Substructural logics with a reflexive transitive closure modality

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Abstract. Reflexive transitive closure modalities represent a number of important notions, such as common knowledge in a group of agents or non-deterministic iteration of actions. Normal modal logics with such modalities are well-explored but weaker logics are not. We add a reflexive transitive closure box modality to the modal non-associative commutative full Lambek calculus with a simple negation. Decidability and weak completeness of the resulting system are established and extensions of the results to stronger substructural logics are discussed. As a special case, we obtain decidability and weak completeness for intuitionistic modal logic with the reflexive transitive closure box.

Keywords: Substructural logics · Modal logic · Reflexive transitive closure · Intuitionistic modal logic

1 Introduction

Modalities semantically interpreted using a reflexive transitive closure of a modal accessibility relation model a number of important notions. For instance, they represent common knowledge in epistemic logic [4] and program iteration in dynamic logic [6].

Normal modal logics with such modalities are well-explored but weaker logics are not. This paper adds a reflexive transitive closure modality to a weak modal substructural logic, namely, the modal non-associative commutative full Lambek calculus with a simple negation. Decidability and weak completeness of the resulting system are established utilising the notion of filtration used by Bílková et al. [1]. Extensions of our theorems to stronger substructural logics are

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also discussed. It is shown that completeness and decidability proofs for intuitionistic logic with common knowlegde [7] follow as a corollary. Our results are expected to find applications in substructural epistemic logics [1, 10] extended with a common knowledge operator and non-classical versions of propositional dynamic logic PDL [11].

The paper is organised as follows. Section 2 introduces our basic modal substructural Lambek calculus and Section 3 adds to it a reflexive transitive closure modality. Decidability and completeness of the resulting system are established in Section 4. Extensions of the results to stronger substructural logics are briefly discussed in Section 5.

2 A modal Lambek calculus

Our basic logic is a modal version of the non-associative commutative full Lambek calculus **DFNLe** [2] extended with a simple negation.[1] This logic is chosen for the sake of syntactic simplicity (one implication and one negation), but also because it is often taken as basic in the literature on substructural epistemic logics [1, 10, 12].

Our results can be established for a non-commutative background logic with a pair of negations as well. As noted in Section 5, however, non-associativity is an important prerequisite of the applicability of the present technique.

The language $\mathcal{L}$ contains a countable set of atomic formulas Var and the set of 0-ary connectives $\{t, \top, \bot\}$, unary connectives $\{\neg, \Box\}$ and binary connectives $\{\wedge, \vee, \rightarrow, \otimes\}$. The set of formulas $Frm(\mathcal{L})$ is defined in the usual manner. The variable p ranges over Var; α, β and φ, ψ, χ etc. range over $Frm(\mathcal{L})$.

Definition 1. *A* $\mathcal{L}$-model *is a tuple* $M = \langle P, \leq, L, C, S, R, [\![\cdot]\!]_M \rangle$ *such that* P *is a non-empty set,* $\leq$ *is a partial order on* P*,* L *is a (upwardly)* $\leq$*-closed subset of* P *(let the set of such subsets be denoted* $\mathrm{Up}(P)$*),* C *and* S *are binary relations on* P*,* R *is a ternary relation on* P *and* $[\![\cdot]\!]_M$ *is a mapping from* $Frm(\mathcal{L})$ *to* 2^P*. It is required that every model satisfies the following conditions:*

$$x' \leq x \implies (Cxy \implies Cx'y) \tag{1}$$
$$x' \leq x \implies (Sxy \implies Sx'y) \tag{2}$$
$$x' \leq x \implies (Rxyz \implies Rx'yz) \tag{3}$$
$$x \leq y \iff (\exists z)(z \in L \ \& \ Rzxy) \tag{4}$$
$$Rxyz \implies Ryxz \tag{5}$$
$$Cxy \implies Cyx \tag{6}$$

[1] Due to space limitations, we do not provide an introduction to substructural logics and their relational semantics. See [9], for example.

