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On entropy rates of dynamical systems and Gaussian processes

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Abstract

A possibility of a relation between the Kolmogorov-Sinai entropy of a dynamical system and the entropy rate of a Gaussian process isospectral to time series generated by the dynamical system is numerically investigated using discrete and continuous chaotic dynamical systems. The results suggest that such a relation as a nonlinear one-to-one function may exist when the Kolmogorov-Sinai entropy varies smoothly with variations of system's parameters, but is broken in critical states near bifurcation points.

1 Entropy rates

Entropy rates will be considered as a tool for quantitative characterization of dynamic processes evolving in time. Let $\{x_i\}$ be a time series, i.e., a series of measurements done on a system in consecutive instants of time $i = 1, 2, \ldots$. The time series $\{x_i\}$ can be considered as a realization of a stochastic process $\{X_i\}$, characterized by the joint probability distribution function $p(x_1, \ldots, x_n)$, $p(x_1, \ldots, x_n) = \Pr\{(X_1, \ldots, X_n) = (x_1, \ldots, x_n)\}$. The entropy rate of $\{X_i\}$ is defined as [1]:

$$h = \lim_{n\to\infty} \frac{1}{n} H(X_1, \ldots, X_n), \tag{1}$$

where $H(X_1, \ldots, X_n)$ is the entropy of the joint distribution $p(x_1, \ldots, x_n)$:

$$H(X_1, \ldots, X_n) = -\sum_{x_1} \ldots \sum_{x_n} p(x_1, \ldots, x_n) \log p(x_1, \ldots, x_n). \tag{2}$$

Alternatively, the time series $\{x_i\}$ can be considered as a projection of a trajectory of a dynamical system, evolving in some measurable state space. As a definition of the entropy rate of a dynamical system, known as the Kolmogorov-Sinai entropy (KSE) [2, 3, 4] we can consider the equation (1), however, the variables X_i should be understood as m-dimensional variables, according to a dimensionality of the dynamical system [5]. If the dynamical system is evolving in a continuous measure space, then any entropy depends on a partition chosen to discretize the space and the KSE is defined as a supremum over all finite partitions [2, 3, 4].

The KSE is a topological invariant, suitable for classification of dynamical systems or their states, and is related to the sum of the system's positive Lyapunov exponents (LE) according to the theorem of Pesin [6].

A number of algorithms (see, e.g., [7, 8, 9, 10] and references therein) have been proposed for estimation of the KSE from time series. Reliability of these estimates, however, is limited [11] by available amount of data, finite precision measurements and noise always present in experimental data. No general approach to estimating the entropy rates of stochastic processes has been established, except of simple cases such as finite-state Markov chains [1]. However, if $\{X_i\}$ is a zero-mean stationary Gaussian process with spectral

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density function $f(\omega)$, its entropy rate h_G, apart from a constant term, can be expressed using $f(\omega)$ as [12, 13, 14]:

$$h_G = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log f(\omega) d\omega. \tag{3}$$

Dynamics of a stationary Gaussian process is fully described by its spectrum. Therefore the connection (3) between the entropy rate of such a process and its spectral density $f(\omega)$ is understandable. The estimation of the entropy rate of a Gaussian process is reduced to the estimation of its spectrum.

If a studied time series was generated by a nonlinear, possibly chaotic, dynamical system, its description in terms of a spectral density is not sufficient. Indeed, realizations of isospectral Gaussian processes are used in the surrogate-data based tests in order to discern nonlinear (possibly chaotic) processes from colored noises [15, 16]. On the other hand, there are results indicating that some characteristic properties of nonlinear dynamical systems may be "projected" into their "linear properties", i.e., into spectra, or equivalently, into autocorrelation functions: Sigeti [17] has demonstrated that there may be a relation between the sum of positive Lyapunov exponents (KSE) of a chaotic dynamical system and the decay coefficient characterizing the exponential decay at high frequencies of spectra estimated from time series generated by the dynamical system. Asymptotic decay of autocorrelation functions of such time series is ruled by the second eigenvalue of the Perron-Frobenius operator of the dynamical system [18, 19]. Lipton & Dabke [20] have also investigated asyptotic decay of spectra in relation to properties of underlying dynamical systems.

