From positive PDL to its non-classical extensions

Igor Sedlár* and Vít Punčochář

The Czech Academy of Sciences, Institute of Computer Science
Pod Vodárenskou věží 271/2, 182 07 Prague 8, Czech Republic
{sedlar,puncochar}@cs.cas.cz

February 14, 2019

Abstract

We provide a complete binary implicational axiomatization of the positive fragment of Propositional Dynamic Logic. The intended application of this result are completeness proofs for non-classical extensions of positive PDL. Two examples are discussed in this article, namely, a paraconsistent extension with modal De Morgan negation and a substructural extension with the residuated operators of the Non-associative Lambek calculus. Informal interpretations of these two extensions are outlined.

1 Introduction

Propositional Dynamic Logic PDL, introduced in [15] following the ideas of [34], is a well known modal logic with applications in formal verification of programs [17], dynamic epistemic logic [2] and deontic logic [28], for example. PDL can be seen more generally as a logic for reasoning about *structured actions* modifying various types of *objects*; for instance, programs modifying states of the computer, information state updates or actions of agents changing the world around them.

The study of PDL and its variants—propositional dynamic logics, PDLs—is a research quite active until today. The bulk of this research, however, concentrates on logics extending the classical propositional calculus; the study of *non-classical* PDLs is largely underdeveloped so far. Among the first contributions to the study of non-classical PDLs were the articles by Leivant [24] and Nishimura [31], studying intuitionistic PDLs. Intuitionistic PDLs were also studied later in [9, 44].[1] Other work in the area concentrates mainly on many-valued PDLs. Liau [25] defines a general framework for fuzzy PDL and

*Corresponding author.

[1]See also [19], where an intuitionistic epistemic logic with common knowledge is studied. This logic can be seen as a $\{;, \cup, ?\}$-free fragment of intuitionistic PDL.

shows that the Łukasiewicz version has potential applications in decision theory. Hughes et al. [18] introduce a Gödel-style fuzzy PDL for reasoning about reliability of actions as means to achieve goals. Běhounek [8] sketches a general framework for PDL with fuzzy actions and outlines its applications in reasoning about costs of program runs. Completeness results are not established in these application-oriented papers. Teheux [41] formulates a finitely-valued Łukasiewicz PDL, provides a complete axiomatization, and applies the logic to modelling the Rényi–Ulam searching game with errors. Madeira et al. [26, 27] take a more general approach by considering PDL using residuated Kleene algebras (axiomatization is not provided). Sedlár [37, 38] provides an axiomatization of paraconsistent four-valued PDLs for reasoning about algorithmic modifications of database-like bodies of information that may be incomplete but also inconsistent.[2]

In this article we take first steps towards a general study of non-classical PDLs. We are ultimately interested in variants of PDL over non-classical extensions of the Distributive Lattice Logic DLL. Instead of studying such extensions individually, we focus first on *positive* PDL, a negation-free PDL-style extension of DLL.[3] The completeness result for positive PDL can then be used as a building block in completeness proofs for combinations of positive PDL with various non-classical extensions of DLL (including paraconsistent logics, but also many distributive substructural logics [35, 33, 16]). Two examples are discussed in this article, namely, a paraconsistent extension of positive PDL with modal De Morgan negation and a substructural extension with the residuated operators of the Non-associative Lambek calculus.

The article is structured as follows. In Section 2, the positive fragment PDL^+ of PDL is defined alongside with the binary implicational proof system $\mathcal{PDL}^+$. Section 3 introduces the canonical $\mathcal{PDL}^+$-structure. In Section 4 the filtration of the canonical structure is defined and the Filtration Theorem is established. The proof contains one of the main technical novelties of the article, namely, a modification of the standard proof of the Filtration Theorem for PDL that relies heavily on the presence of Boolean negation in the language. Filtration is then used to prove completeness of $\mathcal{PDL}^+$ with respect to PDL^+. Section 5 introduces an extension of PDL^+ with De Morgan negation. Completeness is established and an informal interpretation of the framework is outlined. (We study a version of positive PDL with De Morgan negation suitable for reasoning about updates of so-called Belnapian databases, thereby showing that this version of PDL connects naturally with some of the central ideas of Belnap's classic papers [3, 4].) Section 6 studies extensions of PDL^+ with the residuated operators (product, left division and right division) of the Non-associative Lambek calculus. Completeness of four versions of Lambek PDL is established. Three informal interpretations of the framework are outlined, namely, a linguistic interpretation (actions modify linguistic resources),

[2] A related contribution is Wansing's [42] where a version of concurrent PDL with 'negation' of actions—distinct from Boolean complement—is studied.

[3] Our results on positive PDL extend the work of Dunn [11] who provides a binary implicational axiomatization of DLL extended with $\Box$ and $\Diamond$.

an informational interpretation (actions modify bodies of information) and an epistemic interpretation (actions modify information states of agents). Our results are summarized and some limitations of our approach are discussed in Section 7.

2 Positive PDL

The *language* $\mathcal{L}$ contains two classes of expressions, namely, actions and formulas, defined by mutual induction:

$$\mathsf{Act} \quad A := \ a \mid A;A \mid A \cup A \mid A^* \mid X?$$

$$\mathsf{Form} \quad X := \ p \mid X \wedge X \mid X \vee X \mid [A]X \mid \langle A \rangle X$$

where $a \in \mathsf{AAct}$ (a countable set of 'atomic actions') and $p \in \mathsf{Prop}$ (a countable set of propositional variables). We assume familiarity with the informal interpretation of the language; the reader is referred to [17, 164–167].

A *consequence $\mathcal{L}$-pair* (here we use the terminology of Dunn [11]) is an ordered pair of formulas, written as $X \vdash Y$. Logics defined in this article are sets of consequence pairs (not sets of formulas or consequence relations). This approach is standard in positive modal logic [11], general investigations of non-Boolean negations [12] and some of the literature on the Lambek calculus [10, 30].

A *dynamic model* is a couple $M = \langle W_M, [\![\ ]\!]_M \rangle$ where W is a non-empty set (of 'states') and $[\![\ ]\!]_M$ is a function such that

$[\![A]\!]_M$ is a binary relation on W and

$[\![X]\!]_M$ is a subset of W.

It is assumed that $[\![A;B]\!]_M$ is the composition of $[\![A]\!]_M$ and $[\![B]\!]_M$ (in that order); $[\![A \cup B]\!]_M$ is the union of $[\![A]\!]_M$ and $[\![B]\!]_M$; $[\![A^*]\!]_M$ is the reflexive transitive closure $[\![A]\!]_M^*$ of $[\![A]\!]_M$ and $[\![X?]\!]_M$ is the identity relation on $[\![X]\!]_M$. $[\![X \wedge Y]\!]_M$ ($[\![X \vee Y]\!]_M$) is the intersection (union) of $[\![X]\!]_M$ and $[\![Y]\!]_M$. Moreover,

$[\![[A]X]\!]_M = \{w \ ; \forall v : w[\![A]\!]_M v \text{ implies } v \in [\![X]\!]_M\}$ and

$[\![\langle A \rangle X]\!]_M = \{w \ ; \exists v : w[\![A]\!]_M v \text{ and } v \in [\![X]\!]_M\}$.

where $w[\![A]\!]_M v$ means that $\langle w, v \rangle \in [\![A]\!]_M$. We sometimes write $u \vDash_M X$ instead of $u \in [\![X]\!]_M$. The subscript is omitted whenever possible.

A consequence pair $X \vdash Y$ is *valid in a model* M iff $[\![X]\!]_M \subseteq [\![Y]\!]_M$ (notation $X \vdash_M Y$). Positive PDL, PDL^+, is the set of all consequence $\mathcal{L}$-pairs valid in all dynamic models. The subset of PDL^+ comprising only consequence pairs without occurrences of modal operators is the Distributive Lattice Logic, DLL.

The binary implicational proof system (the terminology derives from [13]) $\mathcal{PDL}^+$ contains the following axioms and inference rules ($X \dashv\vdash Y$ is shorthand for '$X \vdash Y$ and $Y \vdash X$'):

Axioms

(1) $X \vdash X$

(2) $X \wedge Y \vdash X$ and $X \wedge Y \vdash Y$

(3) $X \vdash X \vee Y$ and $Y \vdash X \vee Y$

(4) $X \wedge (Y \vee Z) \vdash (X \wedge Y) \vee (X \wedge Z)$

(5) $[A]X \wedge [A]Y \vdash [A](X \wedge Y)$ and $\langle A\rangle(X \vee Y) \vdash \langle A\rangle X \vee \langle A\rangle Y$

(6) $[A](X \vee Y) \vdash [A]X \vee \langle A\rangle Y$ and $\langle A\rangle X \wedge [A]Y \vdash \langle A\rangle(X \wedge Y)$

(7) $[A;B]X \dashv\vdash [A][B]X$ and $\langle A;B\rangle X \dashv\vdash \langle A\rangle\langle B\rangle X$

(8) $[A \cup B]X \dashv\vdash [A]X \wedge [B]X$ and $\langle A \cup B\rangle X \dashv\vdash \langle A\rangle X \vee \langle B\rangle X$

(9) $[A^*]X \dashv\vdash X \wedge [A][A^*]X$ and $X \vee \langle A\rangle\langle A^*\rangle X \dashv\vdash \langle A^*\rangle X$

(10) $[Y?]X \wedge Y \vdash X$ and $Y \wedge X \vdash \langle Y?\rangle X$

(11) $[X?]X \vdash [A][X?]X$

(12) $Z \vdash X \vee [X?]Y$

Rules

(1) $\dfrac{X \vdash Y \quad Y \vdash Z}{X \vdash Z}$

(2) $\dfrac{X \vdash Y \quad X \vdash Z}{X \vdash Y \wedge Z}$ and $\dfrac{Y \vdash X \quad Z \vdash X}{Y \vee Z \vdash X}$

(3) $\dfrac{X \vdash Y}{[A]X \vdash [A]Y}$ and $\dfrac{X \vdash Y}{\langle A\rangle X \vdash \langle A\rangle Y}$

(4) $\dfrac{X \vdash [A]X}{X \vdash [A^*]X}$ and $\dfrac{\langle A\rangle X \vdash X}{\langle A^*\rangle X \vdash X}$

(5) $\dfrac{X \wedge Y \vdash Z}{X \vdash [Y?]Z}$

The notion of a *provable* consequence pair (or theorem; notation: $X \vdash_{\mathcal{PDL}^+} Y$) is defined as usual.