Moreover, the truth-set mapping $[\![\cdot]\!]_M$ *(mapping each formula to the set of states in which the formula is true) is required to satisfy the following conditions:*

$$[\![\top]\!]_M = P,\ [\![\bot]\!]_M = \emptyset \text{ and } [\![t]\!]_M = L \tag{7}$$

$$[\![\varphi \wedge \psi]\!]_M = [\![\varphi]\!]_M \cap [\![\psi]\!]_M \text{ and } [\![\varphi \vee \psi]\!]_M = [\![\varphi]\!]_M \cup [\![\psi]\!]_M \tag{8}$$

$$[\![\neg\varphi]\!]_M = \{x \mid (\forall y)(Cxy \implies y \notin [\![\varphi]\!]_M)\} \tag{9}$$

$$[\![\Box\varphi]\!]_M = \{x \mid (\forall y)(Sxy \implies y \in [\![\varphi]\!]_M)\} \tag{10}$$

$$[\![\varphi \to \psi]\!]_M = \{x \mid (\forall yz)(Rxyz \ \&\ y \in [\![\varphi]\!]_M \implies z \in [\![\psi]\!]_M)\} \tag{11}$$

$$[\![\varphi \otimes \psi]\!]_M = \{x \mid (\exists yz)(Ryzx \ \&\ y \in [\![\varphi]\!]_M \ \&\ z \in [\![\psi]\!]_M)\} \tag{12}$$

A formula φ *is* valid in M *(*$M \Vdash \varphi$*) iff* $L \subseteq [\![\varphi]\!]_M$*;* φ *is* $\mathcal{L}$-valid *(*$\mathcal{L} \Vdash \varphi$*) iff* $M \Vdash \varphi$ *for all* $\mathcal{L}$*-models* M*. A formula* φ entails ψ in M *(*$\varphi \Vdash_M \psi$*) iff* $[\![\varphi]\!]_M \subseteq [\![\psi]\!]_M$*.*

The frame conditions (1)–(3) entail that $[\![\cdot]\!]_M$ is a mapping from *Frm* to $\mathrm{Up}(P)$. This, together with condition (4) implies that $M \Vdash \varphi \to \psi$ iff $\varphi \Vdash_M \psi$.

We do not have space to provide a full informal interpretation of the semantics,[2] but it will perhaps be helpful to think of $x \in P$ as "bodies of information" in some general sense and $\leq$ as "informational containment" ($x \leq y$ means that every piece of information supported by x is supported by y). We can then think of L a set of "logical" bodies of information (i.e. those that support logically valid formulas), C as a relation of compatibility and R as a relation associated with combining bodies of information ($Rxyz$ means, roughly, that the result of combining x and y is at least as strong as z).

Theorem 1. $\mathcal{L} \Vdash \varphi$ *iff* φ *is a theorem of the axiom system* H*, consisting of axioms:*

- $\varphi \to \varphi$
- $\varphi \wedge \psi \to \varphi$ *and* $\varphi \wedge \psi \to \psi$
- $\varphi \to \varphi \vee \psi$ *and* $\psi \to \varphi \vee \psi$
- $\varphi \to \top$ *and* $\bot \to \varphi$
- $\varphi \wedge (\psi \vee \chi) \to (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$
- $\Box\varphi \wedge \Box\psi \to \Box(\varphi \wedge \psi)$
- $\top \to \Box\top$

and inference rules ('//' indicates a two-way rule):

- $\varphi, \varphi \to \psi \,/\, \psi$
- $\varphi \to \psi, \psi \to \chi \,/\, \varphi \to \chi$
- $\chi \to \varphi, \chi \to \psi \,/\, \chi \to (\varphi \wedge \psi)$
- $\varphi \to \chi, \psi \to \chi \,/\, (\varphi \vee \psi) \to \chi$
- $\varphi \to \psi \,/\, \Box\varphi \to \Box\psi$
- $\varphi \to (\psi \to \chi) \,//\, (\psi \otimes \varphi) \to \chi$
- $\varphi \to (\psi \to \chi) \,//\, \psi \to (\varphi \to \chi)$
- $t \to \varphi \,//\, \varphi$
- $\varphi \to \neg\psi \,//\, \psi \to \neg\varphi$

Proof. This is a standard result [1, 5].