2 Numerical study

A possibility of a relation between the KSE of a dynamical system and the entropy rate GPER of a Gaussian process with the same spectrum as the time series, generated by the dynamical system is investigated in this study. A formal application of formula (3) to a spectral density (periodogram) estimated from analyzed time series cannot be considered as an estimate of the KSE of an underlying dynamical system, however, if there was a one-to-one relation between the KSE and the GPER, the GPER could be used for a "relative quantification" [11] of dynamic processes, in particular, it could distinguish and classify different states of chaotic dynamical systems.

The numerical investigation of this hypothetical relationship was performed using these dynamical systems:

The baker transformation [5]:

$$(x_{n+1}, y_{n+1}) = (\lambda x_n, \frac{1}{\alpha} y_n)$$

for $y_n \leq \alpha$, or:

$$(x_{n+1}, y_{n+1}) = (0.5 + \lambda x_n, \frac{1}{1-\alpha}(y_n - \alpha)) \tag{4}$$

for $y_n > \alpha$;
$0 \leq x_n, y_n \leq 1$, $0 < \alpha < 1$, $\lambda = 0.25$;

the logistic map [5]:

$$x_{n+1} = a x_n (1 - x_n); \tag{5}$$

and the continuous Lorenz system [21]:

$$(dx/dt, dy/dt, dz/dt) = (\sigma(y-x), rx - y - xz, xy - bz), \tag{6}$$

$\sigma = 16$, $b = 4$.

Each of the three systems has one positive Lyapunov exponent, equal to the system's Kolmogorov-Sinai entropy, therefore we will use the terms LE and KSE interchangeably.

Changing a parameter of a particular system (α, a, r in the cases of the baker, logistic and Lorenz systems, respectively), time series related to different system states were generated, GPER's were estimated and compared with LE (KSE) related to particular system states. In each system state studied, fifteen time series of length 16,384 samples (the sampling interval was 0.002 in the case of the Lorenz system) were recorded from the first component (x), linearly transformed in order to have zero mean and unit variance[1] and their periodograms[2] computed using the fast Fourier transform (FFT) [22]. To prevent numerical underflow, the periodograms were shifted[3] by +1, i.e., $f(\omega)+1$ was used instead of $f(\omega)$ in Eq. (3). For each considered dynamical state, means and standard deviations (SD's) of the GPER estimates, obtained from the 15 realizations of 16k time series, are reported in this paper.[4]

The positive Lyapunov exponents were *not* estimated from time series, but computed as follows. The KSE/LE of the baker map can be expressed analytically as the function of the parameter α [23, 24]:

$$h(\alpha) = \alpha \log \frac{1}{\alpha} + (1-\alpha) \log \frac{1}{1-\alpha}. \tag{7}$$

For the logistic map the LE was estimated according to its definition [5] as the averaged logarithm of the absolute derivative of the function (5). A recent implementation [25] of the method proposed by Wolf et al. [26] for estimation of the Lyapunov exponents from equations was used for the Lorenz system. The LE's in these two cases were estimated using 300,000 iterations in each state.

3 Results

In Figures 1a-c the results for the baker map (4) are presented: The LE as the analytic function (7) of the parameter α (Fig. 1a), the GPER estimated from time series plotted against α (Fig. 1b), and the GPER plotted against the LE (Fig. 1c). The latter plot demonstrates that in the case of the chaotic baker map (4) the LE/KSE and the GPER are related by *a nonlinear one-to-one function.* Considering precision of the GPER estimates, the same conclusion can be drawn for the Lorenz system (6) for the parameter r varying from 34 to 65 (Figs. 1d-f).

The situation is different in the case of the logistic map (5) (Fig. 2): the basic trends in the dependences of the LE and the GPER on the parameter a (Figs. 2a, 2b, respectively) agree, however, there are clear discrepancies larger than the estimation errors and the functional relation between the GPER and the LE/KSE is lost (Fig. 2c).

Comparing the plots in Fig. 1 and Fig. 2, one can see that in Fig. 1 the LE (KSE) varies smoothly with variations of a system parameter, i.e., the systems change only quantitatively remaining in the chaotic regime, while in the case of the logistic map in Fig. 2 bifurcations into periodic states interrupt the regime of chaotic states. Similarly, the Lorenz system (6) with $r > 65$ enters the bifurcation region (Figs. 3a,b and 3d,e) and deviations from the bijective functional dependence between the KSE/LE and the GPER occur in the LE values related to the bifurcation region (Fig. 3c and 3f).