Dunn's positive modal logic studied in [11] corresponds to a fragment of PDL^+ where the set of actions is $\{a\}$ for some atomic action a, i.e. $[a]$ and $\langle a\rangle$ are the only modal operators in the language (Dunn's proof system is obtained from $\mathcal{PDL}^+$ by means of the obvious omissions). The fragment of $\mathcal{PDL}^+$ without the modal axioms and rules is denoted $\mathcal{DLL}$.

Theorem 2.1 (Soundness)**.** *Every consequence pair provable in $\mathcal{PDL}^+$ is in PDL^+.*

Proof. Most of the cases are straightforward. We show only the fact that the rule $(4_\Diamond)$ preserves validity in each model (the rule is essential in the proof of the Filtration Theorem below). Assume that $\langle A\rangle X \vdash_M X$ and $w \vDash \langle A^*\rangle X$. The latter means that there is a sequence of states $w_1, \ldots, w_n$ such that $w_1 = w$, $w_m[\![A]\!]w_{m+1}$ for each $m < n$ and $w_n \vDash X$. We prove that $w_m \vDash X$ for each $m < n$ as well. It is clear that $w_{n-1} \vDash \langle A\rangle X$, so by the first assumption, $w_{n-1} \vDash X$. We continue in a similar fashion until w_1. □

3 Prime theories and the canonical structure

A *prime* $\mathcal{PDL}^+$*-theory* is a set of formulas Γ such that i) $X \in \Gamma$ and $X \vdash_{\mathcal{PDL}^+} Y$ imply $Y \in \Gamma$; ii) $X, Y \in \Gamma$ implies $X \wedge Y \in \Gamma$; and iii) $X \vee Y \in \Gamma$ implies that $X \in \Gamma$ or $Y \in \Gamma$. We will use variables t, s etc. for prime theories. A prime theory t is *non-trivial* iff $t \neq \emptyset$ and if there is $X \notin t$. An *independent* $\mathcal{PDL}^+$*-pair* is an ordered pair of sets of formulas $\langle\Gamma, \Delta\rangle$ such that there are no finite $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ such that

$$\bigwedge \Gamma' \vdash_{\mathcal{PDL}^+} \bigvee \Delta'$$

Lemma 3.1 (Belnap's Lemma)**.** *If $\langle\Gamma, \Delta\rangle$ is an independent $\mathcal{PDL}^+$-pair, then there is a prime $\mathcal{PDL}^+$-theory $t \supseteq \Gamma$ disjoint from Δ.*

Proof. This holds thanks to the distributive lattice axioms and rules of $\mathcal{PDL}^+$; see [35, 92–95]. □

The *canonical structure* is $C = \langle P, [\![\,]\!]_C\rangle$ where P is the set of all non-empty prime theories and $[\![\,]\!]_C$ is a function such that

$[\![X]\!]_C = \{t \,;\, X \in t\}$ and

$[\![A]\!]_C$ is a binary relation on P such that $t[\![A]\!]_C t'$ iff

1. for all X, $[A]X \in t$ only if $X \in t'$ and
2. for all X, $X \in t'$ only if $\langle A\rangle X \in t$.

We use notation similar to the one used for dynamic models but, importantly, the canonical structure is *not* a dynamic model—although $[\![A]\!]_C^* \subseteq [\![A^*]\!]_C$, the converse inclusion cannot be established (this is a standard fact about PDL, see [17]). However, C is similar to dynamic models in most of the important aspects. For instance, it is easy to show that $[\![X \wedge Y]\!]_C = [\![X]\!]_C \cap [\![Y]\!]_C$ and $[\![X \vee Y]\!]_C = [\![X]\!]_C \cup [\![Y]\!]_C$. The following lemma implies that $[\![[A]X]\!]_C$ and $[\![\langle A\rangle X]\!]_C$ satisfy the standard conditions as well.

Lemma 3.2 (Witness Lemma)**.**

(a) If $[A]X \notin t$, then $t[\![A]\!]_C s$ for some s such that $X \notin s$

(b) If $\langle A\rangle X \in t$, then $t[\![A]\!]_C s$ for some s such that $X \in s$

Proof. Similar to Dunn's proof of Lemma 5.1 in [11]; axiom $[X?]X \vdash [A][X?]X$ is necessary in (a) to show that s is non-empty. □

It is also easy to show that $[\![A \cup B]\!]_C$ ($[\![A;B]\!]_C$) is the union (composition) of $[\![A]\!]_C$ and $[\![B]\!]_C$ and that $[\![X?]\!]_C$ is the identity relation on $[\![X]\!]_C$. In what follows we use $t \vDash_C X$, $t \in [\![X]\!]_C$ and $X \in t$ (for prime theories t) interchangeably (and we often omit the subscript).

Lemma 3.3. *$[\![X]\!]_C \subseteq [\![Y]\!]_C$ only if $X \vdash_{\mathcal{PDL}^+} Y$.*

Proof. This follows from Belnap's Lemma. □

4 Filtration and completeness

$F \subseteq \mathsf{Form}$ is *Fisher–Ladner closed* iff it is closed under subformulas and

- if $[X?]Y \in F$, then $X \in F$
- if $[A \cup B]X \in F$, then $[A]X \in F$ and $[B]X \in F$
- if $[A;B]X \in F$, then $[A][B]X \in F$
- if $[A^*]X \in F$, then $[A][A^*]X \in F$
- variants of the above with $\langle\,\rangle$ instead of $[\,]$

The *Fisher–Ladner closure* $FL(F)$ of F is the least Fisher–Ladner closed superset of F; the Fisher-Ladner closure of $X \vdash Y$ is $FL(\{X, Y\})$. It is clear that F is Fisher–Ladner closed iff $F = FL(F)$.

Fact 4.1. *If F is finite, then $FL(F)$ is finite. $FL(X \vdash Y)$ is bounded by the number of symbols occurring in X, Y (excluding parentheses).*

Proof. See [17], Lemma 6.3. □

Let $F = FL(F)$. We define an equivalence relation $\equiv^F$ on the set of non-empty prime theories as follows:

$$t \equiv^F t' \iff t \cap F = t' \cap F$$

Let t^F be the $\equiv^F$-equivalence class containing t.

The *filtration of C through F* is $M^F = \langle W^F, [\![\,]\!]^F \rangle$ where

- $W^F = \{t^F \,;\, t \in P\}$
- $[\![p]\!]^F = \{t^F \,;\, p \in t\}$ for $p \in F$; for $p \notin F$ we set $[\![p]\!]^F = \emptyset$
- $t^F [\![a]\!]^F s^F$ iff there are $t' \in t^F$ and $s' \in s^F$ such that $t'[\![a]\!]s'$

(Note that $[\![p]\!]^F$ is well-defined as $[\![p]\!]$ is closed under $\equiv^F$ for $p \in F$.) Relations $[\![A]\!]^F$ for complex A and sets $[\![X]\!]^F$ for complex X are defined as in dynamic models. Hence, M^F is a dynamic model by definition.

In what follows, we write $t[\![A]\!]^F s$ instead of $t^F [\![A]\!]^F s^F$ and $t \in [\![X]\!]^F$ or $t \vDash^F X$ instead of $t^F \in [\![X]\!]^F$. Sometimes we also write $\equiv$ instead of $\equiv^F$.

We now turn to the proof of the Filtration Theorem, saying that for all $X \in F$, X is true in a theory t within the canonical structure iff X is true in the F-equivalence class of t in the F-filtration of the canonical structure. Our proof follows the strategy of the standard proof for the full PDL [17] with one important exception—the standard proof uses the fact that a finite set of equivalence classes in the filtration can be defined by a formula (meaning, in the context of a filtration of the canonical structure, that the formula holds precisely in those t such that t^F belongs to the set) and the definition of such a formula uses Boolean negation. This approach is of course not available in the case of PDL^+. However, as shown in the following lemma, we can define complements of sets satisfying certain requirements even without Boolean negation in the language. The lemma, and the subsequent proof of the Filtration Theorem may be considered as the main technical novelties of the article.