[2] See [8], for example.

Example 1. It is easily seen that the modal $\Box$ distributes over $\wedge, \vee$ in the expected way; both

$$\Box(\varphi \wedge \psi) \leftrightarrow (\Box\varphi \wedge \Box\psi) \text{ and } \Box\varphi \vee \Box\psi \rightarrow \Box(\varphi \vee \psi)$$

are valid in every M. However, the "K-axiom"

$$\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$$

is not valid in every M (observe that the set $\{\varphi \mid x \in [\![\varphi]\!]_M\}$ is closed under Modus Ponens only if $Rxxx$). For a similar reason, $\Box$ does not distribute over $\otimes$, i.e.

$$\Box(\varphi \otimes \psi) \rightarrow (\Box\varphi \otimes \Box\psi)$$

is not valid in every M.[3]

3 Adding a reflexive transitive closure modality

The language $\mathcal{L}^*$ extends $\mathcal{L}$ with a unary connective $\Box^*$. The set of formulas $Frm(\mathcal{L}^*)$ of $\mathcal{L}^*$ is defined in the usual manner and all the syntactic metavariables are now taken to range over $Frm(\mathcal{L}^*)$. Γ, Δ etc. range over subsets of $Frm(\mathcal{L}^*)$. Let $\Gamma/U = \{\varphi \mid U\varphi \in \Gamma\}$ for all $U \in \{\neg, \Box, \Box^*\}$.

Definition 2. *A* $\mathcal{L}^*$-model *is* $M = \langle P, \leq, L, C, S, S^*, R, [\![\cdot]\!]_M\rangle$ *where everything is as in Definition 1 and, in addition,*

$$S^* \text{ is the reflexive transitive closure of } S \tag{13}$$

$$[\![\Box^*\varphi]\!]_M = \{x \mid (\forall y)(S^*xy \implies y \in [\![\varphi]\!]_M)\} \tag{14}$$

In general, if $\Gamma^{\downarrow}$ *is closed under subformulas, then a* $\Gamma^{\downarrow}$-model *is a structure that satisfies all the conditions required for* $\mathcal{L}^*$*-models, but the truth-set conditions* (7) – (12) *and* (14) *are required to hold only for* $\Gamma^{\downarrow}$ *as the range of* $[\![\cdot]\!]$.

Lemma 1. *Let* $\varphi \in \Gamma^{\downarrow}$*. If there is a* $\Gamma^{\downarrow}$*-model* M *such that* $M \not\Vdash \varphi$*, then there is a* $\mathcal{L}^*$*-model* M' *such that* $M' \not\Vdash \varphi$*. If* M *is finite then so is* M'*.*

We note that $[\![\Box^*\varphi]\!]_M \in \mathrm{Up}(P)$, so $\varphi \Vdash_M \psi$ iff $M \Vdash \varphi \rightarrow \psi$ for all $\Gamma^{\downarrow}$-models M where $\varphi \rightarrow \psi \in \Gamma^{\downarrow}$.

One immediate consequence of the truth condition for $\Box^*\varphi$ is that our logic is *not compact.* To see this, observe that every finite subset of $\{\neg\Box^*p\} \cup \{p\} \cup \{\Box^n p \mid n \in \omega\}$ is satisfiable, but the set itself is not.

[3] Stated more precisely, counterexamples to the K-axiom can be constructed if the frame property $Syx \implies Rxxx$ fails. Similarly, counterexamples to distributivity of $\Box$ over $\otimes$ can be found if we have Swx and $Ryzx$ but also $Ry'z'w$ and $Sy'u$ with $y \not\leq u$ for some u. Counterexamples to the converse implication can be found if a similar frame condition holds.

