Direction of coupling from phases of interacting oscillators: An information-theoretic approach

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A directionality index based on conditional mutual information is proposed for application to the instantaneous phases of weakly coupled oscillators. Its abilities to distinguish unidirectional from bidirectional coupling, as well as to reveal and quantify asymmetry in bidirectional coupling, are demonstrated using numerical examples of quasiperiodic, chaotic and noisy oscillators, as well as real human cardiorespiratory data.

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Cooperative behavior of coupled complex systems has recently attracted considerable interest from theoreticians as well as experimentalists (see e.g. the monograph [1]), since synchronization and related phenomena have been observed not only in physical, but also in many biological systems. Examples include the cardio-respiratory interaction [2,3] and the synchronization of neural signals [4–9]. In such physiological systems it is not only important to detect synchronized states, but also to identify causal (driver-response) relationships between the systems studied. The problem of coupling direction in generalized synchronization [10] has been treated using amplitudes of the system observables and evaluating their mutual predictability [4,5] or mutual nearest neighbors in reconstructed state spaces [7,11]. Information-theoretic approaches [8,9,12] have also been successfully applied.

Considering weakly coupled oscillators the coupling properties of the systems studied can be inferred from an analysis of the interrelations between the instantaneous phases of the oscillators, $\phi_{1,2}(t)$. These can be estimated from (scalar) observable signals [1,13,14]. Several methods have been proposed for the detection and quantification of phase synchronization from experimental data [1,6,14]. Rosenblum et al. [15,16] have also introduced methods for inferring directionality of coupling, based either on Fourier approximation of phase increments or instantaneous periods as functions of the phases $\phi_{1,2}(t)$, or on mutual predictability of the instantaneous phases $\phi_{1,2}(t)$. Paluš et al. [8] have introduced an information-theoretic framework for the study of generalized synchronization in experimental time series based on evaluation of so-called coarse-grained transinformation rates (CTIRs). In this paper, CTIRs are developed and applied to instantaneous phases $\phi_{1,2}(t)$ of coupled oscillators.

The method introduced in [8] operates with information-theoretic tools such as the well-known mutual information $I(X;Y)$ of two random variables X and Y, given as $I(X;Y) = H(X) + H(Y) - H(X,Y)$, where the entropies $H(X)$, $H(Y)$, $H(X,Y)$ are given in the usual Shannonian sense [8,17]. The conditional mutual information $I(X;Y|Z)$ of the variables X, Y given the variable Z is defined using the conditional entropies [8,17] as

$$I(X;Y|Z) = H(X|Z) + H(Y|Z) - H(X,Y|Z). \quad (1)$$

Consider two time series $\{x(t)\}$ and $\{y(t)\}$ regarded as realizations of two stationary ergodic stochastic processes $\{X(t)\}$ and $\{Y(t)\}$ which represent observables of two possibly coupled systems. Dependence structures between the two processes (time series) can be studied using the simple mutual information $I(y;x_\tau)$, where we use y for $y(t)$ and x_τ for $x(t+\tau)$. $I(y;x_\tau)$ measures the average amount of information contained in the process $\{Y\}$ about the process $\{X\}$ in its future τ time units ahead (τ-future thereafter). This measure, however, as well as other dependence and predictability measures, could also contain information about the τ-future of the process $\{X\}$ contained in this process itself if the processes $\{X\}$ and $\{Y\}$ are not independent, i.e., if $I(x;y) > 0$.

For inferring causality relations, i.e., the directionality of coupling between the processes $\{X(t)\}$ and $\{Y(t)\}$, we need to estimate the "net" information about the τ-future of the process $\{X\}$ contained in the process $\{Y\}$ itself using an appropriate tool – the conditional mutual information $I(y;x_\tau|x)$. It has been shown [8,9] that using $I(y;x_\tau|x)$ and $I(x;y_\tau|y)$ the coupling directionality can be inferred from time series measured in coupled, but not yet fully synchronized systems.