When a system parameter exactly fits a periodic-state value, the periodic state with zero KSE and negative LE occurs, which is indicated also by a very low (but positive) GPER value[5] (cf. plot a with plot b, or plot d with plot e in Fig. 2 and Fig. 3). The functional relation between the KSE/LE and the GPER, however, is broken not only in periodic states, but apparently also at any point near a bifurcation. Only two bifurcations appeared in Fig. 2a, when the plot was obtained by increasing the parameter a from 3.857 to 4 by step $\Delta a = 0.001$. Using smaller step ($\Delta a = 0.0003$), seven periodic states were "hit" (Fig. 2d). In fact, it is impossible to find any "bifurcation free" sub-interval of chaotic states of the logistic map. Note, that there are no bifurcations in the case of the baker map studied in Fig. 1a-c. In the case of the Lorenz system, both situations were observed: A chaotic region with smooth ("bifurcation free") dependence of the KSE/LE on the parameter r for $r \in [34, 65]$ in which a one-to-one relation between the KSE/LE and the

[1] Note, that the GPER (3) is variance-dependent. Therefore all analyzed time series were rescaled to have unit variance so that the GPER should classify the series according to their dynamics, without the influence of the variance.

[2] I.e., discrete estimates of the spectral density obtained as squared magnitudes of the Fourier coefficients. The integral in (3) is then computed as a sum over the 8192 periodogram bins.

[3] This shift is equivalent to an addition of white noise to the original time series and thus it could worsen distinction of system states with similar spectra. On the other hand, presence of a few periodogram bins with magnitude close to zero could bias the GPER estimate downwards and obscure the dependence of GPER on a system parameter.

[4] Stability of GPER estimates obtained from shorter time series or from individual realizations and other technical questions will be discussed elsewhere.

[5] Strictly speaking, the GPER is not defined for periodic states, and its formally estimated values do not reflect behaviour of negative LE – see Fig. 5.

GPER exists (Fig. 1); and for $r > 65$ a regime of chaotic states suddenly interrupted by bifurcations into periodic states, where digressions from the one-to-one functional dependence of the GPER on the KSE/LE occur (Fig. 3).

4 Transients and critical behaviour

Solutions of dynamical systems in the vicinity of bifurcation may have longer transient times than solutions in other states. Could the increased transient time be the reason for the digressions from the one-to-one functional dependence of the GPER on the KSE/LE?[6] Using the logistic map in the range of the parameter a considered in Fig. 2d-f, we have studied variances of the GPER estimates (using the 15 realizations of 16k time series) as well as variances of the LE estimates, after skipping out different numbers of initial iterations considered as the transient time. In this case we used 15 LE estimates from 20,000 iterations each (unlike in the previous section, where the LE estimates from whole 300,000 iterations were used). The SD (standard deviations, square roots of the variances) of the GPER (Fig. 4a,c,e) and LE (Fig. 4b,d,f) estimates as functions of the parameter a are plotted in Fig. 4. When no transient iterations were omitted and the computation of the GPER and the LE started at the beginning of the iteration, the variances of the GPER and LE estimates are very large due to the transients (Fig. 4a,b, note different scales). Starting the LE/GPER estimation after skipping hundred thousand[7] initial iterations led to decrease of the variance of the estimates — SD of LE (Fig. 4d) decreased several times and SD of GPER (Fig. 4c) decreased one order of magnitude. The number of the "transient iterations", i.e. the number of the skipped initial iterations was further increased through 10^6, 10^7, 10^8, up to one billion (10^9, Fig. 4e,f), however, no further changes in the variance of the estimates were observed. Therefore we could conclude that omitting 10^5 initial iterations was enough for transients to disappear and for the system to converge to the attractor in all considered states (all considered values of the parameter a). Larger variances in the vicinity of bifurcations are probably due to typical behaviour (fluctuations) of systems in critical states. Note that the intervals of the increased variance of the GPER are limited to the points located immediately before and after bifurcations, while the variance of the LE rises gradually in wider intervals surrounding the bifurcations. This phenomenon is illustrated in detail in Fig. 5, where the LE, the GPER and their variances are plotted as functions of the parameter a, depicting one of the bifurcations into periodic states.

