Institute of Computer Science
Academy of Sciences of the Czech Republic

Sparse Test Problems for Unconstrained Optimization

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Abstract:

This report contains a description of subroutines which can be used for testing unconstrained optimization codes. These subroutines can easily be obtained either by using the anonymous `ftp` address `ftp://ftp.cs.cas.cz/pub/msdos/opt` (file `TEST25.FOR`) or from the web homepage `http://www.cs.cas.cz/luksan/test.html`. Furthermore, all test problems contained in these subroutines are presented in the analytic form.

Keywords:
unconstrained optimization, test problems

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1 Introduction

This report describes subroutines TIUD25, TIUS25, TFFU25, TFGU25, TFBU25, which contain 82 sparse problems for testing unconstrained optimization codes. Some of these problems were chosen from [19], but additional problems with sparse Hessian matrices are included. All subroutines are written in the standard Fortran 77 language. Their names are derived from the following rule:

- The first letter is T - test subroutines.
- The second letter is either I - initiation, or F - objective function.
- The third letter is either U - initiation of unconstrained problem, or F - computation of the function value, or G - computation of the gradient vector, or B - computation of both the function value and the gradient vector simultaneously.
- The fourth letter is either U - universal subroutine, or D - subroutine for problems with unknown sparsity patterns, or S - subroutine for problems with given sparsity patterns.

The last two digits determine a given collection (the numbering corresponds to the UFO system [20], which contains similar collections).

Initiation subroutines use the following parameters (array dimensions are given in parentheses):

N	input	number of variables,
M	output	number of elements of the sparse Hessian matrix,
X(N)	output	vector of variables,
IH(N+1)	output	pointers of diagonal elements of the sparse Hessian matrix,
JH(M)	output	column indices of nonzero elements of the sparse Hessian matrix,
FMIN	output	lower bound of the objective function value,
XMAX	output	maximum stepsize,
NEXT	input	number of the problem selected,
IERR	output	error indicator (0 - correct data, 1 - N is too small).

Although N is an input parameter, it can be changed by the initiation subroutine when its value does not satisfy the required conditions. For example, most of the problems require N to be even or a multiple of a positive integer.

Evaluation subroutines use the following parameters (array dimensions are given in parentheses):

N	input	number of variables,
X(N)	input	vector of variables,
F	output	value of the objective function,
G(N)	output	gradient of the objective function,
NEXT	input	number of the problem selected.

2 Test problems for general unconstrained optimization

Calling statements have the form

```
CALL TIUD25(N,X,FMIN,XMAX,NEXT,IERR)
CALL TIUS25(N,M,X,IH,JH,FMIN,XMAX,NEXT,IERR)
CALL TFFU25(N,X,F,NEXT)
CALL TFGU25(N,X,G,NEXT)
CALL TFBU25(N,X,F,G,NEXT)
```

with the following significance:

- `TIUD25` – initiation of vector of variables `X`, which has dimension `N`.
- `TIUS25` – initiation of vector of variables `X` and the pattern of the sparse Hessian matrix `IH, JH`.
- `TFFU25` – evaluation of the general objective function value `F` at the point `X`.
- `TFGU25` – evaluation of the general objective function gradient `G` at the point `X`.
- `TFBU25` – evaluation of the general objective function value `F` and gradient `G` at the point `X`.

We seek a minimum of a general objective function $F(x)$ from the starting point $\bar{x}$. For positive integers k and l, we use the notation $\mathrm{div}(k,l)$ for integer division, i.e., maximum integer not greater than k/l, and $\mathrm{mod}(k,l)$ for the remainder after integer division, i.e., $\mathrm{mod}(k,l) = l(k/l - \mathrm{div}(k,l))$. The description of individual problems follows.

Problem 1. Chained Rosenbrock function [7].

$$\begin{aligned} F(x) &= \sum_{i=2}^{n} [100(x_{i-1}^2 - x_i)^2 + (x_{i-1} - 1)^2], \\ \overline{x}_i &= -1.2, \quad \mathrm{mod}(i,2) = 1, \qquad \overline{x}_i = 1.0, \quad \mathrm{mod}(i,2) = 0. \end{aligned}$$

Problem 2. Chained Wood function [7].

$$\begin{aligned} F(x) &= \sum_{j=1}^{k} [100(x_{i-1}^2 - x_i)^2 + (x_{i-1} - 1)^2 + 90(x_{i+1}^2 - x_{i+2})^2 \\ &+ (x_{i+1} - 1)^2 + 10(x_i + x_{i+2} - 2)^2 + (x_i - x_{i+2})^2/10], \\ i &= 2j, \quad k = (n-2)/2, \\ \overline{x}_i &= -3, \quad \mathrm{mod}(i,2) = 1, \quad i \le 4, \qquad \overline{x}_i = -2, \quad \mathrm{mod}(i,2) = 1, \quad i > 4, \\ \overline{x}_i &= -1, \quad \mathrm{mod}(i,2) = 0, \quad i \le 4, \qquad \overline{x}_i = 0, \quad \mathrm{mod}(i,2) = 0, \quad i > 4. \end{aligned}$$