Consider now that the processes $\{X\}$ and $\{Y\}$ can be modelled by weakly coupled oscillators and that their interactions can be inferred by analyzing the dynamics of their instantaneous phases $\phi_1(t)$ and $\phi_2(t)$ [15,16]. The latter can be estimated from the measured time series $\{x(t)\}$ and $\{y(t)\}$, e.g., by application of the discrete Hilbert transform [1,13,14]. Rather than simply substituting the series $\{x(t)\}$ and $\{y(t)\}$ by the phases $\phi_1(t)$ and $\phi_2(t)$ (which are confined in interval $[0,2\pi)$ (or $[-\pi,\pi)$), we consider phase increments

$$\Delta_\tau\phi_{1,2} = \phi_{1,2}(t+\tau) - \phi_{1,2}(t),$$

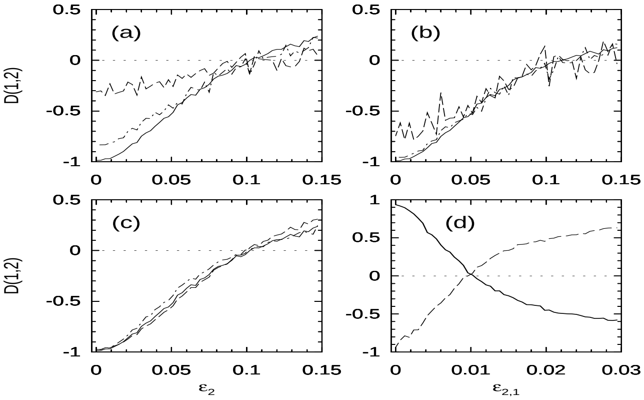

FIG. 1. (a-c) Directionality index $D(1,2)$ for noisy phase oscillators (3) for $\epsilon_1 = 0.1$ as a function of ϵ_2 computed using q=8 (a) and q=4 (b) equiprobable marginal bins and series length N=1k (dashed line), N=8k (dash-and-dotted line) and N=128k samples (full line). (c) $D(1,2)$ for N=128k, q=8, averaged over time lags: 1-5 (dashed line), 1-15 (full line), 1-150 (dash-and-dotted line). Integration step is $\pi/7$. (d) $D(1,2)$ for coupled Rössler systems (4) with $\epsilon_1 = 0.01$ as a function of ϵ_2 (dashed line) and with $\epsilon_2 = 0.01$ as a function of ϵ_1 (full line). N=128k, q=8, lags 1-15.

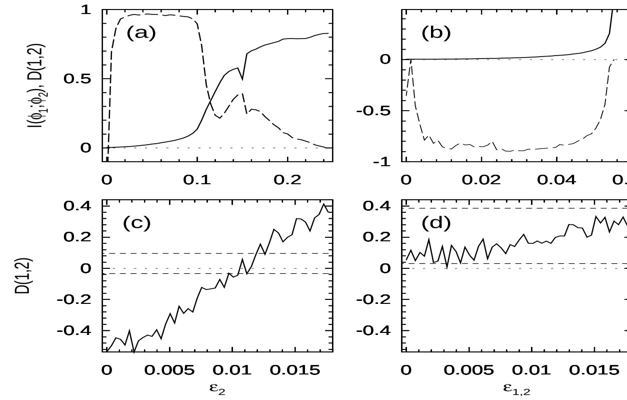

FIG. 2. (a,b) Directionality index $D(1,2)$ (dashed line) and mutual information $I(\phi_1;\phi_2)$ (full line) of the phases of unidirectionally coupled Rössler systems (4) ($\epsilon_1 = 0$) as a function of ϵ_2 (a), and for $\epsilon_2 = 0$ as a function of ϵ_1 (b). N=128k, q=8, lags 1-15. (c,d) $D(1,2)$ for noisy phase oscillators (3) (full line) with $\omega_1 = 0.1$ and $\omega_2 = 1.1$ (1:11) for $\epsilon_1 = 0.01$ as a function of ϵ_2 for lags 1-15 (c) and lags 10, 20, ..., 150 (d). The horizontal dashed lines are ranges of the mean ± 2SD of $D(1,2)$ obtained from the surrogate data.