The results presented above suggest that the discrepancies in the functional relation between the GPER and the KSE (LE) at the vicinity of bifurcations are not due to transients, but probably due to critical behaviour of the system near a bifurcation point. Therefore the linear description (based on the spectral density) is inadequate for systems in critical states.

5 Conclusion

A possibility of a relation between the Kolmogorov-Sinai entropy (KSE) of a dynamical system and the entropy rate (GPER) of a Gaussian process isospectral to time series generated by the dynamical system was numerically investigated using three[8] well-known chaotic dynamical systems. The results obtained suggest that such a relation as a nonlinear one-to-one function may exist when the Kolmogorov-Sinai entropy varies smoothly with variations of system's parameters, but is broken in critical states near bifurcation points. Further theoretical and numerical studies are necessary to establish general conditions for validity of this conclusion. These results could find applications in two areas of the analysis of complex time series: The GPER itself could be used as a computationally cheap tool for classification of different chaotic states of dynamical systems; while discrepancies in the relation between the GPER and the KSE/LE (or other nonlinear entropy-rate equivalent [11]) could be applied for detecting bifurcation onsets in structurally evolving systems.

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[6] The author is grateful to an anonymous referee for posing this interesting question.

[7] Hundred thousand initial iterations were skipped as transient time also in all numerical studies presented in previous section.

[8] Equivalent results have also been obtained from tent, tilted tent, Gaussian and Hénon maps [27].

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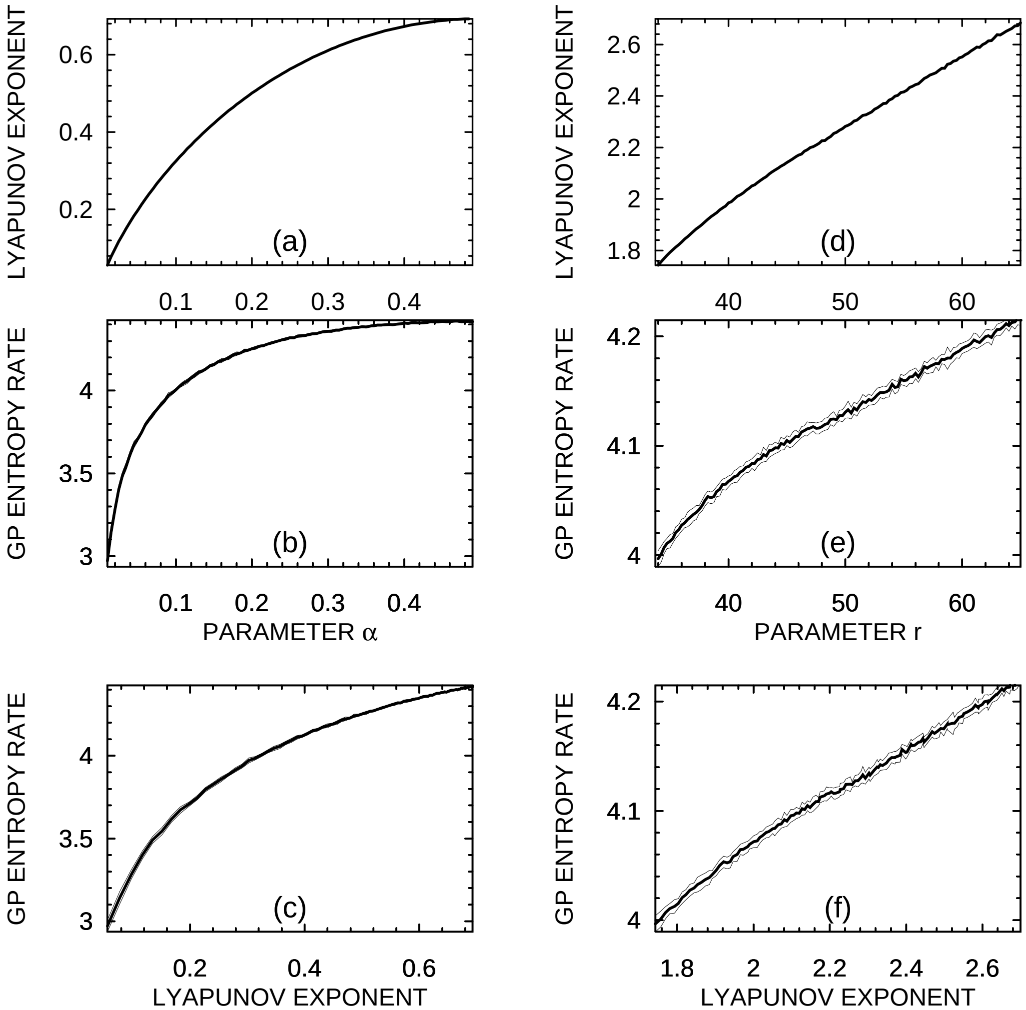

