \in L$ *such that* $Rsxt$*;*
- C *is a symmetric binary relation on* $(S, \sqsubseteq)$ *that is anti-monotonic in both coordinates such that for all* s *there is a unique maximal element* $\bar{s}$ *of the set* $C(s) = \{t \mid Cst\}$ *and*

$$\bar{\bar{s}} = s \quad (5) \qquad Rstu \implies Rs\bar{u}\bar{t} \quad (6)$$

- V *is a function from* Pr *to up-sets in* $(S, \sqsubseteq)$*;*

and E *is a function from* G *to binary relations on* $(S, \sqsubseteq)$ *that are anti-monotonic in the first coordinate and monotonic in the second coordinate.*

Informally, $E(a)$ represents the information about the epistemic state of a provided by situations in the model as follows. For each $s \in S$ there is a body of information, denote it as $s(a)$, such that s provides information that $s(a)$ is the epistemic state of a ($s(a)$ may be empty). $E(a)st$ represents the assumption that $s(a)$ is contained in t. Hence, $E(a)(s) = \{t \mid E(a)st\}$ can be seen as the representation of $s(a)$.

We define $E(A) := \bigcup_{a \in A} E(a)$ and $E^*(A)$ as the reflexive transitive closure of $(\sqsubseteq \cup E(A))$.[2] $E^+(A)$ is the transitive closure of $E(A)$. We will usually write the agent (group) indices in a subscript.

Remark 1. We note that each relevant epistemic model is *fully associative*:

$$Rstuw \iff Rs(tu)w \tag{7}$$

The full associativity condition will be required in our completeness proof, in particular in the steps for $[\varphi][\psi]\chi$. Without common knowledge in the language, these cases are dealt with implicitly using the monotonicity rule R5; however, in the present context they have to be dealt with explicitly. This means that our present approach is limited to logics based on fully associative frames, for example R-based logic we focus on here. An extension of our results to weaker logics is an open problem.

Definition 3. *The* satisfaction relation *between pointed models (that is, pairs of the form $(\mathfrak{M}, s)$ where s is in $\mathfrak{M}$) and $\mathcal{L}$-formulas is induced by V of $\mathfrak{M}$ as follows:*

- $(\mathfrak{M}, s) \models p$ *iff* $s \in V(p)$;
- $(\mathfrak{M}, s) \models \mathrm{t}$ *iff* $s \in L$;
- $(\mathfrak{M}, s) \models \neg\varphi$ *iff* $\forall t$, Cst *implies* $(\mathfrak{M}, t) \not\models \varphi$;
- $(\mathfrak{M}, s) \models \varphi \wedge \psi$ *iff* $(\mathfrak{M}, s) \models \varphi$ *and* $(\mathfrak{M}, s) \models \psi$;
- $(\mathfrak{M}, s) \models \varphi \to \psi$ *iff*, $\forall tu$, *if* $Rstu$ *and* $(\mathfrak{M}, t) \models \varphi$, *then* $(\mathfrak{M}, u) \models \psi$;
- $(\mathfrak{M}, s) \models \varphi \otimes \psi$ *iff* $\exists tu$ *such that* $Rtus$ *and* $(\mathfrak{M}, t) \models \varphi$ *and* $(\mathfrak{M}, u) \models \psi$;
- $(\mathfrak{M}, s) \models B_a\varphi$ *iff* $\forall t$, $E_a st$ *only if* $(\mathfrak{M}, t) \models \varphi$;
- $(\mathfrak{M}, s) \models C_A\varphi$ *iff* $\forall t$, $E^*_A st$ *only if* $(\mathfrak{M}, t) \models \psi$;
- $(\mathfrak{M}, s) \models [\varphi]\psi$ *iff* $(\mathfrak{M}^{[\varphi]}, s) \models \psi$, *where* $\mathfrak{M}^{[\varphi]}$ *differs from* $\mathfrak{M}$ *only in that* $E^{[\varphi]}_a st$ *iff there are* u, v *such that* $E_a su$, $Ruvt$ *and* $(\mathfrak{M}, v) \models \varphi$.

A formula φ is valid in $\mathfrak{M}$ iff $(\mathfrak{M}, s) \models \varphi$ for all $s \in L$. We define $\mathfrak{M}(\varphi) := \{s \mid (\mathfrak{M}, s) \models \varphi\}$.

$\mathfrak{M}^{[\varphi]}$ is the model that results from $\mathfrak{M}$ after the public announcement of φ. Intuitively, $\mathfrak{M}^{[\varphi]}$ results from $\mathfrak{M}$ by extending all situations in $\mathfrak{M}$ with the information that φ has been publicly announced. This transformation of the model $\mathfrak{M}$ does not affect the "non-epistemic" structure consisting of the underlying Routley–Meyer model, but it does affect the epistemic accessibility relations since, intuitively, $s(a)$ for each s and a is modified by the announcement. Our semantics reflects the idea that this modification can be represented using the ternary relation R. We take R to represent the effects of "*merging*" situations: $Rstu$ iff the result of "merging" the information in s with the information in t

[2] Note that this notation is somewhat misleading as $E^*(A)$ does not denote the reflexive transitive closure of $E(A)$.

is contained in u. Crucially, "merging" is not necessarily monotonic.[3] Merging is lifted to sets of situations X, Y (representing arbitrary pieces of information) using the standard construction

$$X \otimes Y = \{u \mid \exists st(s \in X \ \& \ t \in Y \ \& \ Rstu\}\,. \tag{8}$$

After the information that φ *has been announced to* a is added to s, the epistemic state $s(a)$ is transformed into a new epistemic state $s(a)^{[\varphi]}$. Intuitively, announcing φ triggers a "merge" of the epistemic state of the given agent with the information expressed by φ. Using (8), we obtain

$$E_a^{[\varphi]}(s) = E_a(s) \otimes \mathfrak{M}(\varphi) = \{u \mid \exists tv(E_a st \ \& \ (\mathfrak{M}, v) \models \varphi \ \& \ Rtvu\} \tag{9}$$

Hence, $s(a)^{[\varphi]}$ is contained in u iff u contains the result of merging $s(a)$ with the information that φ. This is exactly how $\mathfrak{M}^{[\varphi]}$ is defined.

Proposition 1. *Formulas of the following forms are not valid: 1.* $\neg\varphi \to [\varphi]\,\psi$*; 2.* $[\varphi]\,B_a\varphi$*; 3.* $B_a\varphi \to [\psi]\,B_a\varphi$*.*

We leave the construction of counterexamples to the reader. The fact that $\neg\varphi \to [\varphi]\,\psi$ is not valid means that announcements are not necessarily truthful: announcing a false formula does not lead to "explosion". The failure of $[\varphi]\,B_a\varphi$ means that announced information is not necessarily accepted by agents. This feature is related in spirit to well-known *unsuccessful updates* of classical public announcement logic, but the mechanism underlying the feature is more general than the so-called Moorean phenomena at work in the classical case. The failure of $B_a\varphi \to [\psi]\,B_a\varphi$ shows that announcements in our setting are *non-monotonic:* after an announcement of ψ, agents may abandon previously held beliefs. As mentioned above, all three aspects are typical features of public argumentation.

Definition 4. *For any* $\mathfrak{M}$*,* φ *and non-empty* $A \subseteq G$*:*

- *an* A-path *in* $\mathfrak{M}$ *is a finite sequence* $\langle s_i \mid i < n\rangle$ *of situations in* $\mathfrak{M}$*, for some* $n \geq 0$*, such that for all* $j < n-1$*, either* $s_j \sqsubseteq s_{j+1}$ *or* $E_A s_j s_{j+1}$*;*
- *an* $A^{[\varphi]}$-path *in* $\mathfrak{M}$ *is a finite sequence* $\langle s_i \mid i < n\rangle$ *of situations in* $\mathfrak{M}$*, for some* $n \geq 0$*, such that for all* $j < n-1$*, either* $s_j \sqsubseteq s_{j+1}$ *or* $E_A^{[\varphi]} s_j s_{j+1}$*.*

A path (A*-path or an* $A^{[\varphi]}$*-path) is a path* from s *iff it is non empty and its first element is* s*; it is a path* ending in t *iff it is non-empty and its last element is* t*.*

Note that $(\mathfrak{M}, s) \models C_A\varphi$ iff $(\mathfrak{M}, t) \models \varphi$ for all t such that there is an A-path in $\mathfrak{M}$ starting with s and ending in t; similarly $(\mathfrak{M}, s) \models [\psi]\,C_A\varphi$ iff $(\mathfrak{M}, t) \models [\psi]\,\varphi$ for all t such that there is an $A^{[\psi]}$-path in $\mathfrak{M}$ starting with s and ending in t.

