NUMERICAL STABILITY OF GMRES

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Abstract

In this contribution we consider the GMRES method, the most widely known and used representative of the class of nonsymmetric Krylov subspace method. This method consists of constructing the basis of associated Krylov subspace via the Arnoldi method and then solving the transformed Hessenberg least squares problem at each iteration step. In exact arithmetic the Arnoldi vectors are orthogonal. However, in finite precision computation the orthogonality is lost, which may potentially affect both the convergence rate and the ultimate attainable accuracy of the computed approximate solution. One may therefore want to keep the computed orthogonality as close to the machine precision as possible using proper orthogonal transformations, e.g. Householder orthogonalizations. The price is, unfortunately, too high for most of the applications. The Gram-Schmidt process is a cheaper alternative and its modified version represents the most frequently used compromise. Although, the (classical or modified) Gram-Schmidt orthogonalization may end up with the basis which lost its orthogonality completely, in the GMRES context, however, there is a very important relation between the loss of orthogonality among the Arnoldi vectors and the decrease of the residual of the computed approximation close to its final value. It was proved that, for the modified Gram-Schmidt GMRES, the Arnoldi vectors loose their orthogonality completely only after the residual of the computed approximation is reduced close to its final level of accuracy, which is proportional to the machine precision multiplied by

the condition number of the system matrix. For the classical Gram-Schmidt the corresponding level of limiting accuracy, of course, is significantly different. The modified Gram-Schmidt GMRES however performs almost exactly as well as the Householder implementation and both implementations are backward stable. This suggests that unless the system matrix is extremely ill-conditioned, the use of the Householder or modified Gram-Schmidt GMRES is theoretically well justified. Presented results lead to important conclusions about the parallel implementation and application of the GMRES method. The theoretical analysis has not been finished yet. In the end of our talk we mention some questions related to the rate of convergence of the implementation in the finite precision arithmetic which are open and still need some effort.

References

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