Lemma 4.2 (Defining Formula Lemma)**.** *Take the canonical $\mathcal{PDL}^+$-structure and a finite set $F = FL(F)$. Assume that $D \subseteq P$*

- *is non-empty*
- *is closed under $\equiv^F$*
- $\neg\exists t \in D : F \subseteq t$

Then there is a formula $Out(D)$ such that $u \vDash Out(D)$ iff $u \notin D$.

Proof. Fix a $p \notin F$. Define for all $v \in D$

$$v^+ = \begin{cases} \bigwedge\{X \in F \,;\, v \vDash X\} & \text{if } F \cap v \neq \emptyset \\ [p?]p & \text{otherwise} \end{cases}$$

$$v^- = \bigvee\{X \in F \,;\, v \nvDash X\}$$

$$Out(D) = \bigwedge\{[v^+?]v^- \,;\, v \in D\}$$

Note that even though D is not necessarily finite, there are only finitely many formulas of the form $[v^+?]v^-$ for $v \in D$ since F is finite.

Claim 4.3. $u \vDash [v^+?]v^-$ iff $u \not\equiv v$.

Proof of the Claim. It is clear that $u \vDash [X?]Y$ iff $u \nvDash X$ or $u \vDash Y$ (If $u \nvDash X$, then $u \vDash [X?]Y$ by axiom $Z \vdash X \vee [X?]Y$ and the assumption that u is a non-empty prime theory). If $u \nvDash v^+$, then $u \not\equiv v$ (if v does not contain any formulas from F, then this implication is vacuously true); and similarly, if $u \vDash v^-$, then $u \not\equiv v$. This proves the left-to-right implication of the claim. Conversely, if $u \not\equiv v$, then either $v \cap F \not\subseteq u$ or $u \cap F \not\subseteq v$. In the first case $u \nvDash v^+$, in the second case $u \vDash v^-$. In both cases $u \vDash [v^+?]v^-$. *This proves the Claim.*

We can now complete the proof of the lemma. It is clear that $u \vDash Out(D)$ iff $u \not\equiv v$ for all $v \in D$ (Claim 4.3) iff $u \notin D$. (If $u \in D$ then $\exists v \in D : u \equiv v$ by reflexivity of $\equiv$; if $\exists v \in D : u \equiv v$ then $u \in D$ by closure of D under $\equiv$). □

It would be desirable to prove Lemma 4.2 without using the test modality (e.g. for the sake of applicability of the result to test-free fragments of the language), but we were unable to provide such a proof.

We now state and prove the Filtration Theorem for PDL^+. Because formulas are defined inductively using programs ($[A]X$ and $\langle A\rangle X$) and vice versa ($X?$), the proof is by induction on *subexpressions*. A subexpression of X is either a subformula of X or an action A if X has a subformula of the form $[A]Y$ or $\langle A\rangle Y$; a subexpression of A is either a subaction of A or a formula Y if A contains a subaction of the form $Y?$.

Theorem 4.4 (Filtration Theorem)**.** *Let F be a Fisher–Ladner closed set.*

(a) For all $X \in F$, $t \vDash X$ iff $t \vDash^F X$

(b) If $t[\![A]\!]s$, then $t[\![A]\!]^F s$

(c) For all $[A]X \in F$, if $t[\![A]\!]^F s$ and $t \vDash [A]X$, then $s \vDash X$

(d) For all $\langle A\rangle X \in F$, if $t[\![A]\!]^F s$ and $s \vDash X$, then $t \vDash \langle A\rangle X$

Proof. The main claim of the Filtration Theorem is (a). The other claims are lemmas that help to establish (a) for modal formulas. In proving each claim of the theorem for a particular complex X or A, the induction hypothesis is that all the claims (a) – (d) hold for all proper subexpressions of X and A.

Claim (a). If $X = p$, then the claim holds by definition. The cases for $\wedge, \vee$ are straightforward. Assume that $X = [A]Y$. If $t \nvDash [A]Y$, then $t[\![A]\!]_C s$ and $s \nvDash Y$ for some s (by the Witness Lemma 3.2). A is a subexpression of $[A]Y$, so we may use Claim (b) to infer that $t[\![A]\!]^F s$. Since $Y \in F$ by the definition of a Fisher–Ladner closed set, we may use the induction hypothesis to infer that $s \nvDash^F Y$. But this means that $t \nvDash^F [A]Y$. Conversely, if $t \nvDash^F [A]Y$, then $t[\![A]\!]^F s$ and $s \nvDash^F Y$ for some s. By IH, $s \nvDash Y$. Claim (c) entails that $t \nvDash [A]Y$. The claim in case $X = \langle A\rangle Y$ is established similarly (here Claim (d) is used).

Claim (b). In case of a the claim holds by definition. The cases for $A;B$ and $A \cup B$ are established by simple application of the IH. Now take the case where the program in question is in the form A^* and assume $t[\![A^*]\!]s$. We need to show that $s \in \{v \,;\, t\left([\![A]\!]^F\right)^* v\}$; call this set E.

Claim 4.5. E is closed under $[\![A]\!]$.

Proof of claim. If there is a sequence $t[\![A]\!]^F \cdots [\![A]\!]^F v$ and $v[\![A]\!]u$, then $v[\![A]\!]^F u$ by the induction hypothesis and so $t[\![A]\!]^F \cdots [\![A]\!]^F u$. This proves the claim.

Now we prove that $s \in E$. If $E = P$, then of course $s \in E$. Also, obviously, $t \in E$. Hence, we may assume that both E and its complement are non-empty. Furthermore, there is some $t \in P$ such that $F \subseteq t$ (By Belnap's Lemma, every set of formulas is included in a prime theory.) Call this set t_F. Now t_F is contained either in E or in its complement $\bar{E}$. Call these cases (E1) and (E2), respectively. Both E and $\bar{E}$ are clearly closed under $\equiv$. We reason as follows.

(E1) In this case $\bar{E}$ satisfies the assumptions of the Defining Formula Lemma 4.2, so there is $Out(\bar{E}) = In(E)$ 'defining' E in the sense that $[\![In(E)]\!]_C = E$.

By Claim 4.5 and Lemma 3.3, $In(E) \vdash [A]In(E)$ is provable in $\mathcal{PDL}^+$. Using rule ($4_\Box$) we conclude that $In(E) \vdash [A^*]In(E)$ is provable in $\mathcal{PDL}^+$. Now, since $t \in E$ we have $In(E) \in t$. Hence, $[A^*]In(E) \in t$ and, by the definition of $[\![A^*]\!]$ in the canonical structure, $In(E) \in s$. This means that $s \in E$.

(E2) In this case E satisfies the assumptions of the Defining Formula Lemma 4.2, so there is $Out(E)$ 'defining' $\bar{E}$ in the sense that $[\![Out(E)]\!]_C = \bar{E}$. By Claim 4.5 and Lemma 3.3, $\langle A\rangle Out(E) \vdash_{\mathcal{PDL}^+} Out(E)$. Using rule ($4_\Diamond$) we conclude that $\langle A^*\rangle Out(E) \vdash_{\mathcal{PDL}^+} Out(E)$. Now if $s \notin E$, then $Out(E) \in s$ and, by the definition of $[\![A^*]\!]$ in the canonical structure, $\langle A^*\rangle Out(E) \in t$. This means that $Out(E) \in t$, so $t \notin E$. Hence, we have a contradiction from which we conclude that $s \in E$. This concludes the proof for the case A^*.

To conclude the proof of Claim (b), assume that A is of the form $X?$. If $t[\![X?]\!]s$, then $t \equiv^F s$ and $X \in t$. Since X is a subexpression of $X?$, we may use Claim (a) to infer that $t \vDash^F X$. Hence, $t[\![X?]\!]^F s$.

Claims (c) and (d). The proof is virtually the same as in the case of full PDL, see [17]. □

Remark 4.6. The notion of filtration introduced above derives from the notion of *smallest filtration* in modal logic. It is easy to show that we also could have used the notion of *greatest filtration*, where everything is defined as before, with the exception of defining $t[\![a]\!]^F s$ as

- for all $[a]X \in F$, $[a]X \in t$ only if $X \in s$, and
- for all $\langle a\rangle X \in F$, $X \in s$ only if $\langle a\rangle X \in t$.

An inspection of the proof of the Filtration Theorem 4.4 (and the corresponding parts of [17]) shows that the argument works also if greatest filtration is used.

We will use these facts in Section 5.3 where we study a logic for which smallest filtration is not suitable.

Remark 4.7. If the falsum constant $\bot$ is added to the positive language, the proof of the Filtration Theorem becomes easier. The reason is that formulas $[X?]\bot$ can be used to define Boolean negation and, thus, the Filtration Theorem is established using the same argument as in the case of full PDL. However, one of our main goals in this article is to explore the situation where the standard approach is not available.