Figure 1: *(a–c) Results for the baker map: a) The Lyapunov exponent as the analytic function of the parameter α. b) The GP entropy rates estimated from 15 realizations of 16k time series (mean – thick line, mean$\pm$SD – thin lines, coinciding with the mean) for different values of the parameter α varying from 0.01 to 0.49 by step 0.005. c) Plot of GPER (the same line codes as in b) vs. LE. (d–f) Results for the Lorenz system: d) The positive Lyapunov exponents computed from the Lorenz equations for the parameter r varying from 33.75 to 65 by step 0.25. e) The GP entropy rates estimated from 15 realizations of 16k time series (mean – thick line, mean$\pm$SD – thin lines) for different values of the parameter r varying as in plot d. f) Plot of GPER (the same line codes as before) vs. LE.*

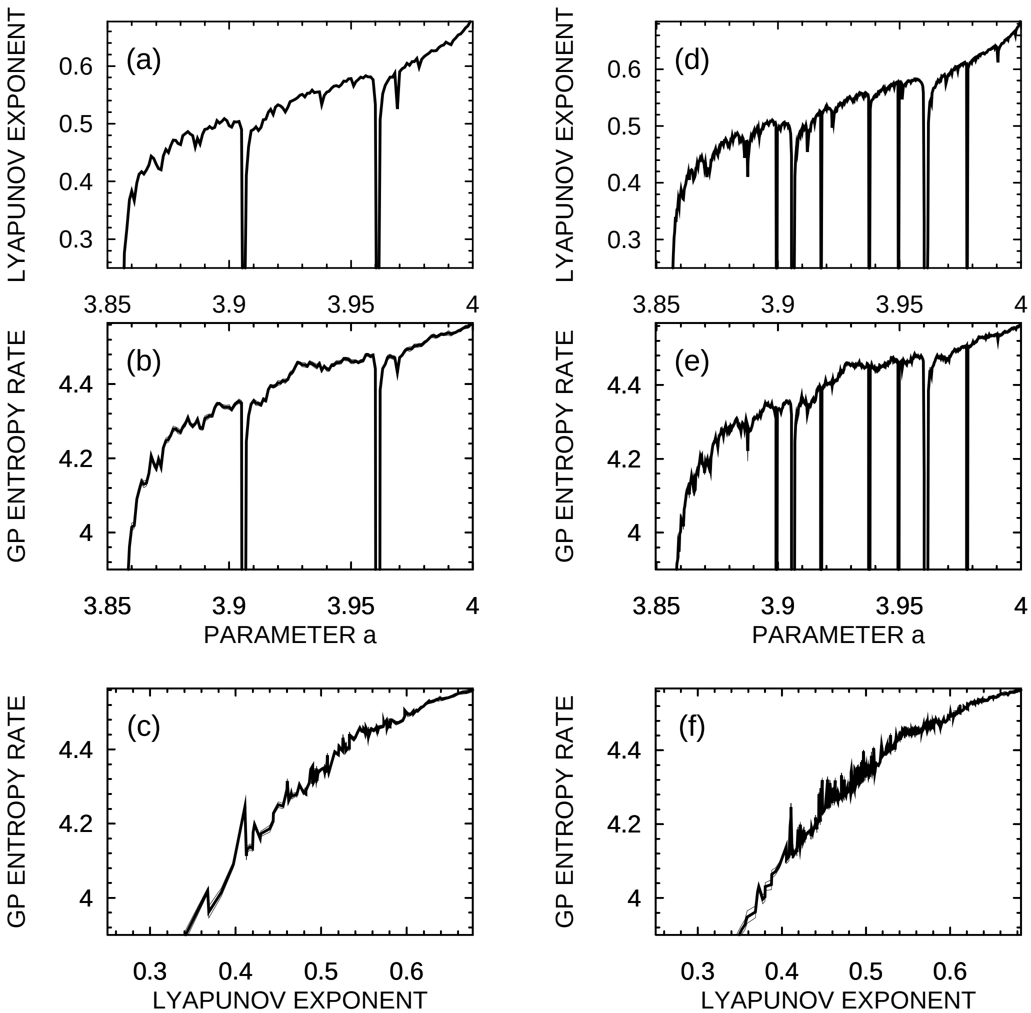

Figure 2: *Results for the logistic map: a) The Lyapunov exponents computed from the map for the parameter a varying from 3.857 to 4 by step 0.001. b) The GP entropy rates estimated from 15 realizations of 16k time series (mean – thick line, mean±SD – thin lines, coinciding with the mean) for different values of the parameter a varying as in plot a. c) Plot of GPER (the same line codes as before) vs. LE. Plots d, e, f: The same as the plots a, b, c, respectively, except of the parameter a varying by step 0.0003.*