[3] This reading is related to a number of interpretations of R popular in the relevant logic literature. For instance, Dunn and Restall point out that "perhaps the best reading [of $Rstu$] is to say that the combination of the pieces of information s and t (not necessarily the union) is a piece of information in u" [8, p. 67]. Restall adds that "a body of information warrants $\varphi \to \psi$ if and only if whenever you *update* that information with new information which warrants φ, the resulting (perhaps new) body of information warrants ψ" [17, p. 362] (notation adjusted).

3 A relational completeness result

In this section we provide a complete axiomatization of the set of formulas valid in all relevant epistemic models. The axiom system RPAC, shown in Figure 1, is a combination of the proof system for the relevant logic R, axioms and rules specifying that B_a are regular and monotonic modalities, the usual axioms and rules for the common knowledge operator, and the so-called *reduction axioms* for the update operator.

The completeness proof will combine the method of "partial filtration", used to prove completeness for versions of Propositional Dynamic Logic based on relevant logics [23,24], the standard canonical model argument for R, and the completeness argument for Public Announcement Logic using reduction axioms. Our proof follows the usual strategy of proving completeness for Public Announcement Logic with common knowledge, but we use a different notion of filtration (the "filtrated model" is infinite, only epistemic accessibility relations are defined in terms of a finite set of formulas) and our models are more general. We note that, unlike in [23,24], our proof does not require the presence of "extensional truth constants" $\top, \bot$. (We will specify the reason for this in Remark 2 below.)

(A1) An axiomatization of R

(A2) $B_a\varphi \wedge B_a\psi \to B_a(\varphi \wedge \psi)$

(A3) $C_A\varphi \leftrightarrow (\varphi \wedge B_a C_A\varphi)$

(A4) $[\varphi]p \leftrightarrow p$

(A5) $[\varphi]\mathrm{t} \leftrightarrow \mathrm{t}$

(A6) $[\varphi]r_{>0}(\psi_1, \ldots, \psi_n) \leftrightarrow r_{>0}([\varphi]\psi_1, \ldots, [\varphi]\psi_n)$

(A7) $[\varphi]B_a\psi \leftrightarrow B_a(\varphi \to [\varphi]\psi)$

(A8) $[\varphi][\psi]\chi \leftrightarrow [\varphi \otimes [\varphi]\psi]\chi$

(R1) $\dfrac{\varphi \quad \varphi \to \psi}{\psi}$

(R2) $\dfrac{\varphi \quad \psi}{\varphi \wedge \psi}$

(R3) $\dfrac{\varphi \to \psi}{B_a\varphi \to B_a\psi}$

(R4) $\dfrac{\varphi \to \psi}{C_A\varphi \to C_A\psi}$

(R5) $\dfrac{\varphi \to \psi}{[\chi]\varphi \to [\chi]\psi}$

(R6) $\dfrac{\varphi \to (\psi \wedge B_A\varphi)}{\varphi \to C_A\psi}$

(R7) $\dfrac{\chi \to B_A(\varphi \to \chi) \quad \chi \to [\varphi]\psi}{\chi \to [\varphi]C_A\psi}$

Fig. 1. The axiom system RPAC.

Lemma 1. *All theorems of* RPAC *are valid in all relevant epistemic models.*

Proof. We prove only the cases for A7 and R7 explicitly. First, $(\mathfrak{M}, s) \not\models [\varphi]B_a\psi$ iff there is t such that $E_a^{[\varphi]}st$ and $(\mathfrak{M}^{[\varphi]}, t) \not\models \psi$ iff there are t, u, v such that $E_a su$ and $Ruvt$ and $(\mathfrak{M}, v) \models \varphi$ and $(\mathfrak{M}, t) \not\models [\varphi]\psi$ iff $(\mathfrak{M}, s) \not\models B_a(\varphi \to [\varphi]\psi)$.

Second, to show that R7 preserves validity, assume that $\mathfrak{M}(\chi) \subseteq \mathfrak{M}(B_a(\varphi \to \chi))$ and $\mathfrak{M}(\chi) \subseteq \mathfrak{M}([\varphi]\psi)$. Let $(\mathfrak{M}, s) \models \chi$. To prove that $(\mathfrak{M}, s) \models [\varphi]C_A\psi$, we prove that for all $A^{[\varphi]}$-paths from s ending in t, $(\mathfrak{M}, t) \models [\varphi]\,\psi$; using the second assumption of the rule, this can be established by showing that $(\mathfrak{M}, t) \models \chi$ for each such t. We show this by induction on the length of $A^{[\varphi]}$-paths from s ending in t. The base case $s = t$ follows directly from our assumptions. Now assume that we have a path $(t_1, \ldots, t_m, t)$ such that $t_1 = s$ and that $(\mathfrak{M}, t_m) \models \chi$. If $t_m \sqsubseteq t$, then we are done. If $E_A^{[\varphi]} t_m t$, then we reason as follows. $(\mathfrak{M}, t_m) \models \chi$ entails that $(\mathfrak{M}, t_m) \models B_A(\varphi \to \chi)$ and $E_A^{[\varphi]} t_m t$ entails that there are u, v such that $E_A t_m u$, $Ruvt$ and $(\mathfrak{M}, v) \models \varphi$. Hence, $(\mathfrak{M}, t) \models \chi$. □

Definition 5. *A set of formulas Γ is* closed *iff $\varphi \in \Gamma$ implies $\psi \in \Gamma$ for all subformulas ψ of φ, and*

- $C_A\varphi \in \Gamma$ *only if* $\{B_A C_A \varphi, B_A \varphi\} \subseteq \Gamma$;
- $[\varphi]\, r_{>0}(\psi_1, \ldots, \psi_n) \in \Gamma$ *only if* $\{[\varphi]\,\psi_i \mid i \leq n\} \subseteq \Gamma$;
- $[\varphi]\, B_A \psi \in \Gamma$ *only if* $B_A(\varphi \to [\varphi]\,\psi) \in \Gamma$;
- $[\varphi]\, C_A \psi \in \Gamma$ *only if* $[\varphi]\, B_A C_A \psi \in \Gamma$ *and* $[\varphi]\,\psi \in \Gamma$;
- $[\varphi][\psi]\,\chi \in \Gamma$ *only if* $[\varphi \otimes [\varphi]\,\psi]\,\chi \in \Gamma$;

A set of formulas is a prime RPAC-theory iff it is closed under forming conjunctions, closed under RPAC-provable implications, and satisfies the property that if $\varphi \vee \psi$ is in the set, then φ or ψ is in the set.

Definition 6. *Let Φ be a finite closed set. The* canonical model for Φ *is a structure $\mathfrak{M}_\Phi = (S, \sqsubseteq, L, R, C, E, V)$ where*

- S *is the set of all prime* RPAC*-theories;*
- $\sqsubseteq$ *is set inclusion;*
- $\Gamma \in L$ *iff Γ contains all theorems of* L;
- $R\Gamma\Delta\Sigma$ *iff, for all $\varphi \to \psi \in \Gamma$, if $\varphi \in \Delta$, then $\psi \in \Sigma$;*
- $C\Gamma\Delta$ *iff, for all $\neg\varphi \in \Gamma$, $\varphi \notin \Delta$;*
- $E(a)\Gamma\Delta$ *iff, for all $B_a\varphi \in \Phi \cap \Gamma$, $\varphi \in \Delta$;*
- $V(p) = \{\Gamma \mid p \in \Gamma\}$.

We define $F(a)\Gamma\Delta$ iff, $\varphi \in \Delta$ for all $B_a\varphi \in \Gamma$. The satisfaction relation is defined just as for epistemic models.

Note that the canonical epistemic accessibility relations $E(a)$ are defined using $B_a\varphi \in \Phi$, not arbitrary $B_a\varphi$; the latter defines the auxiliary relations $F(a)$. As in epistemic models, E_A denotes the union of $E(a)$ for $a \in A$. $E^*(A)$ is the reflexive transitive closure of the union of $\subseteq$ with $E(A)$. Note that $F(a) \subseteq E(a)$ for all a, but the converse inclusion does not hold.

The notation $\vdash_{\mathsf{RPAC}} \varphi$ means that φ is a theorem of RPAC; we will use only $\vdash \varphi$ in this paper. We call a pair of sets of formulas (Γ, Δ) *independent* iff there are no finite non-empty $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ such that $\vdash \bigwedge \Gamma' \to \bigvee \Delta'$.