and the conditional mutual information $I(\phi_1(t);\Delta_\tau\phi_2|\phi_2(t))$ and $I(\phi_2(t);\Delta_\tau\phi_1|\phi_1(t))$, in a shorter notation $I(\phi_1;\Delta_\tau\phi_2|\phi_2)$ and $I(\phi_2;\Delta_\tau\phi_1|\phi_1)$. Now, in analogy with Rosenblum et al. [15,16] we define a directionality index

$$D(1,2) = \frac{i(1\to 2) - i(2\to 1)}{i(1\to 2) + i(2\to 1)}, \tag{2}$$

where the measure $i(1\to 2)$ of how the system 1 drives the system 2 is either equal to the conditional mutual information $I(\phi_1;\Delta_\tau\phi_2|\phi_2)$ for a chosen time lag τ, or to an average of $I(\phi_1;\Delta_\tau\phi_2|\phi_2)$ over a selected range of lags τ. (For motivation for averaging and the concept of the coarse-grained information rates see [8,18].) And in full analogy we define $i(2\to 1)$ using $I(\phi_2;\Delta_\tau\phi_1|\phi_1)$. $D(1,2)$ should be positive if the driving from system 1 to system 2 prevails, and negative for the opposite case.

In order to test how well the directionality index (2) works we start with the same simple model of two coupled phase oscillators as Rosenblum and Pikovsky [15] and Rosenblum et al. [16]:

$$\dot{\phi}_1 = \omega_1 + \epsilon_1 f_1(\phi_1,\phi_2) + \xi_1(t)$$
$$\dot{\phi}_2 = \omega_2 + \epsilon_2 f_2(\phi_2,\phi_1) + \xi_2(t) \tag{3}$$

Using $\omega_{1,2} = 1 \pm 0.1$, $q_{1,2} = 0$, $f_{1,2} = \sin(\phi_{2,1} - \phi_{1,2})$ and mutually independent Gaussian IID noises with zero mean and standard deviation $\sigma = 0.2$ for $\xi_{1,2}$, with a fixed coupling parameter $\epsilon_1 = 0.1$ we generated time series of the phases $\phi_{1,2}(t)$ for fifty different values of the coupling parameter ϵ_2. The directionality indices $D(1,2)$ were obtained from coarse-grained estimates of the conditional mutual information. The latter were obtained by a simple box-counting algorithm based on equiprobable marginal bins (marginal equiquantization [18]). The dependence of $D(1,2)$ on the quantization and the series length can be seen in Figs. 1(a,b), and its dependence on time lags in Fig. 1(c).

Averaging $I(\phi_{1,2};\Delta_\tau\phi_{2,1}|\phi_{2,1})$ over a short range of lags decreases fluctuations of the estimates. For shorter time series (N=1k=1024 samples) more coarse (q=4) estimates have higher variance (Fig. 1b, dashed line), while for q=8 the estimates have a higher bias for weaker coupling (Fig. 1a, dashed line). The results for series lengths N=8k=8192 (Figs. 1(a,b), dash-dotted line) and N=128k=1.3x10^5 samples (Figs. 1(a,b), full line) reflect well the coupling asymmetry and smoothly changes with changing coupling parameter ϵ_2 (c.f. the results in [15], Fig. 3a.)

Let us now consider two coupled Rössler systems, the same as studied in [13,14], but with different coupling coefficients $\epsilon_1 \neq \epsilon_2$:

$$\begin{aligned}\dot{x}_{1,2} &= -\omega_{1,2}y_{1,2} - z_{1,2} + \epsilon_{1,2}(x_{2,1} - x_{1,2}),\\ \dot{y}_{1,2} &= \omega_{1,2}x_{1,2} + 0.15y_{1,2},\\ \dot{z}_{1,2} &= 0.2 + z_{1,2}(x_{1,2} - 10)\end{aligned} \tag{4}$$

The frequencies $\omega_{1,2}$ are defined as $\omega_{1,2} = 1 \pm 0.015$. The phases of the Rössler systems (4) have been obtained using the Hilbert transform by the same way as in [14], where the simple mutual information $I(\phi_1;\phi_2)$ was proposed for detecting phase synchronization. Here we repeat the numerical study of transients to phase synchronization, as in [14], but for unidirectional coupling, i.e., either $\epsilon_1 = 0$ (Fig. 2a), or $\epsilon_2 = 0$ (Fig. 2b). We can see that $D(1,2)$ (dashed line) correctly identifies the driver from the response system [19] before the coupling parameter reaches the synchronization threshold [20]. The latter is detected by a steep increase of $I(\phi_1;\phi_2)$ (full line in Figs. 2a,b) [14].