4 Axiomatization and decidability

Our main result is a weakly complete axiomatization of the set of $\mathcal{L}^*$-valid formulas and a proof that this set is decidable. We use a generalisation of the standard filtration technique [4, 6]. In particular, we build on the notion of filtration used in [1].

Definition 3. *Let H^* be the axiom system obtained from H by adding the axiom and rule schemas shown in Fig. 1 (called* stars*).*

$(*1)\ \Box^*\varphi \wedge \Box^*\psi \to \Box^*(\varphi \wedge \psi)$
$(*2)\ \top \to \Box^*\top$
$(*3)\ \Box^*\varphi \leftrightarrow (\varphi \wedge \Box\Box^*\varphi)$
$(*4)\ \varphi \to \psi \,/\, \Box^*\varphi \to \Box^*\psi$
$(*5)\ \varphi \to \Box\varphi \,/\, \varphi \to \Box^*\varphi$

Fig. 1. The stars.

We note that axiomatizations of normal modal logics with a reflexive transitive closure modality usually contain *induction axiom*

$$(\varphi \wedge \Box^*(\varphi \to \Box\varphi)) \to \Box^*\varphi$$

inhstead of the *loop invariance rule* $(*5)$ (see [6], foe example). These two are equivalent in the classical setting (in the sense that the rule preserves validity iff the axiom is valid), but not so in our framework. In fact, it can be shown that the induction axiom is not valid in every model. The reason is closely related to the failure of the K-axiom pointed out in Example 1.

We write $\vdash \varphi$ if φ is a theorem of H^*, $\varphi \vdash \psi$ if $\vdash \varphi \to \psi$ and $\Gamma \vdash \Delta$ if there are finite $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ such that $\bigwedge \Gamma' \vdash \bigvee \Delta'$.

A set of formulas Γ is a *proper prime theory* iff $\Gamma \neq Frm(\mathcal{L}^*)$ and $\Gamma \vdash \varphi \vee \psi$ only if $\varphi \in \Gamma$ or $\psi \in \Gamma$. Note that, for all proper prime theories Γ, $\Gamma \vdash \varphi$ only if $\varphi \in \Gamma$ since $\Gamma \vdash \varphi$ implies $\Gamma \vdash \varphi \vee \varphi$. Note also that if $\Delta \vdash \varphi$, then $\Delta \subseteq \Gamma$ only if $\varphi \in \Gamma$.

Theorem 2. *If $\Gamma \nvdash \Delta$, then there is a proper prime theory $\Gamma' \supseteq \Gamma$ disjoint from Δ.*

Proof. This is established by using a variant of the Pair Extension Theorem [9, p. 94].

Definition 4. *The* canonical structure

$$M_c = \langle P_c, \leq_c, L_c, S_c, S^*_c, C_c, R_c, [\![\cdot]\!]_c \rangle$$

is defined as follows:

– P_c *is the set of all proper prime theories;*
– $\leq_c$ *is set inclusion;*
– $L_c = \{\Gamma \mid t \in \Gamma\}$;
– $S_c = \{\langle \Gamma, \Delta \rangle \mid \Gamma/\Box \subseteq \Delta\}$;
– $S_c^* = \{\langle \Gamma, \Delta \rangle \mid \Gamma/\Box^* \subseteq \Delta\}$;
– $C_c = \{\langle \Gamma, \Delta \rangle \mid \Gamma/\neg \cap \Delta = \emptyset\}$;
– $R_c = \{\langle \Gamma, \Delta, \Theta \rangle \mid (\forall \varphi \psi)(\varphi \to \psi \in \Gamma \,\&\, \varphi \in \Delta \implies \psi \in \Theta)\}$;
– $[\![\varphi]\!]_c = \{\Gamma \mid \varphi \in \Gamma\}$.