Lemma 2 (Pair Extension). *If (Γ, Δ) is independent, then there is a prime theory $\Sigma \supseteq \Gamma$ disjoint from Δ.*

Proof. This is a corollary of the well-known fact that each non-overlapping filter-ideal pair in a distributive lattice is extended by a non-overlapping prime filter-ideal pair. This result is usually stated for non-empty filters and ideals. However, if Γ is empty, then we can set $\Sigma := \emptyset$ and if Γ is non-empty but Δ is empty, then we set $\Sigma := Fm$. It is clear that both Fm and $\emptyset$ are prime theories. □

Lemma 3. *For all Φ, $\mathfrak{M}_\Phi$ is a relevant epistemic model.*

Definition 7. *For all finite Φ, we define the following:*

- *If Γ is a prime theory not disjoint from Φ, then $\underline{\Gamma_\Phi} := \bigwedge(\Gamma \cap \Phi)$;*
- *if X is a non-empty set of prime theories, then $\underline{X_\Phi} := \bigvee_{\Gamma \in X} \underline{\Gamma_\Phi}$.*

If Φ is clear from the context, then we write just $\underline{\Gamma}$ and $\underline{X}$.

Note that $\underline{X}$ is well-defined even for infinite X since Φ is finite. Note also that $\Gamma \in X$ implies $\underline{X} \in \Gamma$ (since if the assumption holds then $\underline{\Gamma} \to \underline{X}$ is a theorem and obviously $\underline{\Gamma} \in \Gamma$).

Remark 2. Formulas of the form $\underline{\Gamma_\Phi}$ and $\underline{X_\Phi}$ will be used in the proof of the Truth Lemma, in the cases for $C_A\varphi$ and $[\varphi]C_A\psi$. In that particular context, each Γ we will need to "characterize" by $\underline{\Gamma_\Phi}$ will have a non-empty intersection with Φ and each X considered will be non-empty. For this reason, we do not need to account for empty conjunctions and disjunctions and so, unlike in [23,24], we do not need the presence of the "extensional" truth constant $\top$ and $\bot$ in the language.

Definition 8 (Complexity). *We define the following* complexity function $c : Fm \to \mathbb{N}$*:*

1. *$c(p) = 1$ for all $p \in Pr$;*
2. *$c(\mathrm{t}) = 1$;*
3. *$c(f(\varphi_1, \dots, \varphi_n)) = (\sum_{i=1}^{n} c(\varphi_i)) + 1$ for all $f \in \{\neg, \wedge, \to, \otimes, B_a, C_A\}$;*
4. *$c([\varphi]\psi) = (4 + c(\varphi)) \cdot c(\psi)$.*

The closure *of Γ is the smallest closed superset of Γ.*

It can be shown that the closure of any finite set is finite. Our definition of the complexity function is virtually the same as the definition used in [26], for example.

Lemma 4. *For all φ, ψ and χ:*

1. *$c(\varphi) > c(\psi)$ if ψ is a proper subformula of φ;*
2. *$c([\varphi]r_{>0}(\psi_1, \dots, \psi_n) > r_{>0}([\varphi]\psi_1, \dots, [\varphi]\psi_n)$;*
3. *$c([\varphi]B_a\psi) > c(\varphi \to [\varphi]\psi) + 1$;*
4. *$c([\varphi]C_A\psi) > c([\varphi]\psi)$;*
5. *$c([\varphi][\psi]\chi) > c([\varphi \otimes [\varphi]\psi]\chi)$.*

Lemma 5 (Truth Lemma). *For all $\mathfrak{M}$, all finite closed Φ, all formulas $\varphi \in \Phi$ and all prime theories Γ:*

$$\varphi \in \Gamma \iff (\mathfrak{M}_\Phi, \Gamma) \models \varphi\,. \tag{10}$$

Proof. Induction on $c(\varphi)$. The base case is trivial. The rest of the cases are established as follows. Our induction hypothesis is that (10) holds for all φ such that $c(\varphi) < c(\theta)$ for some fixed θ. We prove that (10) holds for θ. We reason by cases.

The cases where the main connective of θ is $r_{>0}$ are established as usual in completeness proofs for R (these cases use the assumption that Φ is closed under subformulas), and the case for $\theta = B_a\varphi$ is established as usual in modal logic.

If $\theta = C_A\varphi$, then we reason as follows. To prove the left-to-right implication, we will use the following facts:

(i) if $C_A\varphi \in \Gamma \cap \Phi$ and $E_A\Gamma\Delta$, then $C_A\varphi \in \Delta$; and
(ii) if $C_A\varphi \in \Gamma \cap \Phi$, then each A-path from Γ ends with Δ such that $C_A\varphi \in \Delta$.

Fact (i) is established as follows: If $C_A\varphi \in \Phi$, then $B_AC_A\varphi \in \Phi$ and so $B_aC_A\varphi \in \Phi$ for all $a \in A$. If $E_A\Gamma\Delta$, then $E_a\Gamma\Delta$ for some $a \in A$. Hence, if $C_A\varphi \in \Gamma$, then $B_aC_A\varphi \in \Gamma$ using A3 and R3. Hence, $C_A\varphi \in \Delta$ by the definition of E_a and properties of prime theories.

Fact (ii) is established by induction on the length of A paths from Γ. The base case of the one-element path is trivial: $C_A\varphi \in \Gamma$ by assumption and $\varphi \in \Gamma$ by A3. Now assume that we have a path $(\Delta_1, \dots, \Delta_m, \Delta)$ such that $\Delta_1 = \Gamma$ and that the claim holds for $(\Delta_1, \dots, \Delta_m)$. Then $C_A\varphi \in \Delta_m$. If $\Delta_m \subseteq \Delta$, then clearly $C_A\varphi \in \Delta$ and so $\varphi \in \Delta$ by A3. If $E_A\Delta_m\Delta$, then $C_A\varphi \wedge \varphi \in \Delta$ by (i).

Now assume that $C_A\varphi \in \Gamma\cap\Phi$ and $E_{C(A)}\Gamma\Delta$. The latter means that there is an A-path from Γ ending with Δ. Hence, $\varphi \in \Delta$ by (ii). Since Δ was arbitrary, we obtain $(\mathfrak{M}, \Gamma) \models C_A\varphi$.

Conversely, let X be the set of Δ such that there is an A-path $(\Gamma, \dots, \Delta)$. Assume that $(\mathfrak{M}, \Delta) \models \varphi$ for all $\Delta \in X$. By the induction hypothesis, $\varphi \in \Delta$ for all $\Delta \in X$. Since $\varphi \in \Phi$, $\underline{\Gamma}$ and indeed $\underline{\Delta}$ for all $\Delta \in X$ and $\underline{X}$ are defined. (We know that at least $\Gamma \in X$.) We prove that

(iii) $\vdash \underline{\Gamma} \to \underline{X}$;
(iv) $\vdash \underline{X} \to \varphi$;
(v) $\vdash \underline{X} \to B_A\underline{X}$.

Using R6, (iv) and (v) entail $\vdash \underline{X} \to C_A\varphi$, which together with (iii) gives $\vdash \underline{\Gamma} \to C_A\varphi$. Hence, $C_A\varphi \in \Gamma$.

Claims (iii) and (iv) are obvious. Claim (v) is established as follows. If $\nvdash \underline{X} \to B_A\underline{X}$, then there are $\Delta \in X$ and Θ such that $\underline{\Delta} \in \Theta$ and $B_A\underline{X} \notin \Theta$. Using the Pair Extension Lemma, there is Σ such that $F_A\Theta\Sigma$ and $\underline{X} \notin \Sigma$. But since $\underline{\Delta} \in \Theta$, $F_A\Theta\Sigma$ implies that $E_A\Delta\Sigma$. ($F_A\Theta\Sigma$ entails that $E_a\Theta\Sigma$ for some $a \in A$. Assume that $B_a\chi \in \Delta\cap\Phi$. Then $\vdash \underline{\Delta} \to B_a\chi$ and so $B_a\chi \in \Theta$. This means that $\chi \in \Sigma$.) Hence, $\Delta \in X$ implies $\Sigma \in X$, contradicting $\underline{X} \notin \Sigma$.

Now we consider the case $\theta = [\varphi]\psi$. We reason by cases, depending on the form of ψ:

Case 1. $\theta = [\varphi]p$. Then $\theta \in \Gamma$ iff $p \in \Gamma$ (using A4) iff $(\mathfrak{M}, \Gamma) \models p$ (using the induction hypothesis) iff $(\mathfrak{M}^{[\varphi]}, \Gamma) \models p$ (using the definition of $\mathfrak{M}^{[\varphi]}$) iff $(\mathfrak{M}, \Gamma) \models [\varphi]p$ (using the definition of $\models$).

Case 2. $\theta = [\varphi]r(\psi_1, \ldots, \psi_n)$. All sub-cases of this form are established using the definition of a closed set (closure under subformulas), Lemma 4, and the fact that update with φ does not modify the underlying Routley–Meyer model $(S, \sqsubseteq, L, R, C, V)$, just the epistemic accessibility relations E_a.