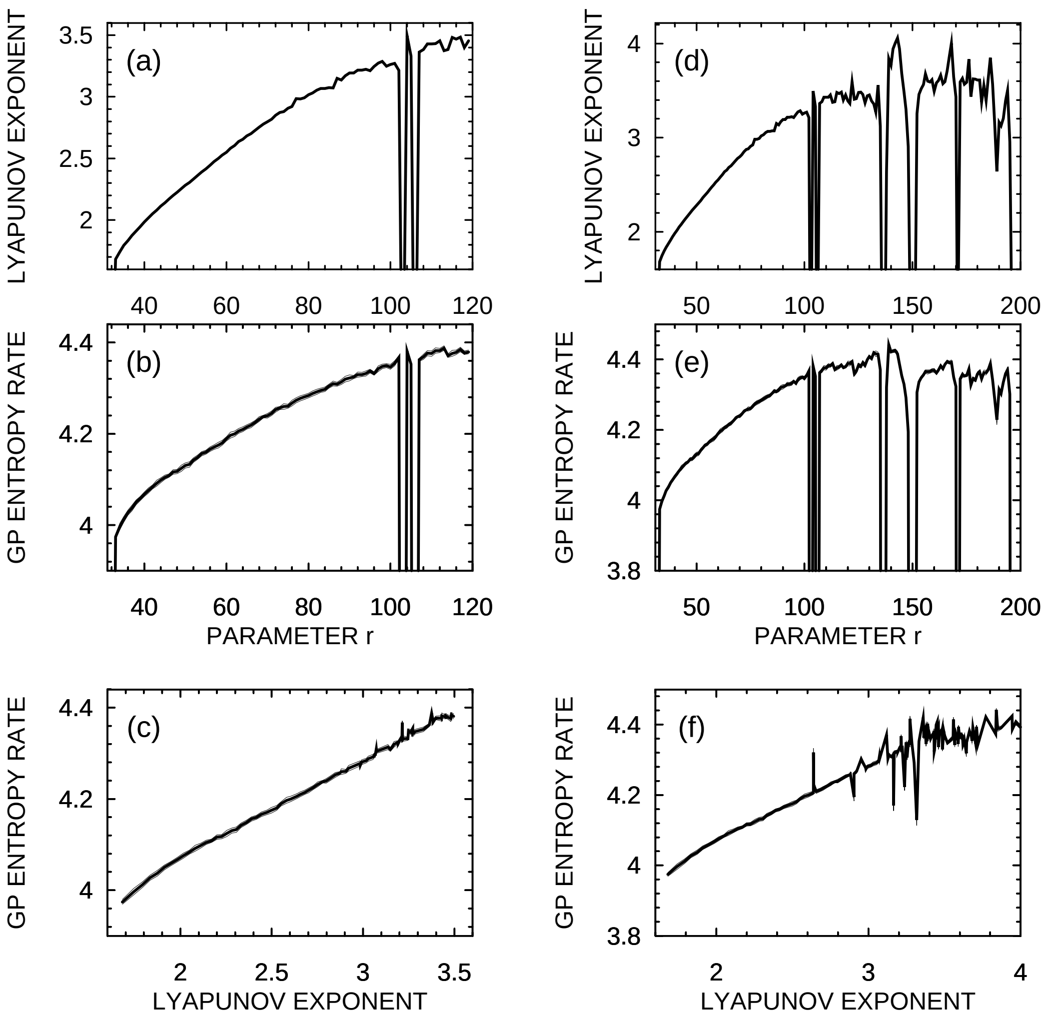

Figure 3: *Further results for the Lorenz system: a) The positive Lyapunov exponents computed from the Lorenz equations for the parameter r varying from 33 to 120 by step 1. b) The GP entropy rates estimated from 15 realizations of 16k time series (mean – thick line, mean±SD – thin lines, coinciding with the mean) for different values of the parameter r varying as in plot a. c) Plot of GPER (the same line codes as before) vs. LE. Plots d, e, f: The same as the