Problem 3. Chained Powel singular function [7].

$$F(x) = \sum_{j=1}^{k} \left[(x_{i-1} + 10x_i)^2 + 5(x_{i+1} - x_{i+2})^2 + (x_i - 2x_{i+1})^4 + 10(x_{i-1} - x_{i+2})^4 \right],$$

$$
\begin{aligned}
i &= 2j, \quad k = (n-2)/2, \\
\overline{x}_i &= 3, \ \operatorname{mod}(i,4) = 1, \quad \overline{x}_i = -1, \ \operatorname{mod}(i,4) = 2, \\
\overline{x}_i &= 0, \ \operatorname{mod}(i,4) = 3, \quad \overline{x}_i = 1, \ \operatorname{mod}(i,4) = 0.
\end{aligned}
$$

Problem 4. Chained Cragg and Levy function [7].

$$
\begin{aligned}
F(x) &= \sum_{j=1}^{k} \left[(\exp(x_{i-1}) - x_i)^4 + 100(x_i - x_{i+1})^6 + \tan^4(x_{i+1} - x_{i+2}) + x_{i-1}^8 + (x_{i+2} - 1)^2 \right], \\
i &= 2j, \quad k = (n-2)/2, \\
\overline{x}_i &= 1, \ i = 1, \quad \overline{x}_i = 2, \ i > 1.
\end{aligned}
$$

Problem 5. Generalized Broyden tridiagonal function [22].

$$
\begin{aligned}
F(x) &= \sum_{i=1}^{n} |(3 - 2x_i)x_i - x_{i-1} - x_{i+1} + 1|^p, \\
p &= 7/3, \ x_0 = x_{n+1} = 0, \\
\overline{x}_i &= -1, \ i \geq 1.
\end{aligned}
$$

Problem 6. Generalized Broyden banded function [22].

$$
\begin{aligned}
F(x) &= \sum_{i=1}^{n} \left| (2 + 5x_i^2)x_i + 1 + \sum_{j \in J_i} x_j(1 + x_j) \right|^p, \\
p &= 7/3, \ J_i = \{ j : j \neq i, \ \max(1, i-5) \leq j \leq \min(n, i+1) \}, \\
\overline{x}_i &= -1, \ i \geq 1.
\end{aligned}
$$

Problem 7. Seven-diagonal generalization of the Broyden tridiagonal function [7].

$$
\begin{aligned}
F(x) &= \sum_{i=1}^{n} |(3 - 2x_i)x_i - x_{i-1} - x_{i+1} + 1|^p + \sum_{i=1}^{n/2} \left| x_i + x_{i+n/2} \right|^p, \\
p &= 7/3, \ x_0 = x_{n+1} = 0, \\
\overline{x}_i &= -1, \ i \geq 1.
\end{aligned}
$$

Problem 8. Sparse modification of the Nazareth trigonometric function.

$$
\begin{aligned}
F(x) &= \frac{1}{n} \sum_{i=1}^{n} \left(n + i - \sum_{j \in J_i} (a_{ij} \sin x_j + b_{ij} \cos x_j) \right)^2, \\
a_{ij} &= 5[1 + \operatorname{mod}(i,5) + \operatorname{mod}(j,5)], \quad b_{ij} = (i+j)/10, \\
J_i &= \{ j : \max(1, i-2) \leq j \leq \min(n, i+2) \} \cup \{ j : |j - i| = n/2 \}, \\
\overline{x}_i &= 1/n, \ i \geq 1.
\end{aligned}
$$

Problem 9. Another trigonometric function.

$$
\begin{aligned}
F(x) &= \frac{1}{n} \sum_{i=1}^{n} \left(i(1 - \cos x_i) + \sum_{j \in J_i} (a_{ij} \sin x_j + b_{ij} \cos x_j) \right), \\
a_{ij} &= 5[1 + \operatorname{mod}(i,5) + \operatorname{mod}(j,5)], \quad b_{ij} = (i+j)/10, \\
J_i &= \{ j : \max(1, i-2) \leq j \leq \min(n, i+2) \} \cup \{ j : |j - i| = n/2 \}, \\
\overline{x}_i &= 1/n, \ i \geq 1.
\end{aligned}
$$

Problem 10. Toint trigonometric function [26].