Case 3. $\theta = [\varphi]B_a\psi$. Then $\theta \in \Gamma$ iff $B_a(\varphi \to [\varphi]\psi) \in \Gamma$ (using A7) iff $(\mathfrak{M}, \Gamma) \models B_a(\varphi \to [\varphi]\psi)$ (using the induction hypothesis, relying on the definition of a closed set and item 3 of Lemma 4) iff for all Σ, Σ' and Δ, $E_A\Gamma\Sigma$ & $R\Sigma\Sigma'\Delta$ and $(\mathfrak{M}, \Sigma') \models \varphi$ only if $(\mathfrak{M}^{[\varphi]}, \Delta) \models \psi$ (definition of $\models$) iff, for all Δ, $E_A^{[\varphi]}\Gamma\Delta$ only if $(\mathfrak{M}^{[\varphi]}, \Delta) \models \psi$ (definition of $E_A^{[\varphi]}$) iff $(\mathfrak{M}^{[\varphi]}, \Gamma) \models B_a\psi$ (definition of $\models$) iff $(\mathfrak{M}, \Gamma) \models [\varphi]B_a\psi$ (definition of $\models$).

Case 4. $\theta = [\varphi]C_A\psi$. We will use the following two facts:

(vi) If $[\varphi]C_A\psi \in \Phi \cap \Gamma$ and $E_A^{[\varphi]}\Gamma\Delta$, then $[\varphi]C_A\psi \in \Delta$ and $[\varphi]\psi \in \Delta$; and
(vii) if $[\varphi]C_A\psi \in \Phi \cap \Gamma$, then each $A^{[\varphi]}$-path from Γ ends with Δ such that $[\varphi]C_A\psi \in \Delta$ and $[\varphi]\psi \in \Delta$.

Fact (vi) is established as follows. If $E_A^{[\varphi]}\Gamma\Delta$, then there is $a \in A$ such that $E_a^{[\varphi]}\Gamma\Delta$, which means that there are Σ_1, Σ_2 such that $E_a\Gamma\Sigma_1$, $R\Sigma_1\Sigma_2\Delta$ and $(\mathfrak{M}, \Sigma_2) \models \varphi$ (note that $E_a^{[\varphi]}$ is defined using $\models$). Using the induction hypothesis (φ is a subformula of θ), we get $\varphi \in \Sigma_2$. Now $[\varphi]C_A\psi \in \Gamma$ entails $[\varphi]B_aC_A\psi \in \Gamma$ and $[\varphi]B_a\psi \in \Gamma$ (using A3 and R5) and this entails $B_a(\varphi \to [\varphi]C_A\psi) \in \Gamma$ and $B_a(\varphi \to [\varphi]\psi) \in \Gamma$ (using A7). It follows from the definition of a closed set that $\{B_a(\varphi \to [\varphi]C_A\psi), B_a(\varphi \to [\varphi]\psi)\} \subseteq \Phi$, and so we can infer that $\varphi \to [\varphi]C_A\psi \in \Sigma_1$ and $\varphi \to [\varphi]\psi \in \Sigma_1$, which also means that $[\varphi]C_A\psi \in \Delta$ and $[\varphi]\psi \in \Delta$.

Fact (vii) is established by induction on the length of $A^{[\varphi]}$-paths from Γ. The base case of a one-element path is trivial: $[\varphi]C_A\psi$ is assumed to be in Γ and $[\varphi]\psi \in \Gamma$ by A3. Now assume that (vii) holds for all $A^{[\varphi]}$-paths of length m, where $2 \leq 1 < n$ and take $(\Delta_1, \ldots, \Delta_n)$, where $\Delta_1 = \Gamma$. We know that $C_A\varphi \in \Delta_{n-1}$. If $\Delta_n \subseteq \Delta_{n-1}$, then we are done. If $E_A^{[\varphi]}\Delta_{n-1}\Delta_n$, then $[\varphi]C_A\varphi \in \Delta_n$ thanks to (vi).

Now assume that $[\varphi]C_A\psi \in \Gamma$ and that $E_A^{[\varphi]}\Gamma\Delta$. The latter means that there is a $A^{[\varphi]}$-path $(\Gamma_1, \ldots, \Gamma_n)$ such that $\Gamma_1 = \Gamma$ and $\Gamma_n = \Delta$. By (vii), $[\varphi]\psi \in \Delta$. By the induction hypothesis, $(\mathfrak{M}, \Delta) \models [\varphi]\psi$ (see item 5 of Lemma 4) and so $(\mathfrak{M}^{[\varphi]}, \Delta) \models \psi$. Since Δ was arbitrary, we obtain $(\mathfrak{M}^{[\varphi]}, \Gamma) \models C_A\varphi$. This means that $(\mathfrak{M}, \Gamma) \models [\varphi]C_A\psi$.

Conversely, let X be the set of Δ such that there is an $A^{[\varphi]}$-path $(\Gamma, \ldots, \Delta)$. Assume that $(\mathfrak{M}, \Gamma) \models [\varphi]C_A\psi$, which means that $(\mathfrak{M}^{[\varphi]}, \Delta) \models \psi$ for all $\Delta \in X$. By the induction hypothesis, $[\varphi]\psi \in \Delta$ for all $\Delta \in X$. Since $[\varphi]\psi \in \Phi$, $\underline{\Gamma}$ and indeed $\underline{\Delta}$ for all $\Delta \in X$ and $\underline{X}$ are defined. (We know that at least $\Gamma \in X$.) We prove that

(viii) $\vdash \underline{\Gamma} \to \underline{X}$;
(ix) $\vdash \underline{X} \to [\varphi]\psi$;
(x) $\vdash \underline{X} \to B_A(\varphi \to \underline{X})$.

Using R7, (ix) and (x) entail that $\vdash \underline{X} \to [\varphi]C_A\psi$, which together with (viii) entails $\vdash \underline{\Gamma} \to [\varphi]C_A\psi$. This means that $[\varphi]C_A\psi \in \Gamma$.

Claim (viii) follows from $\Gamma \in X$. Claim (ix) follows from the assumption that $[\varphi]\psi \in \bigcap X$. Claim (x) is established as follows. If $\nvdash \underline{X} \to B_A(\varphi \to \underline{X})$, then there is $\Delta \in X$ such that $\nvdash \underline{\Delta} \to B_A(\varphi \to \underline{X})$. Hence, by the Pair Extension Lemma, there is Θ such that $\underline{\Delta} \in \Theta$ and $B_A(\varphi \to \underline{X}) \notin \Theta$. The latter means that there is $a \in A$ and Σ such that $F_a\Theta\Sigma$ and $\varphi \to \underline{X} \notin \Sigma$. The latter here means that there are Π, Ω such that $R\Sigma\Pi\Omega$, $\varphi \in \Pi$ and $\underline{X} \notin \Omega$. Using the induction hypothesis, we obtain $(\mathfrak{M}, \Pi) \models \varphi$, and so $E_A^{[\varphi]}\Delta\Omega$ (since $F_A\Theta\Sigma$ and $\underline{\Delta} \in \Theta$ imply that $E_A\Delta\Sigma$). Since $\Delta \in X$, it follows that $\Omega \in X$, contradicting $\underline{X} \notin \Omega$.

Case 5. $\theta = [\varphi][\psi]\chi$. We will use the following fact:

(xi) For $a \in G$ and all formulas α, β: $E_a^{[\alpha][\beta]} = E_a^{[\alpha \otimes [\alpha]\beta]}$.

(xi) needs the assumption of full associativity (7):

$$\begin{aligned}
E_a^{[\alpha][\beta]}\Delta\Sigma &\iff \exists\Sigma_1\Sigma_2(E_a^{[\alpha]}\Delta\Sigma_1 \ \& \ R\Sigma_1\Sigma_2\Sigma \ \& \ (\mathfrak{M}^{[\alpha]}, \Sigma_2) \models \beta) \\
&\iff \exists\Theta_1\Theta_2\Sigma_2(E_a\Delta\Theta_1 \ \& \ R\Theta_1\Theta_2\Sigma_2\Sigma \ \& \\
&\qquad\qquad (\mathfrak{M}, \Theta_2) \models \alpha \ \& \ (\mathfrak{M}, \Sigma_2) \models [\alpha]\beta) \\
&\iff \exists\Theta_1\Theta_2\Sigma_2(E_a\Delta\Theta_1 \ \& \ R\Theta_1(\Theta_2\Sigma_2)\Sigma \ \& \\
&\qquad\qquad (\mathfrak{M}, \Theta_2) \models \alpha \ \& \ (\mathfrak{M}, \Sigma_2) \models [\alpha]\beta) \\
&\iff \exists\Theta_1\Theta_3(E_a\Delta\Theta_1 \ \& \ R\Theta_1\Theta_3\Sigma \ \& \ (\mathfrak{M}, \Theta_3) \models \alpha \otimes [\alpha]\beta) \\
&\iff E_a^{[\alpha \otimes [\alpha]\beta]}\Delta\Sigma
\end{aligned}$$

Now we reason as follows: $\theta \in \Gamma$ iff $[\varphi \otimes [\varphi]\psi]\chi \in \Gamma$ (using A8) iff $(\mathfrak{M}, \Gamma) \models [\varphi \otimes [\varphi]\psi]\chi$ (induction hypothesis, relying on item 6 of Lemma 4 and the definition of a closed set) iff $(\mathfrak{M}^{[\varphi \otimes [\varphi]\psi]}, \Gamma) \models \chi$ iff $(\mathfrak{M}^{[\varphi][\psi]}, \Gamma) \models \chi$ (since $\mathfrak{M}^{[\varphi \otimes [\varphi]\psi]} = \mathfrak{M}^{[\varphi][\psi]}$ by (xi)) iff $(\mathfrak{M}, \Gamma) \models [\varphi][\psi]\chi$.