Theorem 4.8. *If $X \vdash Y$ is in PDL^+, then it is provable in $\mathcal{PDL}^+$.*

Proof. Standard argument. If $X \nvdash_{\mathcal{PDL}^+} Y$, then there is a non-empty prime theory t such that $X \in t$ and $Y \notin t$ by Belnap's Lemma 3.1. Now filter the canonical structure through $F = FL(X \vdash Y)$. By the Filtration Theorem, $t \vDash^F X$ and $t \nvDash^F Y$. But M^F is a dynamic model, so $X \vdash Y \notin PDL^+$. □

Theorem 4.9. *PDL^+ is a decidable set.*

Proof. The theorem follows from decidability of PDL, but we may argue also as follows. The size of M^F is bounded by the size of F (for the exact bound, see [17]). Hence, it is sufficient to check all the models of size within the bound given by the size of $FL(X \vdash Y)$ for a counterexample to $X \vdash Y$. □

5 Non-classical extensions I: Adding De Morgan negation

Extensions of DLL with non-Boolean negation often treat negation semantically as a negative modal operator in Kripke models. Our results concerning PDL^+ can be extended to PDLs over some of these logics straightforwardly.

We focus on one example here, namely, First Degree Entailment, the logic extending DLL with a De Morgan negation $\sim$. Firstly, we sketch the semantics and proof theory for the (non-modal) logic FDE (Sect. 5.1). Then we add the De Morgan negation of FDE to PDL^+ and we establish the corresponding completeness and decidability results (Sect. 5.2). Finally, we discuss a special case of a PDL over FDE suitable for reasoning about the dynamics of 'Belnapian databases' (Sect. 5.3) and establish completeness and decidability of the logic.

5.1 First Degree Entailment

A *De Morgan model* is $M = \langle W, \sim, [\![\,]\!] \rangle$ where $\sim : W \to W$ such that $w^{\sim\sim} = w$ ($\sim$ is a function of period two) and $[\![\,]\!]$ satisfies the conditions for $\wedge, \vee$ and

$$[\![\sim X]\!] = \{w \,;\, w^{\sim} \not\models X\}$$

Validity of consequence pairs in models is defined as before; FDE is the set of all consequence pairs valid in all De Morgan models.

The proof system $\mathcal{FDE}$ extends $\mathcal{DLL}$ with the following axioms and rule:

Axioms

(1) $X \vdash \sim\sim X$ and $\sim\sim X \vdash X$

(2) $\sim X \wedge \sim Y \vdash \sim(X \vee Y)$ and $\sim(X \wedge Y) \vdash \sim X \vee \sim Y$

Rule $\dfrac{X \vdash Y}{\sim Y \vdash \sim X}$

Theorem 5.1. *$X \vdash Y \in FDE$ iff $X \vdash Y$ is provable in $\mathcal{FDE}$.*

Proof. We show the part of the completeness proof establishing that the canonical $\mathcal{FDE}$-model is a De Morgan model. The canonical model is $M_C = \langle P, \sim_C, [\![]\!]_C \rangle$ where P is the set of all prime theories (note that we do not require $t \in P$ to be non-empty—the reason is that $t^{\sim_C}$ below cannot be shown to be non-empty without using propositional constants for truth and falsity); $[\![X]\!]_C = \{t \in P \,;\, X \in t\}$ and

$$t^{\sim_C} = \{X \,;\, \sim X \notin t\}$$

It is sufficient to demonstrate here that $t^{\sim_C}$ is a prime theory for all t and that $t = t^{\sim_C \sim_C}$. We omit the subscript 'C' in the rest of the proof.

(i) $t^{\sim}$ is closed under $\vdash_{\mathcal{FDE}}$. If $X \vdash Y$ is provable, then so is $\sim Y \vdash \sim X$. So if $Y \notin t^{\sim}$ then $\sim Y \in t$ and hence $\sim X \in t$. But then $X \notin t^{\sim}$.

(ii) $t^{\sim}$ is closed under conjunction introduction. If $X \wedge Y \notin t^{\sim}$, then $\sim(X \wedge Y) \in t$. But then $\sim X \vee \sim Y \in t$. This means that $X \notin t^{\sim}$ or $Y \notin t^{\sim}$.

(iii) $t^{\sim}$ is prime. If $X, Y \notin t^{\sim}$, then $\sim X \wedge \sim Y \in t$. Consequently, $\sim(X \vee Y) \in t$ and $X \vee Y \notin t^{\sim}$.

Finally, $t = t^{\sim\sim}$ since $X \in t^{\sim\sim}$ iff $\sim X \notin t^{\sim}$ iff $\sim\sim X \in t$ iff $X \in t$ (by the double negation axioms). □

5.2 PDL over FDE

The language $\mathcal{L}^{\sim}$ extends $\mathcal{L}$ with $\sim$. A *dynamic De Morgan model* is $M = \langle W, \sim, [\![\,]\!]\rangle$ where $\langle W, [\![\,]\!]\rangle$ is a dynamic model, $\sim$ is a function of period two and $[\![\sim X]\!] = \{w \; ; \; w^{\sim} \not\models X\}$. Validity of consequence pairs in models is defined as before; $PDL^{\sim}$ is the set of all consequence pairs valid in all dynamic De Morgan models. The proof system $\mathcal{PDL}^{\sim}$ is the union of $\mathcal{PDL}^{+}$ and $\mathcal{FDE}$.

In the context of $PDL^{\sim}$, we need to re-define the notion of a Fisher–Ladner closed set; we require that if $X \in F$ *and* $X \neq \sim Y$ (for all $Y \in \mathcal{L}^{\sim}$), then $\sim X \in F$.[4] It is clear that if F is finite, then so is $FL(F)$.

Lemma 5.2. *$t \equiv^{F} u$ implies $t^{\sim} \equiv^{F} u^{\sim}$.*

Proof. $X \in t^{\sim}$ iff $\sim X \notin t$. If $X \neq \sim Y$ (for all Y), then $\sim X \in F$ and we may infer that $\sim X \notin t$ iff $\sim X \notin u$ iff $X \in u^{\sim}$. If $X = \sim Y$ (for some Y), then $\sim X \notin t$ iff $\sim\sim Y \notin t$ iff $Y \notin t$ iff $Y \notin u$ iff $\sim\sim Y \notin u$ iff $\sim X \notin u$ iff $X \in u^{\sim}$. □

Theorem 5.3. *$X \vdash Y$ belongs to $PDL^{\sim}$ iff it is provable in $\mathcal{PDL}^{\sim}$.*

Proof. The canonical $\mathcal{PDL}^{\sim}$-structure is defined just as the canonical $\mathcal{PDL}^{+}$-structure where $\sim_C$ is defined as in the proof of Theorem 5.1, with the exception that $t \in P$ are *non-trivial* prime theories.[5]

Claim 5.4. If t is a non-trivial prime theory, then so is $t^{\sim_C} = \{X \; ; \; \sim X \notin t\}$.

Proof of the claim. It suffices to show that $t^{\sim_C}$ is non-empty and not identical to the set of all formulas. If $t^{\sim_C} = \emptyset$, then $\sim[p?]p \in t$. Applying the rule (5) of $\mathcal{PDL}^{+}$ and the theorem $\sim X \wedge p \vdash p$ we get the consequence that $\sim X \vdash [p?]p$ (for all X). But then, using the contraposition rule and the double negation elimination axiom, $X \in t$ for all X. This contradicts the assumption that t is non-trivial. If $t^{\sim_C}$ were the set of all formulas, then $\sim\sim[p?]p \notin t$. But then, using the double negation introduction axiom, $[p?]p \notin t$. This contradicts the assumption that $t \neq \emptyset$. *This proves the Claim.*

It remains to show that the filtration of the canonical structure is a dynamic De Morgan model such that $t \models X$ iff $t \models^{F} X$ for all $X \in F$. We define

$$(t^{F})^{\sim^{F}} := (t^{\sim})^{F}$$

[4] The reason is that otherwise Lemma 5.2 would fail. Note that we cannot simply require that F be closed under prefixing $\sim$ since this would make F infinite.

[5] For the proof of the Filtration Theorem to go through, we need to assume that every $t \in P$ in the canonical structure is non-empty. However, in order to be able to show that $t^{\sim_C}$ is non-empty, we need to assume that t is not the set of all formulas.

The filtration is well defined since $t \equiv^F u$ implies $t^{\sim^F} = u^{\sim^F}$ by Lemma 5.2. If $\sim X \in F$, then $t \models \sim X$ iff $t^{\sim} \not\models X$ (by the definition of $\sim_C$) iff $(t^{\sim})^F \not\models X$ (by the induction hypothesis) iff $t^{\sim^F} \not\models X$ iff $t \models^F \sim X$. Moreover:

$$(t^F)^{\sim^F \sim^F} = \left((t^{\sim})^F\right)^{\sim^F} = (t^{\sim\sim})^F = t^F$$

□

Theorem 5.5. *$PDL^{\sim}$ is decidable.*

5.3 Updating Belnapian databases

According to the informal interpretation of FDE discussed by Belnap [3, 4], states in De Morgan models correspond to *bodies of information* that might be incomplete or inconsistent. We may call such bodies of information *Belnapian databases*. A Belnapian database can be seen as an ordered pair d of sets of statements (represented by propositional atoms); the first set d^+ comprising statements that are considered true and the second set d^- comprising statements that are considered false. Importantly, $d^+ \cap d^-$ may be non-empty and $d^+ \cup d^-$ may not be Prop.