plots a, b, c, respectively, except of the parameter r varying from 33 to 200 by step 1.*

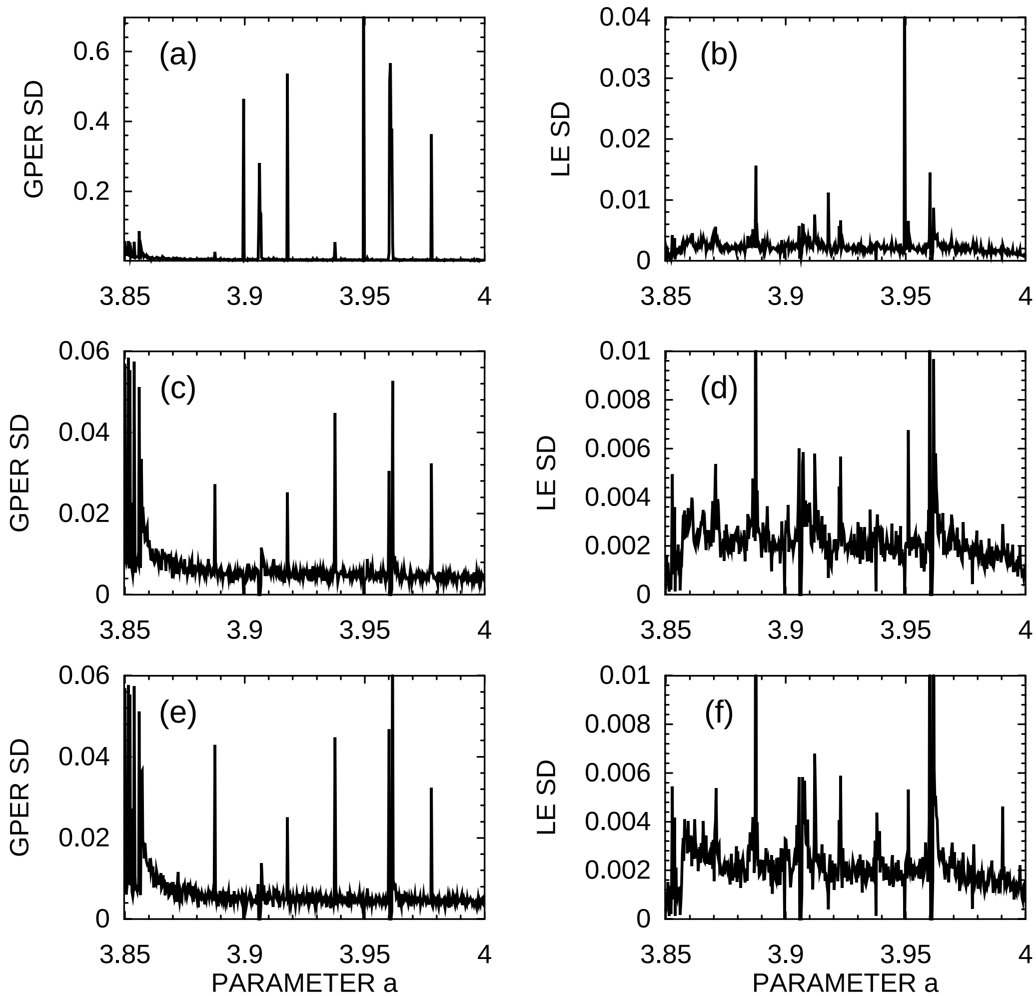

Figure 4: *Standard deviation (square root of variance) of the GPER (a,c,e) and LE (b,d,f) estimates computed from the series generated by the logistic map after skipping zero (a,b), hundred thousand (c,d) and one billion (e,f) initial iterations to avoid influence of transients; plotted as the functions of the parameter a changing in the same range as in Fig. 2d-f.*