□

Theorem 1. *φ is a theorem of* RPAC *iff φ is valid in all relevant epistemic models.*

Proof. Soundness follows from Lemma 1. Completeness is established using the Pair Extension Lemma, the Truth Lemma and Lemma 3: If φ is not a theorem, then neither is $\mathrm{t} \to \varphi$. Take Φ the closure of $\{\mathrm{t} \to \varphi\}$ and consider $\mathfrak{M}_\Phi$, which is a relevant epistemic model by Lemma 3. By the Pair Extension Lemma, there is a $\Gamma \in L$ such that $\varphi \notin \Gamma$. By the Truth Lemma, φ is not valid in $\mathfrak{M}_\Phi$. □

4 Information models

The semantics used in the previous sections builds on the framework introduced in [25], which has been in this paper adjusted to the background logic R and extended with common knowledge. Another approach to a public announcement logic with a substructural basis was developed in [14]. It turns out that these two approaches are closely related, which will be discussed in detail in this section. The merit of the alternative perspective that we will now describe is that it will allow us in the next section to enrich our logic with a further dimension that forms a crucial ingredient of informational dynamics. In particular, this perspective will allow us to express in the object language not only statements but also questions. For definition of the semantic models of this alternative framework we will need the following notion of a situation.

Definition 9. *Let $(P, \leq)$ be a complete lattice, where for any $X \subseteq P$, $\bigsqcup X$ denotes the join of X. An element $s \in P$ is called a* situation *in $(P, \leq)$ iff it is completely join-irreducible, i.e. iff for every $X \subseteq P$, $s = \bigsqcup X$ only if $s = x$, for some $x \in X$. The set of situations in $(P, \leq)$ will be denoted as $Sit(P)$. For any $x \in P$, the set of situations below x, i.e. the set $\{s \in Sit(P) \mid s \leq x\}$, will be denoted as $Sit(x)$.*

Note that if meet distributes over arbitrary joins in a complete lattice $(P, \leq)$ then for every situation $s \in Sit(P)$ and every $X \subseteq P$, if $s \leq \bigsqcup X$ then $s \leq x$ for some $x \in X$.

Definition 10. *An* information model *for G is $\mathfrak{N} = (P, \leq, 1, \cdot, C, \sigma, V)$ such that (a) $\langle P, \leq\rangle$ is a complete lattice; (b) every state from P is identical to the join of a set of situations, that is, for any $x \in P$, $x = \bigsqcup Sit(x)$; (c) 1 is an identity with respect to the binary operation $\cdot$ on P, i.e. $1 \cdot x = x$; (d) $\sqcap$ (i.e. the finite meet) and $\cdot$ distribute over arbitrary joins from both directions; (e) C is symmetric binary relation on P, and $Cx \bigsqcup Y$ iff there is $y \in Y$ such that Cxy; (f) σ is a map that assigns to each agent $a \in G$ a function $\sigma(a)$ (we will often write σ_a) from situations to states; (g) if s, t are situations such that $s \leq t$ then $\sigma_a(s) \leq \sigma_a(t)$; (h) $V(p) \in S$, for every atomic formula p.*

P represents a set of *information states*, $x \leq y$ expresses that the state x is an informational *refinement* of the state y (i.e. x is informationally stronger than y), 1 is called a *logical state*, $x \cdot y$ is called *fusion* of the states x and y, Cxy says that x is *compatible* with y, σ is called an *information state map*, V is a *valuation* that assigns to every atomic formula an informational content, represented as a state in P.

Note that distributivity of fusion over joins implies its monotonicity, i.e. $x_1 \leq x_2$ and $y_1 \leq y_2$ only if $x_1 \cdot y_1 \leq x_2 \cdot y_2$. Moreover, there is the least element 0 in $\mathfrak{N}$ that can be defined as $\bigsqcup \emptyset$.

Definition 11. *A* relevant information model *for G is an information model $\mathfrak{N} = (P, \leq, 1, \cdot, C, \sigma, V)$ for G where (i) fusion is associative, commutative, and $x \leq x \cdot x$, (ii) for every situation $s \in Sit(P)$ there is a state $x \in P$ such that Cxy if and only if $s \leq y$, (iii) for all states $x, y, z \in S$, if $Cx(y \cdot z)$ then $C(x \cdot y)z$.*

Let σ be an inquisitive state map, $A \subseteq G$ a set of agents, and s a situation. We define $\sigma_A(s) = \bigsqcup_{a \in A} \sigma_a(s)$. Moreover, we define:

$$\sigma^*_A(s) = \bigsqcup\{t \in S \mid \; \exists t_1, \ldots, t_n \in Sit(S) \colon t_1 = s,\; t_{i+1} \leq \sigma_A(t_i),\; t_n = t \;\}.$$

The support relation between pointed relevant information models and formulas from $\mathcal{L}$ is defined as follows:

- $(\mathfrak{N}, x) \Vdash p$ iff $x \leq V(p)$,
- $(\mathfrak{N}, x) \Vdash \mathrm{t}$ iff $x \leq 1$,
- $(\mathfrak{N}, x) \Vdash \neg\varphi$ iff, $\forall y$, if Cxy, then $(\mathfrak{N}, y) \nVdash \varphi$
- $(\mathfrak{N}, x) \Vdash \varphi \wedge \psi$ iff $(\mathfrak{N}, x) \Vdash \varphi$ and $(\mathfrak{N}, x) \Vdash \psi$,
- $(\mathfrak{N}, x) \Vdash \varphi \to \psi$ iff, $\forall y$, if $(\mathfrak{N}, y) \Vdash \varphi$, then $(\mathfrak{N}, x \cdot y) \Vdash \psi$,
- $(\mathfrak{N}, x) \Vdash \varphi \otimes \psi$ iff, $\exists yz$, $(\mathfrak{N}, y) \Vdash \varphi$, $(\mathfrak{N}, z) \Vdash \psi$, and $x \leq y \cdot z$,
- $(\mathfrak{N}, x) \Vdash B_a\varphi$ iff, for all $s \in Sit(x)$, $(\mathfrak{N}, \sigma_a(s)) \Vdash \varphi$;
- $(\mathfrak{N}, x) \Vdash C_A\varphi$ iff, for all $s \in Sit(x)$, $(\mathfrak{N}, \sigma^*_A(s)) \Vdash \varphi$;
- $(\mathfrak{N}, x) \Vdash [\varphi]\psi$ iff $(\mathfrak{N}^{[\varphi]}, x) \Vdash \psi$, where $\mathfrak{N}^{[\varphi]}$ differs from $\mathfrak{N}$ only in that $\sigma^{[\varphi]}_a(s) = \bigsqcup\{\sigma_a(s) \cdot y \mid (\mathfrak{N}, y) \Vdash \varphi\}$.

A formula φ is valid in a relevant information model $\mathfrak{N}$ if $(\mathfrak{N}, 1) \Vdash \varphi$. Note that since 1 is an identity for fusion an implication $\varphi \to \psi$ is valid in $\mathfrak{N}$ iff ψ is supported by all states of $\mathfrak{N}$ that support φ.

Lemma 6. *For any relevant information model $\mathfrak{N}$ and any formula φ from $\mathcal{L}$, there is a state $info^{\mathfrak{N}}(\varphi)$ in $\mathfrak{N}$ such that $\mathfrak{N}, x \Vdash \varphi$ iff $x \leq info^{\mathfrak{N}}(\varphi)$.*

Proof. The claim follows from the fact that for any φ from $\mathcal{L}$ the set of states supporting φ contains 0, is downward closed and closed under $\bigsqcup$. This can be shown by induction on φ. We will consider only the inductive steps for modal operators. The inductive steps for $\neg, \wedge, \to$ are discussed in [15]. Support by 0 and downward persistence for the operators $B_a, C_A, [\varphi]$ is straightforward. Let us prove that the set of states supporting $B_a\varphi$ is closed under $\bigsqcup$. Assume $(\mathfrak{N}, x_i) \Vdash B_a\varphi$, for each $i \in I$. Take any $s \in Sit(\bigsqcup_{i \in I} x_i)$. Since s is a situation, we obtain $s \in Sit(x_i)$, for some $i \in I$. It follows that $(\mathfrak{N}, \sigma_a(s)) \Vdash \varphi$. Hence $(\mathfrak{N}, \bigsqcup_{i \in I} x_i) \Vdash B_a\varphi$. The inductive step for C_A is analogous and the step for $[\varphi]$ is straightforward.