$$\begin{aligned}
F(x) &= \frac{1}{n}\sum_{i=1}^{n}\sum_{j\in J_i} a_{ij}\sin(b_{ij}+c_ix_i+c_jx_j),\\
a_{ij} &= 5[1+\mathrm{mod}(i,5)+\mathrm{mod}(j,5)], \quad b_{ij}=(i+j)/10,\\
c_i &= 1+i/10, \quad c_j = 1+j/10,\\
J_i &= \{j:\max(1,i-2)\le j\le \min(n,i+2)\}\cup\{j:|j-i|=n/2\},\\
\overline{x}_i &= 1, \;\; i\ge 1.
\end{aligned}$$

Problem 11. Augmented Lagrangian function [7].

$$\begin{aligned}
F(x) &= \sum_{i\in J}\left\{\exp\left(\prod_{j=1}^{5} x_{i+1-j}\right)+10\left[\left(\sum_{j=1}^{5} x_{i+1-j}^2-10-\lambda_1\right)^2\right.\right.\\
&+ \left.\left.(x_{i-3}x_{i-2}-5x_{i-1}x_i-\lambda_2)^2+(x_{i-4}^3+x_{i-3}^3+1-\lambda_3)^2\right]\right\},\\
\lambda_1 &= -0.002008, \quad \lambda_2 = -0.001900, \quad \lambda_3 = -0.000261,\\
J &= \{i,\ \mathrm{mod}(i,5)=0\},\\
\overline{x}_i &= -2,\ \mathrm{mod}(i,5)=1,\ i\le 2, \quad \overline{x}_i=-1,\ \mathrm{mod}(i,5)=1,\ i>2,\\
\overline{x}_i &= 2,\ \mathrm{mod}(i,5)=2,\ i\le 2, \quad \overline{x}_i=-1,\ \mathrm{mod}(i,5)=2,\ i>2,\\
\overline{x}_i &= 2,\ \mathrm{mod}(i,5)=3, \quad \overline{x}_i=-1,\ \mathrm{mod}(i,5)=4,\\
\overline{x}_i &= -1,\ \mathrm{mod}(i,5)=0.
\end{aligned}$$

Problem 12. Generalization of the Brown function 1 [7].

$$\begin{aligned}
F(x) &= \sum_{i=2}^{n}\left[(x_{i-1}-3)^2+(x_{i-1}-x_i)^2+\exp(20(x_{i-1}-x_i))\right]\\
\overline{x}_i &= 0,\ \mathrm{mod}(i,2)=1, \quad \overline{x}_i = -1,\ \mathrm{mod}(i,2)=0.
\end{aligned}$$

Problem 13. Generalization of the Brown function 2 [7].

$$\begin{aligned}
F(x) &= \sum_{i=2}^{n}\left[(x_{i-1}^2)^{(x_i^2+1)}+(x_i^2)^{(x_{i-1}^2+1)}\right],\\
\overline{x}_i &= -1,\ \mathrm{mod}(i,2)=1, \quad \overline{x}_i = 1,\ \mathrm{mod}(i,2)=0.
\end{aligned}$$

Problem 14. Discrete boundary value problem [22].

$$\begin{aligned}
F(x) &= \sum_{i=1}^{n}\left[2x_i-x_{i-1}-x_{i+1}+h^2(x_i+ih+1)^3/2\right]^2,\\
h &= 1/(n+1), \quad x_0 = x_{n+1}=0,\\
\overline{x}_i &= ih(1-ih), \;\; i\ge 1.
\end{aligned}$$

Problem 15. Discretization of a variational problem [26].

$$\begin{aligned}
F(x) &= 2\sum_{i=1}^{n}\left[(x_i(x_i-x_{i+1})]/h+2h\sum_{i=0}^{n}[(\exp(x_{i+1})-\exp(x_i))/(x_{i+1}-x_i)\right],\\
h &= 1/(n+1), \quad x_0 = x_{n+1}=0,\\
\overline{x}_i &= ih(1-ih), \;\; i\ge 1.
\end{aligned}$$

Problem 16. Banded trigonometric problem.

$$\begin{aligned} F(x) &= \sum_{i=1}^{n} i\left[(1-\cos x_i)+\sin x_{i-1}-\sin x_{i+1}\right], \\ x_0 &= x_{n+1}=0, \\ \overline{x}_i &= 1, \quad i \geq 1. \end{aligned}$$

Problem 17. Variational problem 1 [9].

This problem is a finite difference analogue of a variational problem defined as a minimization of the functional

$$F(x)=\int_0^1 \left[\frac{1}{2}\dot{x}^2(t)+\exp(x(t))-1\right] dt,$$

where $x(0)=0$ and $x(1)=0$. We use the trapeziodal rule together with 3-point finite differences on a uniform grid having $n+1$ internal nodes. The starting point is given by the formula $\overline{x}_i=x(t_i)=ih(1-ih)$, where $h=1/(n+1)$.