$PDL^{\sim}$ can be seen as a formalization of reasoning about structured actions that modify Belnapian databases. A natural example of such an action, discussed at length by Belnap himself, is addition of new information to the database.

Example 5.6 (Belnap [3, 4])**.** Take a database comprising information about the World Series winners in the 1970s, with no entry concerning 1971 yet. Elizabeth enters the information that the Pittsburgh Pirates won the Series in 1971 and that the Baltimore Orioles did not.[6] Let p stand for 'Pittsburgh Pirates won in 1971', q for 'Baltimore Orioles won in 1971' and r for 'Oakland Athletics won in 1971'. Let $\langle d_0^+, d_0^- \rangle$ be the database before Elizabeth's action. Her entry modifies the database as follows:

d_1^+			d_1^-		
d_0^+	$\cup$	$\{p\}$	d_0^-	$\cup$	$\{q\}$

Hence, p is considered true in database d_1, q is considered false and there is no information about r.

Suppose, however, that Sam, not knowing about the addition by Elizabeth, enters the information that Orioles won in 1971 (and the Pirates did not). The database now looks as follows:

d_2^+			d_2^-		
d_0^+	$\cup$	$\{p, q\}$	d_0^-	$\cup$	$\{p, q\}$

[6] We may see the latter either as being explicitly added by Elizabeth or as being automatically supplied by the database on the basis of the information about the Pirates. The distinction does not matter for the purposes of our example.

Hence, p is considered both true and false in d_2 as is q. Yet, there is still no information about r.

In this section we take a look at an extension of $PDL^{\sim}$ suitable for formalizing such additions.[7] (A related logic, with Boolean negation instead of $\sim$ is studied in [40].)

Let $\mathcal{L}^{\pm}$ be a variant of $\mathcal{L}^{\sim}$ where the set of atomic actions is replaced by

$$\{+p\ ;\ p \in \mathsf{Prop}\} \cup \{-p\ ;\ p \in \mathsf{Prop}\}$$

Intuitively, the action $+p$ represents adding the information that p is true to a database; the action $-p$ represents adding the information that p is false.

Belnap thought of $+p$ as a function such that $(+p)(d) = d_{+p}$ where $d^{+}_{+p} = d^{+} \cup \{p\}$ and $d^{-}_{+p} = d^{-}$ (and similarly for $-p$—$d^{-}_{-p} = d^{-} \cup \{p\}$ and $d^{+}_{-p} = d^{+}$), but we shall represent the action more generally as a relation. Interestingly enough, Belnap's definition of updates by complex formulas invokes some of the action operators present in $\mathcal{L}$; see [4, 22–24]. The *Belnapian fragment* of $\mathcal{L}^{\pm}$ is defined as follows:

$$P ::= p \mid {\sim}P \mid P \wedge P \mid P \vee P$$

(P, Q are used as metavariables ranging over formulas of the Belnapian fragment). Based on Belnap's discussion of complex updates, we define

- $+({\sim}P) := -P$ and $-({\sim}P) := +P$
- $+(P \wedge Q) := (+P); (+Q)$ and $-(P \wedge Q) := (-P) \cup (-Q)$
- $+(P \vee Q) := (+P) \cup (+Q)$ and $-(P \vee Q) := (-P); (-Q)$

We define $\pm P := (+P) \cup (-P)$.

A *dynamic Belnapian model* is a dynamic De Morgan model for $\mathcal{L}^{\pm}$ satisfying the following conditions:

(B1) $w[\![+p]\!]v \implies v \in [\![p]\!]$

(B2) $w[\![+p]\!]v[\![\pm q]\!]u \implies w[\![+p]\!]u$

(B3) $w[\![-p]\!]v \implies v^{\sim} \notin [\![p]\!]$

(B4) $w[\![-p]\!]v[\![\pm q]\!]u \implies w[\![-p]\!]u$

(for all $p, q \in \mathsf{Prop}$).

Example 5.7. A dynamic Belnapian model related to Example 5.6 is shown in Figure 1. The states displayed on the bottom of the picture are $w_0, \ldots, w_4$ (from left to right), the states displayed on the top are $u_0, \ldots, u_4$. The variable p is displayed on the left of the comma next to a state x iff $x \in [\![p]\!]$ and similarly

[7] Other examples are briefly discussed in [37] where axiomatization and decidability of an extension of $PDL^{\sim}$ with an implication connective and the falsum constant is established. This logic corresponds to a PDL-style extension of the Belnapian modal logic BK introduced by Odintsov and Wansing [32].

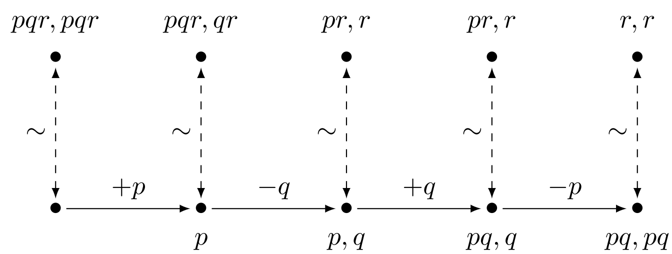

Figure 1: A dynamic Belnapian model.

for the other variables. For the sake of presentation, we display p on the right of the comma iff $\sim p$ holds in the given state and similarly for the other variables. (For the sake of simplicity, we state only the assumptions concerning p, q and r explicitly.)

Now w_1 results from adding p to w_0 and so, using (B1), p holds in w_1. Similarly, w_2 results from adding the information that q is false to w_1 and so, using (B3), q is considered false in w_2 (i.e. q is not considered true in $(w_2)^\sim = u_2$). The step from w_0 to w_1 was concerned with p and the step from w_1 to w_2 was concerning q; therefore, the information about p provided by w_2 should, by (B2) and (B1), be the same as the information provided by w_0. Therefore, p holds in w_2. Note also that, by the same reasoning, p holds in all the remaining states. Once information is added, it remains in the database.

By the definitions of the complex update arrows we have that, for example, w_1 is related via $+(\sim q)$ to w_2 and that w_0 is related via $+(p \wedge \sim q)$ to w_2. Hence, going back to Example 5.6, we may see w_2 informally as representing a database resulting from w_0 by adding the information that the Pirates won *and* that the Orioles *did not*.

Theorem 5.8. *In dynamic Belnapian models, variants of (B1) – (B4) with p, q replaced by P, Q hold as well.*

Proof. A simple proof by induction which is omitted here. □

Conditions (B1) – (B4) represent the minimal requirements necessary for $+p$ and $-p$ to express adding p to the positive or negative part of a database, respectively. If we add information that p is true to a database, then $p \in d^+$ of any resultant database d (B1); p still belongs to the positive part after any further additions (B1+B2). Similarly, if information that p is false is added, then $p \in d^-$ of any resultant d (B3); p still belongs to the negative part after any further additions (B3+B4). There are other conditions that might seem natural, even necessary, on this interpretation. We provide a minimal formalization here and leave a more comprehensive discussion of the topic for another occasion.

Let $PDL^\pm$ be the set of consequence $\mathcal{L}^\pm$-pairs valid in all dynamic Belnapian models. Let $\mathcal{PDL}^\pm$ be the extension of $\mathcal{PDL}^\sim$ by the following axioms:

(±1) $X \vdash [+P]P$ and $X \vdash [-P]{\sim}P$

(± 2) $[+P]X \vdash [+P][\pm Q]X$ and $[-P]X \vdash [-P][\pm Q]X$

(± 3) $\langle +P\rangle\langle \pm Q\rangle X \vdash \langle +P\rangle X$ and $\langle -P\rangle\langle \pm Q\rangle X \vdash \langle -P\rangle X$

(Note that we do not use the Rule of Substitution.)

The canonical $PDL^{\pm}$-structure is defined in the same way as the canonical $\mathcal{PDL}^{\sim}$-structure (for $\{+p, -p\ ;\ p \in \mathsf{Prop}\}$ as the set of atomic actions, of course).

Definition 5.9. We use the notion of a Fisher–Ladner closed set from Section 5.2, extended by the following clauses:

- If $p, q \in F$ and $[+p]X \in F$, then $[+p][\pm q]X \in F$
- If $p, q \in F$ and $\langle +p\rangle X \in F$, then $\langle +p\rangle\langle \pm q\rangle X \in F$
- Same clauses for $-p$ instead of $+p$

It is clear that $FL(F)$ is finite if F is.

Definition 5.10. The filtration of the canonical $\mathcal{PDL}^{\pm}$-structure through a Fisher–Ladner closed set F is $M^F = \langle W^F, \sim^F, [\![\ ]\!]^F\rangle$, where W^F and $\sim^F$ are defined as in Section 5.2 and $[\![\ ]\!]^F$ is defined as follows. For $p \in F$,

- $[\![p]\!]^F = \{t^F\ ;\ p \in t\}$
- $t[\![+p]\!]^F s$ iff (i) for all $[+p]X \in F$, $[+p]X \in t$ only if $X \in s$; and (ii) for all $\langle +p\rangle X \in F$, $X \in s$ only if $\langle +p\rangle X \in t$
- $t[\![-p]\!]^F s$ iff (i) for all $[-p]X \in F$, $[-p]X \in t$ only if $X \in s$; and (ii) for all $\langle -p\rangle X \in F$, $X \in s$ only if $\langle -p\rangle X \in t$

For $p \notin F$, $[\![p]\!]^F = [\![+p]\!]^F = [\![-p]\!]^F = \emptyset$.