It is a standard observation that the canonical structure is not a $\mathcal{L}^*$-model, for it fails to meet condition (13). In general, S_c^* contains the reflexive transitive closure $(S_c)^*$ of S_c, but it is not identical to it.[4] Nevertheless, the conditions (1) – (12) and (14) are met [1, 9]. (For instance, let us check condition (14). The left-to-right inclusion is trivial. The right-to-left inclusion is established by a variant of the Witness Lemma [9, p. 255]. If $\Box^* \varphi \notin \Gamma$, then $\Gamma/\Box^* \nvdash \varphi$. Hence, by the Pair Extension Theorem, there is $\Delta \in P_c$ such that $S_c^* \Gamma \Delta$ and $\Delta \notin [\![\varphi]\!]_c$.)

Definition 5. *The* closure *of* φ, $cl(\varphi)$, *is the smallest set of formulas such that*

– $\{\varphi, t\} \subseteq cl(\varphi)$;
– $\psi \in cl(\varphi)$ *for all subformulas* ψ *of* φ;
– *if* $\Box^* \psi \in cl(\varphi)$, *then* $\Box\Box^* \psi \in cl(\varphi)$.

For every φ, *let* $\Gamma \preceq_\varphi \Delta$ *iff* $(\Gamma \cap cl(\varphi)) \subseteq \Delta$ *and* $\Gamma \sim_\varphi \Delta$ *iff* $\Gamma \preceq_\varphi \Delta$ *and* $\Delta \preceq_\varphi \Gamma$. *Moreover, let* $[\Gamma]_\varphi = \{\Delta \mid \Gamma \sim_\varphi \Delta\}$.

It is plain that $cl(\varphi)$ is finite for all φ.

Definition 6. *Fix a formula* φ. *The* φ-filtration *of the canonical structure is a structure* $M_\varphi = \langle P_\varphi, \leq_\varphi, L_\varphi, S_\varphi, S_\varphi^*, C_\varphi, R_\varphi, [\![\cdot]\!]_\varphi \rangle$ *defined as follows:*

– $P_\varphi = \{[\Gamma]_\varphi \mid \Gamma \in P_c\}$;
– $[\Gamma]_\varphi \leq_\varphi [\Delta]_\varphi$ *iff* $\Gamma \preceq_\varphi \Delta$;
– $L_\varphi = \{[\Gamma]_\varphi \mid (\exists \Gamma' \preceq_\varphi \Gamma)(\Gamma' \in L_c)\}$;
– $S_\varphi = \{\langle [\Gamma]_\varphi, [\Delta]_\varphi \rangle \mid (\exists \Gamma' \succeq_\varphi \Gamma)(S_c \Gamma' \Delta)\}$;
– $S_\varphi^* = (S_\varphi)^*$;
– $C_\varphi = \{\langle [\Gamma]_\varphi, [\Delta]_\varphi \rangle \mid (\exists \Gamma' \succeq_\varphi \Gamma, \exists \Delta' \succeq_\varphi \Delta)(C_c \Gamma' \Delta')\}$;
– $R_\varphi = \{\langle [\Gamma]_\varphi, [\Delta]_\varphi, [\Theta]_\varphi \rangle \mid (\exists \Gamma' \succeq_\varphi \Gamma, \exists \Delta' \succeq_\varphi \Delta)(R_c \Gamma' \Delta' \Theta)\}$;
– *for all* $\alpha \in cl(\varphi)$, $[\![\alpha]\!]_\varphi = \{[\Gamma]_\varphi \mid \alpha \in \Gamma\}$; *for* $\alpha \notin cl(\varphi)$, $[\![\alpha]\!]_\varphi = \emptyset$.

The crucial difference between the canonical structure and its filtration (in addition to the fact that the latter is finite) is the fact that, in a φ-filtration, S_φ^* is *defined* to be the reflexive transitive closure of S_φ. However, one needs to check that the φ-filtration of the canonical structure is a $cl(\varphi)$-model. In what follows, we drop the subscript 'φ' whenever possible.