Keeping the coupling parameters before the synchronization threshold we can repeat the same study as with the oscillators (3), when one coupling parameters was kept constant, i.e. $\epsilon_2 = 0.01$, and the other, ϵ_1 varies from zero to 0.03 (dashed line, Fig. 1d), and vice versa (full line, Fig. 1d). The directionality of coupling was exactly revealed also in this example of chaotic systems, which were not studied yet from this point of view.

Let us return to the phase oscillators (3). We have also studied the noise-free quasiperiodic system, as well as more complex noisy cases with larger differences in the natural frequencies, or with asymmetric coupling, as treated in [16]. In all cases the directionality index $D(1,2)$ identified the correct coupling direction.

Since we intend to study cardiorespiratory interactions during paced respiration, when the ratio of natural frequencies can be rather large, we have studied the systems (3) with such frequency ratios as $\omega_1 : \omega_2 = 1 : 11$ (Fig. 2c,d). For relatively short time lags τ the directionality index $D(1,2)$ detects the correct coupling directionality for the majority of the coupling parameter values (Fig. 2c, the full line), while for long time lags (Fig. 2d) the directionality detection ability of $D(1,2)$ is lost. Looking back at the conditional mutual information $I(\phi_1;\Delta_\tau\phi_2|\phi_2)$ and $I(\phi_2;\Delta_\tau\phi_1|\phi_1)$ we can find that their values are very low, comparable with variance of their estimates. This leads to a large bias and variance of the directionality index $D(1,2)$. Therefore we need to establish significance of $D(1,2)$ values by a statistical test.

We use the concept of surrogate data (see [14] and references therein). In this case the surrogate data can be a set of realizations (with different random initial conditions) of phases of uncoupled oscillators (3). Estimating the conditional mutual information and the directionality indices for these surrogate data sets we can assess the fluctuations of these quantities for uncoupled data without any directionality of coupling. To present these fluctuations, we illustrate the ranges of the mean $\pm$ 2SD (standard deviations) of $D(1,2)$ for the surrogates by the dashed lines in Figs. 2c,d. In the case of large time lags (Fig. 2d) the surrogates confirm the extremely large fluctuations of $D(1,2)$ and its bias to positive values. As a result, the directionality index of the coupled oscillators does not differ significantly from $D(1,2)$ of the surrogates, so in this case no directionality can be inferred (Fig. 2d). In the case of small lags, fluctuations and bias of $D(1,2)$ are much smaller, although not negligible (Fig. 2c, dashed lines). The range of surrogate $D(1,2)$ fluctuations disqualifies values of $D(1,2)$ for the coupled oscillators for a small interval around the symmetry point $\epsilon_2 = 0.01$. Since we can see that the fluctuations of

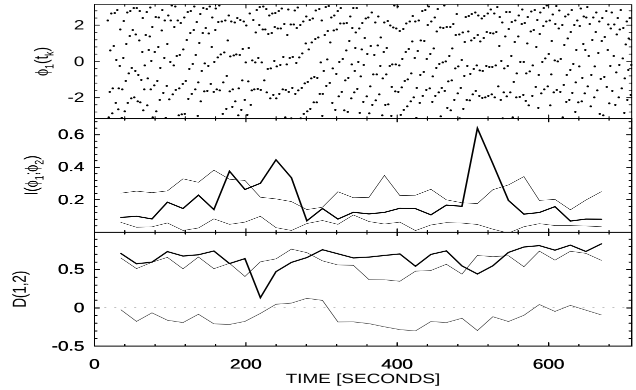

FIG. 3. Synchrogram (top panel), mutual information $I(\phi_1;\phi_2)$ (middle panel) and the directionality index $D(1,2)$ (bottom panel) for the phases of human cardiorespiratory data (respiration and heartbeat, full thick lines). The ranges of surrogate mean$\pm$2SD for $I(\phi_1;\phi_2)$ and $D(1,2)$ are depicted by thin lines in respective panels.