This is the largest filtration of the canonical $\mathcal{PDL}^{\pm}$-structure; a variant of the Filtration Theorem 4.4 holds for this kind of filtration (see Remark 4.6). It is easily seen that the arguments in Section 5.2 remain sound if largest filtration is used.

Theorem 5.11. *M^F is a finite dynamic Belnapian model.*

Proof. (B1) If $t[\![+p]\!]^F s$, then $p \in F$. It is clear that $[+p]p \in t$ ($t \neq \emptyset$, use axiom (± 1)) and so, by the definition of $[\![+p]\!]^F$, $p \in s$.

(B2) If $t[\![+p]\!]^F u[\![\pm q]\!]^F s$, then $p, q \in F$. Assume that $[+p]X \in t$ for some $[+p]X \in F$; we prove $X \in s$. Using (± 2) we obtain $[+p][\pm q]X \in t$ and so $[\pm q]X \in u$ ($[+p][\pm q]X \in F$ by Def. 5.9). Since also $[\pm q]X \in F$, $X \in s$ by the definition of $[\![+q]\!]^F$ and $[\![-q]\!]^F$.

Next, assume that $\langle +p\rangle X \in F$ and $X \in s$. We have to show that $\langle +p\rangle X \in t$. By Def. 5.9, $\langle +p\rangle\langle \pm q\rangle X \in F$, so $\langle \pm q\rangle X \in u$ and $\langle +p\rangle\langle \pm q\rangle X \in t$. Using axiom ($\pm 3$), we get $\langle +p\rangle X \in t$. (B4) is established similarly.

(B3) If $t[\![-p]\!]^F s$, then $p \in F$ and $\sim p \in F$. But $[-p]\sim p \in t$ by axiom (± 1), so $\sim p \in t$. The rest is established as in Theorem 5.3. □

Theorem 5.12. *$X \vdash Y$ is provable in $\mathcal{PDL}^{\pm}$ iff it belongs to $PDL^{\pm}$.*

Proof. Soundness of $\mathcal{PDL}^{\pm}$ with respect to dynamic Belnapian models is established by a standard argument that is omitted here. If $X \vdash Y$ is not provable, then it is not valid in the canonical structure and so, by the Filtration Theorem and Theorem 5.11, $X \vdash Y$ is invalid in a dynamic Belnapian model. □

Theorem 5.13. *$PDL^{\pm}$ is decidable.*

6 Non-classical extensions II: Lambek PDL

Another non-classical extension of PDL^{+} to which our results on PDL^{+} apply is related to the Full Distributive Non-associative Lambek calculus, $DFNL$. In Sect. 6.1 we outline the semantics and proof theory of $DFNL$ and some of its extensions. Then we discuss combinations of PDL^{+} with $DFNL$ and its extensions (Sect. 6.2); in Sect. 6.3 we outline some of the informal interpretations of these logics.

6.1 DFNL and extensions

The Full Non-associative Lambek calculus FNL [20, 16, 7] is an extension of Lambek's non-associative NL [23] with lattice connectives $\wedge, \vee$.[8] The distributive version, *DFNL*, assumes that $\wedge, \vee$ distribute over each other.

The language of *DFNL* contains operators $\wedge, \vee, \backslash, /, \cdot$ and the set of propositional atoms Prop; formulas and consequence pairs are defined in the usual way.

A Lambek model (see [10] and [21, 22], for example) is $M = \langle W, R, [\![\ ]\!]_M \rangle$ where $W \neq \emptyset$, R is a ternary relation on W and $[\![\ ]\!]_M$ is a function from the set of formulas to subsets of W satisfying the usual conditions for $\wedge, \vee$ and, in addition,

$$[\![X{\cdot}Y]\!]_M = \{w \,;\, \exists u_1 u_2 : Ru_1u_2w \text{ and } u_1 \in [\![X]\!]_M \text{ and } u_2 \in [\![Y]\!]_M\}$$

$$[\![X\backslash Y]\!]_M = \{w \,;\, \forall u_1 u_2 : \text{if } Ru_1wu_2 \text{ and } u_1 \in [\![X]\!]_M, \text{ then } u_2 \in [\![Y]\!]_M\}$$

$$[\![Y/X]\!]_M = \{w \,;\, \forall u_1 u_2 : \text{if } Rwu_1u_2 \text{ and } u_1 \in [\![X]\!]_M, \text{ then } u_2 \in [\![Y]\!]_M\}$$

Validity of consequence pairs is defined as before; *DFNL* is the set of all consequence pairs valid in all Lambek models.

Informally, states in Lambek models represent *linguistic resources* and formulas represent *types* of these resources (see [21, 29, 30]). The literature mentions several specific kinds of linguistic resources, including structured expressions, signs or pieces of multidimensional linguistic information. The ternary R represents (non-deterministic) *merge* of resources (specific readings of this derive from the particular interpretation of states; examples include concatenation

[8] An extension of NL with $\wedge$ was already considered by Lambek in [23] but he did not investigate it in detail.

of expressions or pooling of linguistic information). In the most general setting R is arbitrary so, for example, merging u with v might not have the same result as merging v with u. A resource is of type $X{\cdot}Y$ iff it is a result of merging a resource of type X with a resource of type Y. We often write XY instead of $X{\cdot}Y$. A resource is of type $X\backslash Y$ iff whenever it is merged with a resource of type X, the result is of type Y. A resource is of type Y/X iff whenever a resource of type X is merged with it, the result is of type Y. (The distinction between these two cases is clearer when R is interpreted in terms of concatenation of expressions—$X\backslash Y$ is the type of expression that result in an expression of type Y when concatenated *from the left* with an expression of type Y; similarly for Y/X and concatenation *from the right*.) Consequence pairs $X \vdash Y$ say that each resource of type X is of type Y.

The proof system $\mathcal{DFNL}$ is obtained from $\mathcal{DLL}$ by adding (see [10]):

Axioms

(1) $X{\cdot}(X\backslash Y) \vdash Y$ and $(Y/X){\cdot}X \vdash Y$

(2) $Y \vdash X\backslash(X{\cdot}Y)$ and $Y \vdash (Y{\cdot}X)/X$

Rules

(1) $\dfrac{X_1 \vdash Y_1 \quad X_2 \vdash Y_2}{X_1{\cdot}X_2 \vdash Y_1{\cdot}Y_2}$

(2) $\dfrac{X_1 \vdash Y_1 \quad X_2 \vdash Y_2}{Y_1\backslash X_2 \vdash X_1\backslash Y_2}$

(3) $\dfrac{X_1 \vdash Y_1 \quad X_2 \vdash Y_2}{X_2/Y_1 \vdash Y_2/X_1}$

Theorem 6.1. *$X \vdash Y$ is in DFNL iff it is provable in $\mathcal{DFNL}$.*

Proof. A straightforward adaptation of the argument in [35, 253–7]. □

$DFNL_e$ is the set of consequence pairs valid in every *commutative* Lambek model, i.e. every model where

$$Ru_1u_2w \implies Ru_2u_1w \tag{e}$$

$\mathcal{DFNL}_e$ adds to $\mathcal{DFNL}$ the 'commutativity axiom' $XY \vdash YX$. In commutative models, merging u with v amounts to the same thing as merging v with u (so commutative models are usually not interpreted in terms of expressions and their concatenation). It is well known that \ and / turn out to be equivalent in $DFNL_e$; therefore only one connective $\to$ is used.

$DFNL_w$ is the set of consequence pairs valid in every *weakly contractive* Lambek model, i.e. every model where

$$Rwww \tag{w}$$

$\mathcal{DFNL}_w$ extends $\mathcal{DFNL}$ with the 'weak contraction axiom' $X \vdash XX$. The logic $DFNL_{ew}$ and the proof system $\mathcal{DFNL}_{ew}$ are defined as expected.

Theorem 6.2. *$X \vdash Y$ belongs to $DFNL_x$ iff it is provable in $\mathcal{DFNL}_x$ for all $x \in \{e, w, ew\}$.*

Proof. A straightforward adaptation of the arguments used in [35]. □

6.2 PDL over DFNL and extensions

The dynamic Lambek language $\mathcal{L}^{\backslash}$ adds $\backslash, \cdot, /$ to $\mathcal{L}$. A *dynamic Lambek model* is $M = \langle W, R, [\![\ ]\!]_M \rangle$ where $\langle W, [\![\ ]\!]_M \rangle$ is a dynamic model and R is a ternary relation on W. It is assumed that $[\![XY]\!]_M$, $[\![X\backslash Y]\!]_M$ and $[\![X/Y]\!]_M$ are defined as in Lambek models. $PDL^{\backslash}$ is the set of consequence $\mathcal{L}^{\backslash}$-pairs valid in all dynamic Lambek models. $\mathcal{PDL}^{\backslash}$ is the union of $\mathcal{PDL}^{+}$ and $\mathcal{DFNL}$.

