Translation of Sequent Calculus into Natural Deduction for Sentential Calculus with Identity

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Providing translations between different proof methods for a chosen logic allows us to comprehend it better and examine its properties. It enables a closer investigation into characteristics and dependencies between various proof systems. Proofs of axioms alone bring insights in the field of computational domain of proof construction, such as their complexity. The fact that natural deduction and sequent calculus have properties that – even though they are expressed differently – stem from the same ideas, is studied through various methods of translation between the two. Reformulating proof method for a given logic in terms of the other might bring worthwile insights, especially if the logic in question is itself non-classical. Translating sequent calculi to natural deduction is by no means pioneering. The first approach was proposed by Gentzen in *Untersuchungen über das logische Schließen* [3]. Further ideas were provided by Ebner and Schlaipfer [2], Gilbert [4], Prawitz [5] and others. Main concern while translating above systems either way, should be ensuring – or rather retaining – soundness, completeness, and provability. Comparing computational properties of proofs built using two different, even though related, proof methods, is in turn useful from the point of proof theory and theoretical computer science. One of the methods of isomorphic translation between the systems is provided by Negri and von Plato (2001). An example of translation from ND to SC (based on the conjunction elimination):

$$\dfrac{\Gamma \Rightarrow A \wedge B \qquad \dfrac{A^m, B^n, \Delta \ \Rightarrow C}{A \wedge B, \Delta \Rightarrow C}}{\Gamma, \Delta \Rightarrow C}\ SC$$

$$\wr$$

$$\dfrac{\Gamma \rightarrow \ A \wedge B \quad A^m, B^n, \Delta \ \rightarrow C}{\Gamma, \Delta \ \rightarrow \ C}\ ND$$

We apply this strategy to obtain a set of ND rules for Sentential Calculus with Identity, using established sequent calculus for SCI [1]. An exemplary translation of $L^2_{\approx}$ from SCI to ND sequent calculus style would then be:

$$\frac{A \approx B,\ \Gamma \Rightarrow \Delta, B \quad A \approx B, A,\ \Gamma \Rightarrow \Delta}{A \approx B\ , \Gamma \Rightarrow \Delta}$$

$$\wr$$

$$\frac{A \approx B \to B, \Delta \quad A \to \Delta \quad A \approx B \to \Delta}{A \approx B \to \Delta}$$

If we were to adapt Negri's operational interpretation of a sequent[1] which expresses derivability relation between formulae in the antecedent and the derived succedent, we can easily translate a given sequent to ND rule, assuming sequent's antecedent to reflect ND's assumption, and sequent's succedent to correspond with a conclusion in ND. In the particular case of $L^2_{\approx}$ the sequent rule is comprised of two premisses – first of which postulates that B can be derived from $A \approx B$, and the second expresses that Δ (with Δ denoting an undetermined set of formulae) can be derived from both $A \approx B$ and A. Eventually, the premisess allow us to cut both A and B from the conclusion. Furthermore, we can merge two premisses into one by using classical left implication rule and end up with a sequent of the following form: $A \approx B, B \to A, \Gamma \Rightarrow \Delta$.

$$\cfrac{\cfrac{A \approx B,\ \Gamma \Rightarrow \Delta, B \quad A \approx B, A,\ \Gamma \Rightarrow \Delta}{A \approx B, B \to A, \Gamma \Rightarrow \Delta}}{A \approx B\ , \Gamma \Rightarrow \Delta}$$

$$\wr$$

$$\frac{A \approx B \to \Delta \quad (B \to A) \to \Delta}{A \approx B \to \Delta}$$

Eventually, $L^2_{\approx}$ translated from ND sequent calculus style to actual ND would then take form as below:

$$\frac{A \approx B \to \Delta \quad (B \to A) \to \Delta}{A \approx B \to \Delta} \qquad \rightsquigarrow \qquad \frac{A \approx B \quad \begin{matrix}[B \to A] \\ \vdots \\ C\end{matrix}}{C}$$

[1] We can distinguish two interpretations of a sequent – operational and denotational, where the latter one states that conjunction of formulae in the antecedent implies disjunction of formulae in the succedent.

The rule above expresses the following: if we were to assume $A \approx B$ and we can derive a given formula C from $B \to A$, we can conclude that C holds. We do not discharge $A \approx B$ as it reappears in the conclusion of the SC rule. $B \to A$ is removed from the conclusion, therefore we have to discharge it. We apply the same strategy to the remaining rules.

When comparing proofs of axioms specific for SCI we can notice that that ND system for SCI significantly reduced the complexity of derivations[2]. Let us consider the following derivation of axiom $(A \approx B) \to (B \to A)$:

$$\dfrac{\dfrac{\dfrac{A \approx B, B \Rightarrow B \qquad A \approx B, B, A \Rightarrow A}{A \approx B, B \Rightarrow A}\, L^2_{\approx}}{A \approx B \Rightarrow B \to A}\, R_{\to}}{\Rightarrow (A \approx B) \to (B \to A)}\, R_{\approx}$$

In order to arrive at a derivation consisting of leaves labelled with axioms we are forced to use three inference rules. However, if we were to form a derivations using the obtained ND rules, we can prove it in one step less, by using $\approx_2$ rule corresponding to the axiom, and implication introduction rule:

$$\dfrac{\dfrac{[A \approx B]^1 \quad [(A \to \perp) \approx (B \to \perp)]}{(A \to \perp) \approx (B \to \perp)}\, \approx_2}{(A \approx B) \to ((A \to \perp) \approx (B \to \perp))}\, \to I, 1$$

Similar observations can be made in regards to the remaining axioms.

Further work in the field will include right SCI-specific rules and other extensions of SCI, mostly its constructive version – namely intuitionistic and minimal version. It could be worthwhile to move our investigations to the area of automated theorem proving and attempt at implementing an algorithm for performing automated translations The language of our choice will be Haskell as it allows to define new data types and introduce certain functions in a way strikingly similar to their theoretic equivalents.

[2] However, any conclusive and more general claims are withheld at the moment due to further examination needed.

References

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