$D(1,2)$ estimates for the coupled systems (full line in Fig. 2c) are of similar magnitude to the surrogate mean$\pm$2SD range, comparison of the directionality index obtained from the studied data with its surrogate range save us from making an unreliable inference of the directionality.

In any experimental application, estimation of the directionality index should be accompanied by an assessment of its significance. Otherwise an incorrect directionality could be concluded due either to variance or bias in the estimate of the directionality index. The surrogate data test is one possible approach. In many practical applications, however, it is the only available one. Various types of bivariate surrogate data useful in the study of coupled systems are discussed in [14]. A special type related to a specific application is presented below, where we analyze data from human cardiorespiratory interactions.

The cardiorespiratory coupling during spontaneous and paced respiration was analyzed in a group of young healthy subjects. The data were noninvasively recorded for 12 minutes, while the subjects were lying comfortably. The cardiac activity was assessed by recording the electrocardiogram (ECG) and a piezoelectric sensor was used to measure excursions of the thorax and hence the respiratory activity. A sampling rate of 400 Hz was used for both signals. (For details of measurements see [21]) The phases of cardiac activity were estimated using the marked events method, by marking R-peaks. The phases of the respiratory oscillations were obtained by application of Hilbert transform to the respiratory signal. The results will be presented in detail elsewhere; here we briefly illustrate the potential of the proposed approach. The directionality index $D(1,2)$ (1 – respiratory, 2 – cardiac system) was estimated in moving 40-second windows with 50% overlap, using 4 quantization levels and time lags from 20 to 200, increased by 20 (samples).

In the same windows, but using 16 quantization levels, the simple mutual information $I(\phi_1;\phi_2)$ of the instantaneous phases $\phi_1(t),\phi_2(t)$ was calculated in order to assess presence of phase synchronization [14]. Significance levels for both $D(1,2)$ and $I(\phi_1;\phi_2)$ were established using sets of 30 realizations of surrogate data. The latter were constructed by random permutations of R-R intervals, thus producing artificial heartbeat data with the same frequency histograms as the original data. Due to the randomization of the R-peak positions, however, any possible association with the respiratory rhythm was destroyed. The respiratory data remained unchanged, so that the significance levels depend on the character of the respiratory dynamics in each window.

The synchrogram [2], the mutual information $I(\phi_1;\phi_2)$ and the directionality index $D(1,2)$ for an example of spontaneous respiration are illustrated in Fig. 3. Two episodes of phase synchronization between the heartbeat and respiratory rhythms, visible in the synchrogram as almost horizontal lines (at times of approximately 240 and 500 seconds), are detected by $I(\phi_1;\phi_2)$ (thick line in Fig. 3, middle panel) lying outside from the surrogate range (mean$\pm$2SD of the surrogates, depicted by thin lines). The necessity of establishing the significance level (here as the mean plus two standard deviations of the surrogate set) is obvious – even relatively large positive values of $I(\phi_1;\phi_2)$ do not necessarily reflect the presence of synchronization (but the bias and variance of estimates) unless $I(\phi_1;\phi_2)$ is larger than the significance level given by the surrogate mean and variance.

The same holds also for the values of the directionality index $D(1,2)$ (thick line in Fig. 3, bottom panel), estimates of which are severely biased towards positive values, as confirmed by the surrogate mean$\pm$ 2SD ranges (thin lines). Nevertheless, in a large part of the recording, $D(1,2)$ is larger than its significance level, indicating that the respiration is driving the cardiac system, as was recently reported by Rosenblum et al. [16]. It is also noticeable that $D(1,2)$ falls into the surrogate range, i.e., no directionality can be inferred, in the two synchronous intervals [20], detected by $I(\phi_1;\phi_2)$ as well as seen in the synchrograms (Fig. 3).

Note that the tools introduced above have a firm mathematical basis in information theory, and their coarse-grained estimates can be computed more efficiently [18,22] than measures used by other authors.

In conclusion, an information-theoretic approach for detecting the directionality of coupling from the phases of interacting oscillators has been proposed and tested. Its ability to reveal and quantify possible asymmetry in the coupling has been demonstrated using both numerical and real data examples. The problem of assessing the significance of estimated directionality indices is discussed for the first time in this context and solutions were proposed.

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