The state $info^{\mathfrak{N}}(\varphi)$, the existence of which is guaranteed by the previous lemma, represents the informational content of the formula φ in $\mathfrak{N}$. (If no confusion arises the superscript will be omitted.) Its existence allows us to characterize update of states in the following simplified way.

Lemma 7. $\sigma^{[\varphi]}_a(s) = \sigma_a(s) \cdot info(\varphi)$.

The semantics based on relevant information models is closely related to the semantics based on relevant epistemic models. In order to spell out the exact connection we will use the notion of duality of two semantic frameworks introduced in [15]. Let S_1 and S_2 be two semantic frameworks based respectively on some classes of models C_1 and C_2 and equipped with semantic clauses determining which formulas are valid in which models. We say that models $\mathfrak{M} \in C_1$ and $\mathfrak{N} \in C_2$ are $\mathcal{L}$-equivalent if they validate exactly the same formulas from $\mathcal{L}$. We say that the semantic system S_2 is a dual counterpart of S_1 w.r.t. the language $\mathcal{L}$ if there are two maps $f : C_1 \to C_2$ and $g : C_2 \to C_1$ such that (a) every $\mathfrak{M} \in C_1$ is $\mathcal{L}$-equivalent to $f(\mathfrak{M})$, and (b) for every $\mathfrak{M} \in C_1$ and $\mathfrak{N} \in C_2$ it holds that $g(f(\mathfrak{M}))$ is isomorphic to $\mathfrak{M}$ and $f(g(\mathfrak{N}))$ is isomorphic to $\mathfrak{N}$.

The main goal of this section is to show that the semantics based on relevant information models is a dual counterpart of the semantics based on relevant epistemic models. (As already pointed out, the added value of this dual counterpart is that it will allow us to capture not only statements but also questions.) This result extends the results from [15] that established similar duality between information models and Routley–Meyer models for a basic propositional language involving only the operators $\{\wedge, \to, \neg\}$.

We will now describe an operation that transforms relevant epistemic models into relevant information models. We will use the following notation: Let $(S, \sqsubseteq)$ be a partial order and $s \in S$. Then UpS denotes the set of all up-sets of S, and $s^{\uparrow}$ denotes the set $\{t \in S \mid s \sqsubseteq t\}$. Note that the sets of the form $s^{\uparrow}$ are exactly the situations in the complete lattice $(UpS, \subseteq)$.

Definition 12. *Let $\mathfrak{M} = (S, \sqsubseteq, L, R, C, E, V)$ be a relevant epistemic model. We define a corresponding structure $\mathfrak{M}^i = (P^i, \leq^i, 1^i, \cdot^i, C^i, \sigma^i, V^i)$, where $P^i = UpS$; $\leq^i = \subseteq$; $1^i = L$; $x \cdot^i y = \{s \in S \mid \exists t \in x, u \in y$ such that $Rtus\}$; $C^i xy$ iff there are $s \in x$ and $t \in y$ such that Cst; $\sigma^i_a(s^{\uparrow}) = E_a(s)$; $V^i = V$.*

Lemma 8. *If $\mathfrak{M}$ is a relevant epistemic model then $\mathfrak{M}^i$ is a relevant information model.*

Theorem 2. *$\mathfrak{M}^i$ is $\mathcal{L}$-equivalent to $\mathfrak{M}$, for every relevant epistemic model $\mathfrak{M}$.*

Proof. Let $\mathfrak{M} = (S, \sqsubseteq, L, R, C, E, V)$ be a relevant epistemic model, and $\mathfrak{M}^i = (P^i, \leq^i, 1^i, \cdot^i, C^i, \sigma^i, V^i)$ the corresponding relevant information model. We have to show that for any θ from $\mathcal{L}$, $(\mathfrak{M}^i, L) \Vdash \theta$ iff for all $s \in L$, $(\mathfrak{M}, s) \models \theta$. We prove something more general:

(*) For any $X \in UpS$, $(\mathfrak{M}^i, X) \Vdash \theta$ iff, for all $s \in X$, $(\mathfrak{M}, s) \models \theta$.

This can be proved by induction on θ. We will go only through the inductive steps for the operators B_a, C_A, and $[\varphi]$. Take any $X \in UpS$. As the induction hypothesis, assume that the claim (*) holds for some given formulas φ, ψ.

Assume that $\theta = B_a\varphi$. Then it holds: $(\mathfrak{M}^i, X) \Vdash B_a\varphi$ iff, for all $s \in X$, $(\mathfrak{M}^i, \sigma^i_a(s^{\uparrow})) \Vdash \varphi$ iff, for all $s \in X$, $(\mathfrak{M}^i, E_a(s)) \Vdash \varphi$ iff (by induction hypothesis), for all $s \in X$ and for all $t \in E_a(s)$, $(\mathfrak{M}, t) \models \varphi$ iff, for all $s \in X$, $(\mathfrak{M}, s) \models B_a\varphi$.

Assume that $\theta = C_A\varphi$. It holds that $(\sigma^i_A)^*(s^{\uparrow}) = E^*_A(s)$ so we can proceed in the same way as in the case of $B_a\varphi$.

Finally assume that $\theta = [\varphi]\psi$. It can be shown that $(\mathfrak{M}^i)^{[\varphi]} = (\mathfrak{M}^{[\varphi]})^i$. So, we can proceed as follows: $(\mathfrak{M}^i, X) \Vdash [\varphi]\psi$ iff $((\mathfrak{M}^i)^{[\varphi]}, X) \Vdash \psi$ iff $((\mathfrak{M}^{[\varphi]})^i, X) \Vdash \psi$ iff, for all $s \in X$, $(\mathfrak{M}^{[\varphi]}, s) \models \psi$ iff, for all $s \in X$, $(\mathfrak{M}, s) \models [\varphi]\psi$.

Now we will define an inverse operation that transforms relevant information models into relevant epistemic models.

Definition 13. *Let* $\mathfrak{N} = (P, \leq, 1, \cdot, C, \sigma, V)$ *be a relevant information model. We define a corresponding structure* $\mathfrak{N}^e = (S^e, \sqsubseteq^e, L^e, R^e, C^e, E^e, V^e)$*, where* $S^e = Sit(P)$*;* $s \sqsubseteq^e t$ *iff* $t \leq s$*;* $L^e = Sit(1)$*;* $R^e stu$ *iff* $u \leq s \cdot t$*;* $C^e st$ *iff* Cst*;* $E^e_a(s) = Sit(\sigma_a(s))$*;* $V^e(p) = Sit(V(p))$*.*

Lemma 9. *If* $\mathfrak{N}$ *is a relevant information model then* $\mathfrak{N}^e$ *is a relevant epistemic model.*

Lemma 10. *Let* $\mathfrak{M} = (S, \sqsubseteq, L, R, C, E, V)$ *be a relevant epistemic model and let* $\mathfrak{M}^{ie} = (S^{ie}, \sqsubseteq^{ie}, L^{ie}, R^{ie}, C^{ie}, E^{ie}, V^{ie})$*. Then the map assigning to any* $s \in S$ *the set* $s^\uparrow$ *is a bijection between* S *and* S^{ie}*. Moreover, the following holds: (a)* $s \sqsubseteq t$ *iff* $s^\uparrow \sqsubseteq^{ie} t^\uparrow$*, (b)* $s \in L$ *iff* $s^\uparrow \in L^{ie}$*, (c)* $Rstu$ *iff* $R^{ie}s^\uparrow t^\uparrow u^\uparrow$*, (d)* Cst *iff* $C^{ie}s^\uparrow t^\uparrow$*, (e)* $E_a st$ *iff* $E^{ie}_a s^\uparrow t^\uparrow$*, (f)* $s \in V(p)$ *iff* $s^\uparrow \in V^{ie}(p)$*.*

Proof. For an illustration, we will just show how to prove (c) and (e). The proof of (c) goes as follows: $R^{ie}s^\uparrow t^\uparrow u^\uparrow$ iff $u^\uparrow \leq^i s^\uparrow \cdot^i t^\uparrow$ iff $u^\uparrow \subseteq s^\uparrow \cdot^i t^\uparrow$ iff $u \in s^\uparrow \cdot^i t^\uparrow$ iff $\exists v \in s^\uparrow \exists w \in t^\uparrow$: $Rvwu$ iff $Rstu$. (e) can be proved in the following way: $E_a st$ iff $t \in \sigma^i_a(s^\uparrow)$ iff $t^\uparrow \leq^i \sigma^i_a(s^\uparrow)$ iff $E^{ie}_a s^\uparrow t^\uparrow$.