Theorem 6.3. *$X \vdash Y$ belongs to $PDL^{\backslash}$ iff it is provable in $\mathcal{PDL}^{\backslash}$.*

Proof. The canonical structure is defined as the canonical $\mathcal{PDL}^{+}$-structure with $R_C t_1 t_2 s$ iff, for all $X_1 \in t_1$ and $X_2 \in t_2$, $X_1 {\cdot} X_2 \in s$ (we will write R instead of R_C). We need to show that the filtration of the canonical structure is a dynamic Lambek model such that $t \vDash X$ iff $t \vDash^F X$ for all $X \in F$ where the main connective is in $\{\backslash, \cdot, /\}$. We define

$$R^F tuv \iff \exists t'u'v' : t' \equiv^F t \text{ and } u' \equiv^F u \text{ and } v' \equiv^F v \text{ and } Rt'u'v'$$

Let $t \vDash XY$ for $XY \in F$, i.e. $\exists s_1, s_2$ such that Rs_1s_2t and $s_1 \vDash X$ and $s_2 \vDash Y$. By IH and the definition of R^F, $R^F s_1 s_2 t$, $s_1 \vDash^F X$ and $s_2 \vDash^F Y$. Hence, $t \vDash^F XY$. Conversely, if $t \vDash^F XY$, then $\exists s_1, s_2$ such that $R^F s_1 s_2 t$, $s_1 \vDash^F X$ and $s_2 \vDash^F Y$. Hence, $Rs'_1 s'_2 t'$ where $u' \in u^F$ for $u \in \{s_1, s_2, t\}$, $s'_1 \vDash X$ and $s'_2 \vDash Y$. Hence, $t' \vDash XY$. But $XY \in F$, so $t \vDash XY$. The cases for $\backslash$ and $/$ are established similarly. □

Theorem 6.4. *$PDL^{\backslash}$ is decidable.*

$\mathcal{L}_{\to}$ is obtained from $\mathcal{L}$ by replacing $\backslash$ and $/$ with a single binary $\to$. PDL_e is the set of all consequence $\mathcal{L}_{\to}$-pairs valid in all commutative dynamic Lambek models, i.e. dynamic Lambek models satisfying (e) where $[\![X \to Y]\!]$ is defined as $[\![X\backslash Y]\!]$. $\mathcal{PDL}_e$ is obtained from $\mathcal{PDL}^{\backslash}$ by 1) erasing the axioms and rules with $/$, 2) replacing $\backslash$ with $\to$ and 3) adding the commutativity axiom $XY \vdash YX$.

PDL_w is the set of consequence $\mathcal{L}$-pairs valid in every weakly contractive dynamic Lambek model, i.e. every dynamic Lambek model satisfying (w). $\mathcal{PDL}_w$ is obtained from $\mathcal{PDL}^{\backslash}$ by adding the weak contraction axiom $X \vdash XX$.

$\mathcal{PDL}_{ew}$ is the set of all consequence $\mathcal{L}_{\to}$-pairs valid in all commutative weakly contractive dynamic Lambek models. $\mathcal{PDL}_{ew}$ is obtained by adding the weak contraction axiom to $\mathcal{PDL}_e$.

Theorem 6.5. *$X \vdash Y$ is in PDL_x iff it is provable in $\mathcal{PDL}_x$, for all $x \in \{e, w, ew\}$.*

Proof. It is sufficient to show that the filtration of the canonical structure satisfies (e) and/or (w) in the corresponding cases. We show this for (e) only, the case of (w) is analogous.

First, the canonical structure itself satisfies (e) as can be shown easily using the fact that $XY \vdash YX$ is an axiom. If R^F is defined in the smallest-filtration way ($R^F tsu$ iff there are t', s', u' from the respective equivalence classes such that $Rt's'u'$), then (e) is clearly satisfied by R^F. □

Theorem 6.6. *$DFNL_x$ is decidable for all $x \in \{e, w, ew\}$.*

6.3 Interpretations of Lambek PDLs

In this section we outline three interpretations of dynamic Lambek models which point to potential applications of $PDL^{\setminus}$. We leave a more thorough discussion of these interpretations and applications for another occasion.

6.3.1 The linguistic interpretation

According to the linguistic interpretation, states in Lambek models are seen as structured expressions and $[\![A]\!]$ represent relations between these expressions. Formulas represent types of expressions (see Section 6.1). An expression is of type $[A]X$ iff it is in relation A only with expressions of type X; an expression is of type $\langle A \rangle X$ iff it is in relation A with some expressions of type X.

An example of a structure of this kind is provided by *term rewriting systems* [1], consisting of a set of 'terms' and a 'rewrite' relation $\longrightarrow$ between terms. In our setting, the rewrite relation corresponds to an arbitrary atomic action a. Many important properties of term rewriting systems are expressed using the reflexive transitive closure $\overset{*}{\longrightarrow}$ of the rewrite relation; in our setting a^*.

Hence, $PDL^{\setminus}$ can be seen as a formalism potentially useful in the context of structured multi-dimensional rewriting systems (with atomic actions representing basic rewrite relations and the composition and choice operations representing inner structure of complex rewrite relations).

6.3.2 The informational interpretation

According to the informational interpretation of Lambek models, states are seen as bodies of information (or information states) and R represents *merging* of information states; we may read $Ruvw$ as "merging u with v might result in information state w". A similar interpretation of relational Lambek models is the basis of several applications of substructural logics in epistemic logic; see [5, 36, 39] where versions of $DFNL_e$ are used. ([39] argues for the need to use *non-associative commutative* structures, but in that paper an operational version—where R is a binary operation—is used; [5, 36] use relational models.)

$PDL^{\setminus}$ and its extensions can be seen as formalisms for reasoning about structured modifications of information states, with actions representing types of such modifications.

6.3.3 The epistemic interpretation

Members of AAct are sometimes interpreted not as atomic actions, but as *agents*. It turns out that, on this interpretation, well-known *group-epistemic* operators such as group knowledge ("everyone in the group G knows that") or common knowledge are expressible by PDL-style complex actions. (For the sake of brevity, we assume familiarity with group-epistemic logic; the reader is referred to [14], for example).

- If AAct is seen as a set of agents, then subsets of AAct represent groups of agents; we may read $[a]X$ as "agent a has information that X".
- For finite $G \subseteq \mathsf{AAct}$, $[\bigcup_{a_i \in G} a_i]X$ means that everyone in the group G has information that X (we write $[E_G]X$).
- For finite $G \subseteq \mathsf{AAct}$, $[(\bigcup_{a_i \in G} a_i)^*]X$ means that X is common knowledge in group G (we write $[C_G]X$).

Similarly as on the informational interpretation, states in Lambek models can be seen as bodies of information. The set $\{v \; ; \; u[\![a]\!]v\}$ may be seen as the set of potential information states of agent a according to u (i.e. states that might be the information state of a for all that u says about a). On this interpretation, $u \vDash [a]X$ means that, according to u, a has information that X, and similarly for group and common knowledge. Accordingly, $u \vDash [a;b]X$ means that, according to u, a has information that b has information that X.

We may read R as an update relation on information states; $Rvuw$ means that updating u by v might result in w. Hence, $u \vDash X \backslash Y$ means that updating u with any state supporting X results in a state supporting Y. Consequently, $u \vDash [a](X \backslash Y)$ means that, according to u, if a's information state (whatever that might be) is updated with (a state supporting) X, then the resulting state of a will support Y. Similarly, $u \vDash [E_G](X \backslash [C_G]Y)$ means that, according to u, every agent in G has an information state such that if the state is updated by X, then the resulting state will support the information that Y is common knowledge in G (we may also say that updating with X will lead to Y *being* common knowledge—the state resulting from updating by X, for each agent in G, will support Y and $[E^n_G]Y$ for all $n \in \omega$).

The epistemic interpretation invites an extension of Lambek PDL with a non-Boolean negation, representing the fact that information states may support negative information (as opposed to not supporting positive information). A natural combination is a combination of $PDL^{\backslash}$ with $PDL^{\sim}$. It is not hard to see that a complete axiomatization for such a combination is obtained by pooling $\mathcal{PDL}^{\sim}$ and $\mathcal{PDL}^{\backslash}$; we leave a deeper study of this combination for another occasion. (For more about adding non-Boolean negation to Lambek calculus, see [43]; for categorial grammar with negative information, see [6]).

Adding a negation to Lambek PDL provides a deeper motivation for the epistemic interpretation—information states in Lambek models with De Morgan negation, for example, are potentially incomplete and inconsistent (yet non-trivial). Hence, we obtain a more realistic generalization of information states as modelled by classical epistemic logic. This is a basis for a group-epistemic logic with dynamic features (recall R) better suited for modelling dynamic epistemic scenarios involving inconsistent information.

7 Conclusion

The main contribution of this article was a modification of the standard completeness argument for PDL yielding an approach suitable for PDLs without Boolean negation in their language. In this manner, we obtained a complete axiomatization of the positive fragment of PDL and combinations of this fragment with some non-classical logics, namely, First Degree Entailment FDE and the Non-associative Lambek calculus NL. Some extensions of these logics were studied as well.