[4] The reason is that if $\Box^* \varphi \in \Gamma$, then $\varphi, \Box^n \varphi \in \Gamma$ for all $n \in \omega$ by $(*3)$, but the converse implication cannot be established (our axiomatization is finitary).

Theorem 3. *For all φ, the φ-filtration of the canonical structure is a $cl(\varphi)$-model.*

Proof. The relation $\leq_\varphi$ is obviously a partial order on P_φ. The fact that C_φ, R_φ and L_φ satisfy the conditions (1), (6), (3), (5) and (4), respectively, and that L_φ is closed under $\leq_\varphi$ are established similarly as in [1]. (13) holds by definition and (2) is established as follows. If $S[\Gamma][\Delta]$ then $S_c\Gamma'\Delta$ for some $\Gamma' \succeq \Gamma$. But if $[\Theta] \leq [\Gamma]$ then $\Theta \preceq \Gamma$ and, consequently, $\Theta \preceq \Gamma'$. Hence, $S[\Theta][\Delta]$.

It remains to be shown that $[\![\cdot]\!]_\varphi$ satisfies the conditions (7) – (12) and (14) when applied to $\psi \in cl(\varphi)$. The cases where the main connective of ψ is in $\{\top, \bot, t, \neg, \wedge, \vee, \rightarrow, \otimes\}$ are established as in [1].

Next, assume that $\psi = \Box\alpha$. We have to show that

$$\Box\alpha \in \Gamma \iff (\forall[\Delta])(S[\Gamma][\Delta] \implies \alpha \in \Delta)$$

($\alpha \in \Delta$ means $[\Delta] \in [\![\alpha]\!]$) Assume first that $\Box\alpha \in \Gamma$ and $S[\Gamma][\Delta]$. It follows that $S_c\Gamma'\Delta$ for some $\Gamma' \succeq \Gamma$. But then $\Box\alpha \in \Gamma'$ and, by the definition of S_c, $\alpha \in \Delta$. Conversely, if $\Box\alpha \notin \Gamma$, then the Witness Lemma [9, p. 255] entails that there is Δ such that $S_c\Gamma\Delta$ and $\alpha \notin \Delta$. But it is plain that $S_c\Gamma\Delta$ only if $S[\Gamma][\Delta]$.

Finally, assume that $\psi = \Box^*\alpha$. We have to show that

$$\Box^*\alpha \in \Gamma \iff (\forall[\Delta])(S^*[\Gamma][\Delta] \implies \alpha \in \Delta)$$

If $\Box^*\alpha \notin \Gamma$ then, by a variation of the Witness Lemma, there is Δ such that $\alpha \notin \Delta$ and $S_c^*\Gamma\Delta$. It is sufficient to show that there is Θ such that $S^*[\Gamma][\Theta]$ and $\Theta \preceq \Delta$.

Let

$$E = \{\Phi \mid (\exists\Theta)(S^*[\Gamma][\Theta] \mathbin{\&} \Theta \preceq \Phi)\}$$

(E is closed under $\leq_c$, but $E' = \{\Phi \mid S^*[\Gamma][\Phi]\}$ is not. Recall that $(S_c)^* \subseteq S_c^*$, but not necessarily vice versa.) We have to show that $\Delta \in E$. For all $[\Phi] \in P$, define

$$\psi_{[\Phi]} = \bigwedge\{\alpha \in cl(\varphi) \mid \alpha \in \Phi\}$$

and

$$\psi_E = \bigvee\{\psi_{[\Phi]} \mid S^*[\Gamma][\Phi]\}.$$

Note that ψ_E is well-defined since P_φ is finite. We establish two claims.

Claim 1. E is closed under S_c, i.e., if $\Phi \in E$ and $S_c\Phi\Psi$, then $\Psi \in E$. If $\Phi \in E$, then there is Θ such that $S^*[\Gamma][\Theta]$ and $\Theta \preceq \Phi$. It follows from $S_c\Phi\Psi$ and $\Theta \preceq \Phi$ that $S[\Theta][\Psi]$. But S^* is the reflexive transitive closure of S, so it follows that $S^*[\Gamma][\Psi]$. Hence, $\Psi \in E$.

