An Investigation into Intuitionistic Logic with Identity

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I propose a constructive interpretation of Suszko's propositional identity operator [5, 1] along with two sequent calculi (labelled and non-labelled) for the logic ISCI. ISCI is an extension of propositional intuitionistic logic by the set of axioms which characterizes propositional identity operator $\approx$.

($\approx_1$) $A \approx A$

($\approx_2$) $(A \approx B) \supset ((A \supset \bot) \approx (B \supset \bot))$

($\approx_3$) $(A \approx B) \supset (B \supset A)$

($\approx_4$) $((A \approx B) \wedge (C \approx D)) \supset ((A \otimes C) \approx (B \otimes D))$

In the context of classical logic the question of equivalency of two formulas reduces to the question whether these formulas have the same logical value. This is not the case with the propositional identity—two formulas may be equivalent yet not identical in Suszko's sense. The philosophical motivation behind SCI was related to the ontology of situations - in classical logic, there are only two situations: *Truth* and *Falsity*. Truth (Falsity) is described by any true (false) proposition. According to Suszko, this is unfortunate, and could be remedied by allowing new *identity* connective, which describes the fact that two propositions denote the same situation. In intuitionism, we are not interested in propositions being true or false but in *constructions* which prove them. Equivalence of two formulas A and B means that the truth values of this formulas are the same in every world, i.e., whenever A is provable, B is provable as well, and *vice versa*. But we can think of a stronger notion, which says that the classes of constructions proving A and B are *exactly* the same.

Our first aim is to show completeness of the Hilbert-style system for ISCI relative to Kripke semantics. This is done by beans of canonical models. Our second objective is to construct sequent calculus for ISCI. There is a number of strategies of building sequent calculi or natural deduction systems for axiomatic theories based on a certain logic (see for example [6, 4, 3, 2]). The strategy I am interested in enables one to turn each axiom of a given axiomatic system into a rule of a corresponding sequent calculus in such a way, that all structural rules—the cut rule in particular—are admissible in the generated calculus. Obtained rules correspond to the initial axiom (where P_i and Q_j are atomic formulas):

$$P_1 \wedge \ldots \wedge P_m \to Q_1 \vee \ldots \vee Q_n$$

in either of the following manners:

$$\frac{Q_1, P_1, \ldots, P_m, \Gamma \Rightarrow \Delta \quad \ldots \quad Q_n, P_1, \ldots, P_m, \Gamma \Rightarrow \Delta}{P_1, \ldots, P_m, \Gamma \Rightarrow \Delta}\ L$$

$$\frac{\Gamma \Rightarrow \Delta, Q_1, \ldots, Q_n, P_1 \quad \ldots \quad \Gamma \Rightarrow \Delta, Q_1, \ldots, Q_n, P_n}{\Gamma \Rightarrow \Delta, Q_1, \ldots, Q_n}\ R$$

We propose two sequent calculi for propositional intuitionistic logic with identity based on the left strategy along with cut-elimination theorems and admissibility of other structural rules. The future work will cover the construction and the analysis of natural deduction system for ISCI along with the typed lambda calculus corresponding to it, which will put more light on the constructive interpretation of Suszko's propositional identity connective.

Keywords: *Non-Fregean logics, intuitionistic logic, admissibility of cut, propositional identity, congruence*

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