The study of other non-classical PDLs will be the topic of future research. For some such logics, additional modifications of the standard completeness argument will be required. Firstly, an argument not involving filtrations through a finite set of formulas will be required for PDLs based on logics without the finite model property (for instance, the relevant logic R). Secondly, PDLs based on logics requiring partially ordered relational models (e.g. some extensions of DLL with modal negation, some superintuitionistic logics or Hilbert-style presentations of many substructural logics) are a challenge if *both* box and diamond modalities (with the usual interpretation) are in the language. The reason is that in these models the partial order is assumed to interact with the accessibility relations and it is often unknown whether these interactions are preserved by (some kind of) filtration. There is also an interesting issue related to the interpretation of the test operator in partially ordered models. In such models, $[\![X]\!]$ needs to be an upper set for all X, but $[\![[Y?]X]\!]$ is not necessarily an upper set if $[\![Y?]\!]$ is defined in the standard manner as the identity relation on $[\![Y]\!]$. An alternative definition of $[\![Y?]\!]$ needs to be used,[9] but then the complex actions involving the test operator obtain a rather different reading. Instead of "Test whether Y holds in the present state", the action $Y?$ corresponds to "Move to any successor of the present state in which Y holds".

Acknowledgements

This work was supported by the Czech Science Foundation grant GJ18-19162Y for the project *Non-classical logical models of information dynamics*. The useful suggestions of three anonymous referees are gratefully acknowledged.

References

[1] F. Baader and T. Nipkow. *Term Rewriting and All That.* Cambridge University Press, 1998.

[9] Nishimura [31] assumes that $[\![Y?]\!] = \{\langle w, u\rangle \; ; \; w \leq u \models Y\}$.

[2] A. Baltag and L. S. Moss. Logics for epistemic programs. *Synthese*, 139(2):165–224, Mar 2004.

[3] N. Belnap. A useful four-valued logic. In J. M. Dunn and G. Epstein, editors, *Modern Uses of Multiple-Valued Logic*, pages 5–37. Springer Netherlands, Dordrecht, 1977.

[4] N. Belnap. How a computer should think. In G. Ryle, editor, *Contemporary Aspects of Philosophy*. Oriel Press Ltd., 1977.

[5] M. Bílková, O. Majer, and M. Peliš. Epistemic logics for sceptical agents. *Journal of Logic and Computation*, 26(6):1815–1841, 2016.

[6] W. Buszkowski. Categorial grammars with negative information. pages 107–126. de Gruyter Berlin, 1996.

[7] W. Buszkowski and M. Farulewski. Nonassociative Lambek Calculus with additives and context-free languages. In O. Grumberg, M. Kaminski, S. Katz, and S. Wintner, editors, *Languages: From Formal to Natural: Essays Dedicated to Nissim Francez on the Occasion of His 65th Birthday*, pages 45–58. Springer Berlin Heidelberg, Berlin, Heidelberg, 2009.

[8] L. Běhounek. Modeling costs of program runs in fuzzified propositional dynamic logic. In F. Hakl, editor, *Doktorandské dny '08*, pages 6 – 14, Prague, 2008. ICS AS CR and Matfyzpress.

[9] J. Degen and J. Werner. Towards intuitionistic dynamic logic. *Logic and Logical Philosophy*, 15(4):305–324, 2006.

[10] K. Došen. A brief survey of frames for the Lambek calculus. *Mathematical Logic Quarterly*, 38(1):179–187, 1992.

[11] J. M. Dunn. Positive modal logic. *Studia Logica*, 55(2):301–317, 1995.

[12] J. M. Dunn. A comparative study of various model-theoretic treatments of negation: A history of formal negation. In D. M. Gabbay and H. Wansing, editors, *What is Negation?*, pages 23–51. Springer, 1999.

[13] J. M. Dunn and G. M. Hardegree. *Algebraic Methods in Philosophical Logic*. Oxford University Press, Oxford, 2001.

[14] R. Fagin, J. Y. Halpern, Y. Moses, and M. Y. Vardi. *Reasoning About Knowledge*. MIT Press, 1995.

[15] M. J. Fischer and R. E. Ladner. Propositional dynamic logic of regular programs. *Journal of Computer and System Sciences*, 18:194–211, 1979.

[16] N. Galatos, P. Jipsen, T. Kowalski, and H. Ono. *Residuated Lattices: An Algebraic Glimpse at Substructural Logics*. Elsevier, 2007.

[17] D. Harel, D. Kozen, and J. Tiuryn. *Dynamic Logic*. MIT Press, 2000.

[18] J. Hughes, A. Esterline, and B. Kimiaghalam. Means-end relations and a measure of efficacy. *Journal of Logic, Language and Information*, 15(1):83–108, Jul 2006.

[19] G. Jäger and M. Marti. Intuitionistic common knowledge or belief. *Journal of Applied Loigc*, 18(C), 2016.

[20] M. Kanazawa. The Lambek calculus enriched with additional connectives. *Journal of Logic, Language, and Information*, 1(2):141–171, 1992.

[21] N. Kurtonina. *Frames and Labels. A Modal Analysis of Categorial Inference*. PhD Thesis, Utrecht University, 1994.

[22] N. Kurtonina. Categorial inference and modal logic. *Journal of Logic, Language, and Information*, 7:399–411, 1998.

[23] J. Lambek. On the calculus of syntactic types. In R. Jakobson, editor, *Structure of Language and Its Mathematical Aspects*, pages 166–178. AMS, Providence, 1961.

[24] D. Leivant. Proof theoretic methodology for propositional dynamic logic. In J. Díaz and I. Ramos, editors, *Formalization of Programming Concepts*, pages 356–373, Berlin, Heidelberg, 1981. Springer Berlin Heidelberg.

[25] C.-J. Liau. Many-valued dynamic logic for qualitative decision theory. In N. Zhong, A. Skowron, and S. Ohsuga, editors, *New Directions in Rough Sets, Data Mining, and Granular-Soft Computing*, pages 294–303, Berlin, Heidelberg, 1999. Springer Berlin Heidelberg.

[26] A. Madeira, R. Neves, and M. A. Martins. An exercise on the generation of many-valued dynamic logics. *Journal of Logical and Algebraic Methods in Programming*, 85(5, Part 2):1011–1037, 2016. Articles dedicated to Prof. J. N. Oliveira on the occasion of his 60th birthday.

[27] A. Madeira, R. Neves, M. A. Martins, and L. S. Barbosa. A dynamic logic for every season. In C. Braga and N. Martí-Oliet, editors, *Formal Methods: Foundations and Applications*, pages 130–145. Springer International Publishing, 2015.

[28] J.-J. C. Meyer. A different approach to deontic logic: deontic logic viewed as a variant of dynamic logic. *Notre Dame Journal of Formal Logic*, 29(1):109–136, 12 1987.

[29] M. J. Moortgat. Multimodal linguistic inference. *Journal of Logic, Language and Information*, 5(3):349–385, Oct 1996.

[30] R. Moot and C. Retoré. *The Logic of Categorial Grammars*. Springer, 2012.

[31] H. Nishimura. Semantical analysis of constructive PDL. *Publications of the Research Institute for Mathematical Sciences*, 18(2):847–858, 1982.

[32] S. Odintsov and H. Wansing. Modal logics with Belnapian truth values. *Journal of Applied Non-Classical Logics*, 20(3):279–301, 2010.

[33] F. Paoli. *Substructural logics: A Primer*. Kluwer, Dordrecht, 2002.

[34] V. Pratt. Semantical considerations on Floyd-Hoare logic. In *7th Annual Symposium on Foundations of Computer Science*, pages 109–121. IEEE Computing Society, 1976.

[35] G. Restall. *An Introduction to Substrucutral Logics*. Routledge, London, 2000.

[36] I. Sedlár. Substructural epistemic logics. *Journal of Applied Non-Classical Logics*, 25(3):256–285, 2015.

[37] I. Sedlár. Propositional dynamic logic with Belnapian truth values. In *Advances in Modal Logic. Vol. 11*, London, 2016. College Publications.

[38] I. Sedlár. Non-classical PDL on the cheap. In P. Arazim and T. Lávička, editors, *The Logica Yearbook 2016*, pages 239–256, London, 2017. College Publications.

[39] S. Sequoiah-Grayson. Epistemic closure and commutative, nonassociative residuated structures. *Synthese*, 190(1):113–128, 2013.

[40] P. Spruit, R. Wieringa, and J.-J. Meyer. Axiomatization, declarative semantics and operational semantics of passive and active updates in logic databases. *Journal of Logic and Computation*, 5(1):27–70, 1995.

[41] B. Teheux. Propositional dynamic logic for searching games with errors. *Journal of Applied Logic*, 12(4):377–394, 2014.

[42] H. Wansing. On the negation of action types: Constructive concurrent PDL. In P. Hájek, L. Valdes-Villanueva, and D. Westerstøahl, editors, *Logic Methodology and Philosophy of Science. Proceedings of the Twelfth International Congress*, pages 207–225. King's College Publications, 2005.

[43] H. Wansing. A note on negation in categorial grammar. *Logic Journal of the IGPL*, 15(3):271–286, 2007.

[44] D. Wijesekera and A. Nerode. Tableaux for constructive concurrent dynamic logic. *Annals of Pure and Applied Logic*, 135(1):1 – 72, 2005.