Claim 2. $E = [\![\psi_E]\!]_c$. First, assume that $\Phi \in E$, i.e., there is Ψ such that $S^*[\Gamma][\Psi]$ and $\Psi \preceq \Phi$. Now $\Psi \preceq \Phi$ implies $\psi_{[\Psi]} \in \Phi$ (proper prime theories are closed under forming conjunctions). But $\Psi \in E$, so $\psi_{[\Psi]} \vdash \psi_E$. Consequently, $\psi_E \in \Phi$, i.e., $\Phi \in [\![\psi_E]\!]_c$. Conversely, assume that $\psi_E \in \Phi$. Φ is a prime theory, so $\psi_{[\Theta]} \in \Phi$ for some Θ such that $S^*[\Gamma][\Theta]$. It follows that $\Theta \preceq \Phi$. Hence, $\Phi \in E$.

The two claims imply that $[\![\psi_E]\!]_c \subseteq [\![\Box\psi_E]\!]_c$. (If $\psi_E \in \Delta$, then $\Delta \in E$ by Claim 2. But then, if $S_c\Delta\Theta$ for some Θ, then $\Theta \in E$ by Claim 1. By Claim 2, if

$S_c \Delta\Theta$, then $\psi_E \in \Theta$. Hence, $\Box\psi_E \in \Delta$.) Consequently, $\psi_E \to \Box\psi_E \in \bigcap\{\Phi | \Phi \in L_c\}$. Now since $\varphi \to \Box\varphi \,/\, \varphi \to \Box^*\varphi$ is an inference rule of H^*, $\psi_E \to \Box^*\psi_E \in \bigcap\{\Phi | \Phi \in L_c\}$ and $[\![\psi_E]\!]_c \subseteq [\![\Box^*\psi_E]\!]_c$. Now we show that $S_c^* \Gamma\Delta$ implies that there is Θ such that $S^*[\Gamma][\Theta]$ and $\Theta \preceq \Delta$. It is plain that $\Gamma \in E$. By Claim 2, $\Gamma \in [\![\psi_E]\!]_c$ and, consequently, $\Gamma \in [\![\Box^*\psi_E]\!]_c$. Now $\Delta \in [\![\psi_E]\!]_c$ since $S_c^*\Gamma\Delta$. In other words, $\Delta \in E$. But this means that there is Θ such that $S^*[\Gamma][\Theta]$ and $\Theta \preceq \Delta$.

The final thing to show is that if $\Box^*\alpha \in \Gamma$ and $S^*[\Gamma][\Delta]$, then $\alpha \in \Delta$. Our assumption $S^*[\Gamma][\Delta]$ entails that either $[\Gamma] = [\Delta]$ or there is $n \geq 1$ such that $S[\Gamma][\Delta_1] \cdots [\Delta_n] = [\Delta]$. If $[\Gamma] = [\Delta]$, then $\Gamma \sim \Delta$ and $\Box^*\alpha \in \Delta$. Since $\vdash \Box^*\alpha \to \alpha$, $\alpha \in \Delta$ and we are done. Assume that there is $n \geq 1$ such that $S[\Gamma][\Delta_1] \cdots [\Delta_n] = [\Delta]$. We show by induction that for all $n \geq 1$, Δ_n contains α and $\Box^*\alpha$. $\Box^*\alpha \in \Gamma$ entails that $\Box\Box^*\alpha \in \Gamma$ (as $\vdash \Box^*\alpha \to \Box\Box^*\alpha$). By the clause $\psi = \Box\beta$ established above, $S[\Gamma][\Delta_1]$ entails that $\Box^*\alpha \in \Delta_1$, so $\alpha \in \Delta_1$ as well. Let us now assume that the claim holds for $k < n$. We show that it holds for $k+1$ as well. Assume that $\alpha, \Box^*\alpha \in \Delta_k$ and $S[\Delta_k][\Delta_{k+1}]$. Again, $\Box\Box^*\alpha \in \Delta_k$ and $\Box^*\alpha \in \Delta_{k+1}$ by the clause $\psi = \Box\beta$ and, consequently, $\alpha \in \Delta_{k+1}$.