Lemma 11. *Let* $\mathfrak{N} = (P, \leq, 1, \cdot, C, \sigma, V)$ *be a relevant information model and let* $\mathfrak{N}^{ei} = (P^{ei}, \leq^{ei}, 1^{ei}, \cdot^{ei}, C^{ei}, \sigma^{ei}, V^{ei})$*. The map assigning to any* $x \in P$ *the set* $Sit(x)$ *is a bijection between* P *and* P^{ei}*. Moreover, the following holds: (a)* $x \leq y$ *iff* $Sit(x) \leq^{ei} Sit(y)$*, (b)* $Sit(1) = 1^{ei}$*, (c)* $Sit(x \cdot y) = Sit(x) \cdot^{ei} Sit(y)$*, (d)* Cxy *iff* $C^{ei}Sit(x)Sit(y)$*, (e)* $Sit(\sigma_a(s)) = \sigma^{ei}_a(Sit(s))$*, (f)* $Sit(V(p)) = V^{ei}(p)$*.*

Lemmas 10 and 11 lead directly to the following theorem.

Theorem 3. $\mathfrak{M}$ *is isomorphic to* $\mathfrak{M}^{ie}$*, for every relevant epistemic model* $\mathfrak{M}$*, and* $\mathfrak{N}$ *is isomorphic to* $\mathfrak{N}^{ei}$*, for every relevant information model* $\mathfrak{N}$*.*

Theorems 3 and 2 together show that the the semantics based on relevant information models is indeed a dual counterpart of the semantics based on relevant epistemic models. As a consequence, RPAC is also complete with respect to relevant information models.

5 Questions

In this section we will extend the language $\mathcal{L}$ so that it will be possible to express not only statements but also questions. To this end, we will borrow some techniques from inquisitive semantics (see, e.g., [7,5]). Let $\mathcal{L}^{inq}$ denote the language $\mathcal{L}$ enriched with a binary connective $\vee\!\!\!\vee$ called *inquisitive disjunction*. The operator $\vee\!\!\!\vee$ can be embedded arbitrarily under any operator with the exception of the public announcement modality. We will assume that $[\varphi]\nu$ is in the language $\mathcal{L}^{inq}$ only if φ is a formula of the language $\mathcal{L}$ (but ν may contain inquisitive disjunction).

The formulas of the language $\mathcal{L}$ will be called *declarative*. The connective $\vee\!\!\!\vee$ produces questions from declarative formulas. Given declarative formulas φ, ψ, the formula $\varphi \vee\!\!\!\vee \psi$ expresses the question *whether* φ *or* ψ. This can be contrasted with $\varphi \vee \psi$ which expresses the statement *that* φ *or* ψ. (Recall that $\varphi \vee \psi$ is defined as $\neg(\neg\varphi \wedge \neg\psi)$). The support condition for this additional connective is defined as follows:

$(\mathfrak{N}, x) \Vdash \nu \vee\!\!\!\vee \mu$ iff $(\mathfrak{N}, x) \Vdash \nu$ or $(\mathfrak{N}, x) \Vdash \mu$.

This captures the idea that an information state resolves a question if it provides some answer to the question. Compare the clause to the support condition for the declarative disjunction $\vee$ spelled out in the following lemma.

Lemma 12. *$(\mathfrak{N}, x) \Vdash \nu \vee \mu$ iff for all $s \in Sit(x)$, $(\mathfrak{N}, s) \Vdash \nu$ or $(\mathfrak{N}, s) \Vdash \mu$.*

In the language $\mathcal{L}^{inq}$ we can express directly, for example, that the agent's information state resolves the question ν ($B_a\nu$), that the question is resolved by the common knowledge in a group ($C_A\nu$), or that ν would be resolved after the public announcement of φ ($[\varphi]\,\nu$).

It is obvious from the semantic clause for inquisitive disjunction that the set of states supporting $\nu \vee\!\!\!\vee \mu$ is the union of the set of states supporting ν and the set of states supporting μ. As a consequence, Lemma 6 cannot be formulated for the whole language $\mathcal{L}^{inq}$. Sets of states supporting formulas of $\mathcal{L}^{inq}$ are not in general closed under join, though they are always downward closed and nonempty (contain always the least element).

Let $\mathsf{InqRPAC}$ be the logic consisting of all formulas from the language $\mathcal{L}^{inq}$ that are valid in all relevant information models. In formulation of an axiomatic system for $\mathsf{InqRPAC}$ we have to be careful to specify which schemata of RPAC are semantically valid for the whole language $\mathcal{L}^{inq}$ and which must be restricted to the declarative language $\mathcal{L}$. In particular, we can take the axiomatization of RPAC as specified in Fig. 1 and assume that we can substitute any formulas of $\mathcal{L}^{inq}$ for the variables in the axiom and rule schemata, with the exception of the variables occurring in the scope of the public announcement modality and in the double negation axiom (which is among the axioms of R), the reduction axiom for the belief modality (A7), and the rules (R6) and (R7). That is, the exception concerns the public announcement modality and the following axioms and rules:

(DN) $\neg\neg\varphi \to \varphi$

(A7) $[\varphi]\,B_a\psi \leftrightarrow B_a(\varphi \to [\varphi]\,\psi)$

(R6) $\dfrac{\nu \to (\varphi \wedge B_A\nu)}{\nu \to C_A\varphi}$

(R7) $\dfrac{\nu \to B_A(\varphi \to \nu) \quad \nu \to [\varphi]\,\psi}{\nu \to [\varphi]\,C_A\psi}$

To secure soundness we must assume that only declarative formulas can be substituted for φ and ψ in these four schemata (but any formula of $\mathcal{L}^{inq}$ can be substituted for ν). Let us illustrate the reason behind this restriction on the rule R6. Assume that $\nu \to (\varphi \wedge B_A\nu)$ is valid in a relevant information model $\mathfrak{N}$ and assume that $\mathfrak{N}, x \Vdash \nu$. Then $\mathfrak{N}, x \Vdash \varphi \wedge B_A\nu$. Let $s \in Sit(x)$. Then φ is supported by every state in $\{t \in P \mid \exists t_1, \ldots, t_n \in Sit(P)\colon t_1 = s,\, t_{i+1} \leq \sigma_A(t_i),\, t_n = t\ \}$.

Since we assume that φ is in $\mathcal{L}$, it is supported also by the join of this set, i.e. by $\sigma^*_A(s)$. Since this holds for every $s \in Sit(x)$, we obtain $\mathfrak{N}, x \Vdash C_A\varphi$ as desired. Hence, $\nu \to C_A\varphi$ is valid in $\mathfrak{N}$.

The system for InqRPAC consists of the system for RPAC, adjusted to the language $\mathcal{L}^{inq}$ in the just described way, and extended with the axioms in Fig. 2 that characterize inquisitive disjunction. Note that while ν, μ, θ range over arbitrary formulas of $\mathcal{L}^{inq}$, φ (in the axioms Inq7 and Inq10) is again restricted to formulas of $\mathcal{L}$. The axioms Inq1-Inq3 are the standard (introduction and elimination) axioms for disjunction. The axioms Inq4-Inq10 specify how the other operators of the language distribute over inquisitive disjunction. Note that the inverse implications of Inq4-Inq10 are provable from the other axioms.

(Inq1) $\nu \to \nu \mathbin{\vee\!\!\!\vee} \mu$
(Inq2) $\mu \to \nu \mathbin{\vee\!\!\!\vee} \mu$
(Inq3) $((\nu \to \theta) \wedge (\mu \to \theta)) \to ((\nu \mathbin{\vee\!\!\!\vee} \mu) \to \theta)$
(Inq4) $\neg(\nu \mathbin{\vee\!\!\!\vee} \mu) \to (\neg\nu \wedge \neg\mu)$
(Inq5) $(\theta \wedge (\nu \mathbin{\vee\!\!\!\vee} \mu)) \to ((\theta \wedge \nu) \mathbin{\vee\!\!\!\vee} (\theta \wedge \mu))$
(Inq6) $(\theta \otimes (\nu \mathbin{\vee\!\!\!\vee} \mu)) \to ((\theta \otimes \nu) \mathbin{\vee\!\!\!\vee} (\theta \otimes \mu))$
(Inq7) $(\varphi \to (\nu \mathbin{\vee\!\!\!\vee} \mu)) \to ((\varphi \to \nu) \mathbin{\vee\!\!\!\vee} (\varphi \to \mu))$ (for declarative φ)
(Inq8) $B_a(\nu \mathbin{\vee\!\!\!\vee} \mu) \to (B_a\nu \vee B_a\mu)$
(Inq9) $C_A(\nu \mathbin{\vee\!\!\!\vee} \mu) \to (C_A\nu \vee C_A\mu)$
(Inq10) $[\varphi](\nu \mathbin{\vee\!\!\!\vee} \mu) \to ([\varphi]\nu \mathbin{\vee\!\!\!\vee} [\varphi]\mu)$ (for declarative φ)

Fig. 2. The axioms for inquisitive disjunction in the system for InqRPAC.