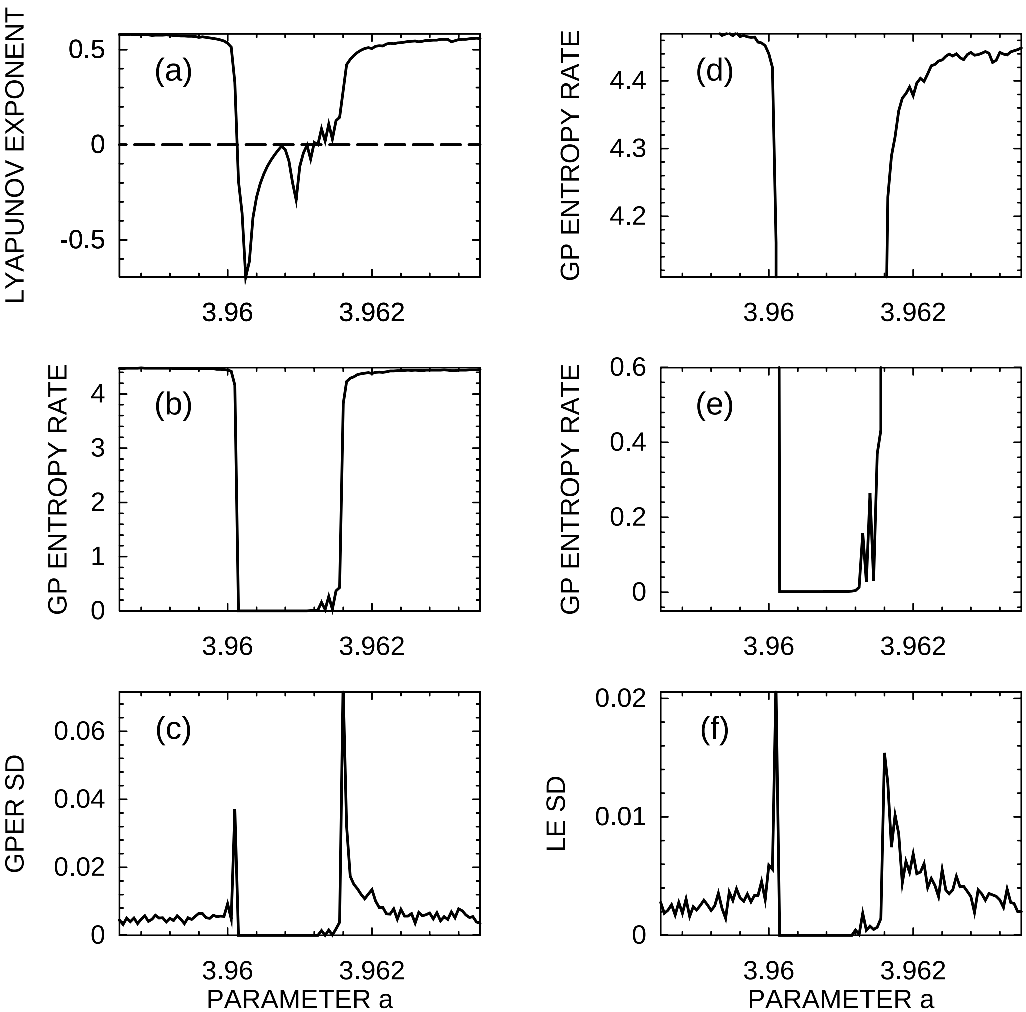

Figure 5: *Detailed illustration of one of the bifurcations of the logistic map. Lyapunov exponent (a), GP entropy rate (b,d,e), standard deviation of the GPER estimate (c) and standard deviation of the LE estimate (f); plotted as functions of the parameter a. Upper and lower parts of the plot b are zoomed in the plots d and e, respectively.*