Theorem 4. *If $\nvdash \varphi$, then there is a finite $\mathcal{L}^*$-model M such that $M \nVdash \varphi$.*

Proof. If $\nvdash \varphi$, then $t \nvdash \varphi$. By the Pair Extension Theorem [9], there is a proper prime theory $\Gamma \in L_c$ such that $\varphi \notin \Gamma$. The φ-filtration M_φ of the canonical structure is a finite $cl(\varphi)$-model by Theorem 3. Moreover, $[\Gamma] \in L_\varphi$ and $[\Gamma] \notin [\![\varphi]\!]_\varphi$. So, $M_\varphi \nVdash \varphi$. By Lemma 1, there is a finite $\mathcal{L}^*$-model M such that $M \nVdash \varphi$.

Theorem 5. *H^* is a sound and weakly complete axiomatisation of the set of formulas valid in every $\mathcal{L}^*$-model. This set is decidable.*

Proof. Soundness of H^* is left to the reader. Weak completeness follows from Theorem 4. Decidability follows from the fact that any φ-filtration of the canonical structure is finite (and, in fact, bounded by the size of φ).

5 Extensions

It is easily seen that our results can be extended to stronger substructural logics.

Theorem 6. *Let **L** be a substructural logic (in $\mathcal{L}$) axiomatised by $H(\mathbf{L})$ and characterised by a class of models $Mod(\mathbf{L})$. Assume that $M \in Mod(\mathbf{L})$ iff M satisfies a set of frame conditions $Con(\mathbf{L})$ such that the $H(\mathbf{L})$-canonical structure and the φ-filtration (for arbitrary φ) of the canonical structure both satisfy $Con(\mathbf{L})$. Then the extension of **L** by a reflexive transitive closure modality is decidable and axiomatised by $H(\mathbf{L})$ plus the stars.*

Proof. If it is assumed that the φ-filtration of the canonical structure satisfies all the relevant frame conditions, then the fact that $[\![\cdot]\!]_\varphi$ satisfies the conditions (7) – (12) and (14) when applied to $\psi \in cl(\varphi)$ is established exactly as in the proof of Theorem 3 above. But this means that the φ-filtration of the canonical structure is a finite $cl(\varphi)$-model. The rest of the argument is as before.

This observation also hints at potential limitations of the present technique. In general, if a frame condition is not preserved under forming filtrations (i.e. if the canonical structure satisfies the condition, then its φ-filtration for arbitrary φ does so as well) then the present technique cannot be applied to logics complete with respect to models satisfying the frame condition. For instance, the frame condition corresponding to associativity[5]

$$Rxyv \ \& \ Rvzw \implies (\exists u)(Rxuw \ \& \ Ryzu)$$

is not preserved by standard notions of filtration such as the one we have used in the present paper.

If one is interested in adding a reflexive transitive closure modality to modal intuitionistic logic (as in [7]), however, the problem with associativity can be avoided by interpreting $\to$ directly in terms of $\leq$:

$$[\![\varphi \to \psi]\!] = \{x \mid (\forall y)(x \leq y \implies (y \in [\![\varphi]\!] \implies y \in [\![\psi]\!]))\}$$

An inspection of our proof of Theorem 3 reveals that, after adding the stars to any axiomatization of intuitionistic logic with $\Box$ [3], our argument can be repeated without modification.

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[5] Note that $(\varphi \otimes \psi) \otimes \chi \to \varphi \otimes (\psi \otimes \chi)$ is valid in M if the model satisfies this frame condition. On the other hand, (5) entails the validity of $\varphi \otimes \psi \to \psi \otimes \varphi$.

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