Lemma 13. *The system for* InqRPAC *is sound with respect to all relevant information models.*

We can prove completeness of the system InqRPAC using a strategy that is common in inquisitive logic (see, e.g., [15]). In our specific setting this strategy amounts to the reduction of completeness for InqRPAC to completeness for RPAC (which was proved in Section 3). Such a reduction is possible due to two characteristic properties of the logic that needs to be proved: (1) disjunctive normal form, and (2) disjunction property.

Lemma 14. *For any formula ν of $\mathcal{L}^{inq}$ there are formulas $\varphi_1, \ldots, \varphi_n$ of $\mathcal{L}$ such that $\nu \leftrightarrow (\varphi_1 \mathbin{\vee\!\!\!\vee} \ldots \mathbin{\vee\!\!\!\vee} \varphi_n)$ is a theorem of* InqRPAC.

Proof. This is a straightforward extension of a standard theorem in inquisitive logic. (For more details, see, e.g., [13].) One can proceed by induction on the complexity of ν using the distributive axioms Inq4-Inq10 and their converses.

Lemma 15. *Let ν, μ be formulas of $\mathcal{L}^{inq}$. Then $\nu \mathbin{\vee\!\!\!\vee} \mu$ is valid in every relevant information model only if ν is valid in every relevant information model or μ is valid in every relevant information model.*

Proof. To prove this claim, we will adapt to our current setting a technique developed in [13]. For any relevant information models $\mathfrak{N}_1 = (P_1, \leq_1, 1_1, \cdot_1, C_1, \sigma_1, V_1)$ and $\mathfrak{N}_2 = (P_2, \leq_2, 1_2, \cdot_2, C_2, \sigma_2, V_2)$ we can define their product $\mathfrak{N}_1 \times \mathfrak{N}_2 = (P, \leq, 1, \cdot, C, \sigma, V)$ where $P = P_1 \times P_2$ (the cartesian product of P_1 and P_2); $(x, y) \leq (v, w)$ iff $x \leq_1 v$ and $y \leq_2 w$; $1 = (1_1, 1_2)$; $(x, y) \cdot (v, w) = (x \cdot_1 v, y \cdot_2 w)$; $C(x, y)(v, w)$ iff $C_1 xv$ or $C_2 yw$; $\sigma_a((s, t)) = (\sigma_1(a)(s), \sigma_2(a)(t))$; $V(p) = (V_1(p), V_2(p))$. It can be shown that $\mathfrak{N}_1 \times \mathfrak{N}_2$ is again a relevant information model. Assume that 0_1 is the least element of $\mathfrak{N}_1$ and 0_2 is the least element of $\mathfrak{N}_2$. The following holds for any formula ν of $\mathcal{L}^{inq}$:

(a) $(\mathfrak{N}_1 \times \mathfrak{N}_2, (x, 0_2)) \Vdash \nu$ iff $(\mathfrak{N}_1, x) \Vdash \nu$,
(b) $(\mathfrak{N}_1 \times \mathfrak{N}_2, (0_1, y)) \Vdash \nu$ iff $(\mathfrak{N}_2, y) \Vdash \nu$.

These claims can be proved by induction on the complexity of ν. For an illustration, let us consider the inductive step for $[\varphi]$. Assume that (a) and (b) hold for some φ from $\mathcal{L}$ and μ from $\mathcal{L}^{inq}$. Using the induction hypothesis it can be shown that $(\mathfrak{N}_1 \times \mathfrak{N}_2)^{[\varphi]} = \mathfrak{N}_1^{[\varphi]} \times \mathfrak{N}_2^{[\varphi]}$. Using this fact we can prove the inductive step for $[\varphi]$ as follows: $(\mathfrak{N}_1 \times \mathfrak{N}_2, (x, 0_2)) \Vdash [\varphi]\mu$ iff $((\mathfrak{N}_1 \times \mathfrak{N}_2)^{[\varphi]}, (x, 0_2)) \Vdash \mu$ iff $(\mathfrak{N}_1^{[\varphi]} \times \mathfrak{N}_2^{[\varphi]}, (x, 0_2)) \Vdash \mu$ iff $(\mathfrak{N}_1^{[\varphi]}, x) \Vdash \mu$ iff $(\mathfrak{N}_1, x) \Vdash [\varphi]\mu$. The step for (b) is analogous.

Assuming that we have proved (a) and (b) we can prove disjunction property as follows. Assume that there is a relevant information model $\mathfrak{N}_1$ in which ν is not valid, and a relevant information model $\mathfrak{N}_2$ in which μ is not valid. Then $(\mathfrak{N}_1, 1_1) \nVdash \nu$ and $(\mathfrak{N}_2, 1_2) \nVdash \mu$. If follows from (a) and (b) that $(\mathfrak{N}_1 \times \mathfrak{N}_2, (1_1, 0_2)) \nVdash \nu$ and $(\mathfrak{N}_1 \times \mathfrak{N}_2, (0_1, 1_2)) \nVdash \mu$. By persistence, $(\mathfrak{N}_1 \times \mathfrak{N}_2, (1_1, 1_2)) \nVdash \nu \mathbin{\vee\!\!\!\vee} \mu$ and thus $\nu \mathbin{\vee\!\!\!\vee} \mu$ is not valid in $\mathfrak{N}_1 \times \mathfrak{N}_2$.

Theorem 4. *ν is a theorem of* InqRPAC *iff ν is valid in all relevant information models.*

Proof. The left-to-right direction amounts to Lemma 13. For the right-to-left direction assume that ν is valid in all relevant information models. Then, by disjunctive normal form (Lemma 14), there are $\varphi_1, \ldots, \varphi_n$ in $\mathcal{L}$ such that $\nu \leftrightarrow (\varphi_1 \mathbin{\vee\!\!\!\vee} \ldots \mathbin{\vee\!\!\!\vee} \varphi_n)$ is a theorem of InqRPAC. By soundness (Lemma 13), $\varphi_1 \mathbin{\vee\!\!\!\vee} \ldots \mathbin{\vee\!\!\!\vee} \varphi_n$ is valid in all relevant information models. By disjunction property (Lemma 15), for some i, φ_i is valid in all relevant information models. By duality between relevant information models and relevant epistemic models for $\mathcal{L}$ (Theorems 3 and 2) we obtain that φ_i is valid in every relevant epistemic model. By completeness of RPAC w.r.t. relevant epistemic models (Theorem 1), φ_i is a theorem of RPAC. Since InqRPAC is an extension of RPAC, φ_i is also a theorem of InqRPAC. It follows that $\varphi_1 \mathbin{\vee\!\!\!\vee} \ldots \mathbin{\vee\!\!\!\vee} \varphi_n$ is a theorem of InqRPAC. Hence, ν is a theorem of InqRPAC.

The semantics based on information models allows us to equip agents not only with information states but also with issues.[4] Then one can define also

[4] In [6] issues were introduced in the context of standard inquisitive epistemic logic based on classical logic. In [16] issues were introduced in the semantics of substructural inquisitive epistemic logic.

a public utterance of questions and the inquisitive analogues of the modalities B_a and C_A.[5] This would be an interesting further extension of the language $\mathcal{L}^{inq}$. However, the methods employed in this paper cannot be directly applied to this extension. The reason is that completeness for such a language cannot be straightforwardly reduced to completeness of its non-inquisitive fragment, which was possible in the case of the language $\mathcal{L}^{inq}$. The inquisitive analogue of B_a was studied in the context of substructural inquisitive logics in [16]. The investigation of the inquisitive analogue of C_A is left for future research.

6 Conclusion

This paper can be seen as a further expansion of our previous work on modal and inquisitive substructural logics [13,14,15,16,24,25]. In particular, we have extended the relevant logic R with various epistemic operators and studied their interactions. The main novelty of the paper is the incorporation of common knowledge into this rich context. The main result of the paper is a completeness of R, extended with a belief modality, public announcement and common knowledge, with respect to a suitable relational semantics (Theorem 1). We also considered an alternative semantics that allowed us to express also questions in the object language and we presented a completeness proof also for this enriched language (Theorem 4).

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[5] The inquisitive analogue of B_a is a standard modality in inquisitive epistemic logic usually denoted as E_a (see [6]). The inquisitive analogue of C_A was introduced semantically in [5] in the context of standard inquisitive epistemic logic without an axiomatic characterization.

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