Problem 18. Variational problem 2 [9].

This problem is a finite difference analogue of a variational problem defined as a minimization of the functional

$$F(x)=\int_0^1 \left[\dot{x}^2(t)-x^2(t)-2tx(t)\right] dt,$$

where $x(0)=0$ and $x(1)=0$. We use the trapeziodal rule together with 3-point finite differences on a uniform grid having $n+1$ internal nodes. The starting point is given by the formula $\overline{x}_i=x(t_i)=ih(1-ih)$, where $h=1/(n+1)$.

Problem 19. Variational problem 3 [9].

This problem is a finite difference analogue of a variational problem defined as a minimization of the functional

$$F(x)=\int_0^1 \left[\dot{x}^2(t)+x^2(t)+2x(t)\exp(2t)\right] dt,$$

where $x(0)=1/3$ and $x(1)=\exp(2/3)$. We use the trapeziodal rule together with 3-point finite differences on a uniform grid having $n+1$ internal nodes. The starting point is given by the formula $\overline{x}_i=x(t_i)=(ih\exp(2)+1)/3$, where $h=1/(n+1)$.

Problem 20. Variational problem 4 [9] (Calvar 3 [10]).

This problem is a finite difference analogue of a variational problem defined as a minimization of the functional

$$F(x)=\int_0^1 \left[\exp(-2x^2(t))(\dot{x}^2(t)-1)\right] dt,$$

where $x(0)=1$ and $x(1)=0$. We use the trapeziodal rule together with 3-point finite differences on a uniform grid having $n+1$ internal nodes. The starting point is given by the formula $\overline{x}_i=x(t_i)=1-ih$, where $h=1/(n+1)$.

Problem 21. Variational problem 5 [9] (Calvar 1 [10]).

This problem is a finite difference analogue of a variational problem defined as a minimization of the functional

$$F(x) = \int_0^1 \left[x^2(t) + \dot{x}(t) \arctan \dot{x}(t) - \log \sqrt{1 + \dot{x}^2(t)} \right] dt,$$

where $x(0) = 1$ and $x(1) = 2$. We use the trapeziodal rule together with 3-point finite differences on a uniform grid having $n+1$ internal nodes. The starting point is given by the formula $\overline{x}_i = x(t_i) = ih + 1$, where $h = 1/(n+1)$.

Problem 22. Variational problem Calvar 2 [10].

This problem is a finite difference analogue of a variational problem defined as a minimization of the functional

$$F(x) = \int_0^1 \left[100(x(t) - \dot{x}^2(t))^2 + (1 - \dot{x}(t))^2 \right] dt,$$

where $x(0) = 0$ and $x(1) = 0$. We use the trapeziodal rule together with 3-point finite differences on a uniform grid having $n+1$ internal nodes. The starting point is given by the formula $\overline{x}_i = x(t_i) = ih(1 - ih)$, where $h = 1/(n+1)$.

Problem 23. Extended Rosenbrock function [22].

$$\begin{aligned}
F(x) &= \frac{1}{2} \sum_{k=1}^{n} f_k^2(x), \\
f_k(x) &= 10(x_k^2 - x_{k+1}) \quad , \quad \mathrm{mod}(k,2) = 1 \\
f_k(x) &= x_{k-1} - 1 \quad , \quad \mathrm{mod}(k,2) = 0 \\
\bar{x}_l &= -1.2, \quad \mathrm{mod}(l,2) = 1, \qquad \bar{x}_l = 1.0, \quad \mathrm{mod}(l,2) = 0
\end{aligned}$$

Problem 24. Extended Powell singular function [22].

$$\begin{aligned}
F(x) &= \frac{1}{2} \sum_{k=1}^{n} f_k^2(x), \\
f_k(x) &= x_k + 10x_{k+1} \quad , \quad \mathrm{mod}(k,4) = 1, \\
f_k(x) &= \sqrt{5}\,(x_{k+2} - x_{k+3}) \quad , \quad \mathrm{mod}(k,4) = 2, \\
f_k(x) &= (x_{k+1} - 2x_{k+2})^2 \quad , \quad \mathrm{mod}(k,4) = 3, \\
f_k(x) &= \sqrt{10}\,(x_k - x_{k+3})^2 \quad , \quad \mathrm{mod}(k,4) = 0, \\
\bar{x}_l &= 3, \quad \mathrm{mod}(l,4) = 1, \qquad \bar{x}_l = -1, \quad \mathrm{mod}(l,4) = 2, \\
\bar{x}_l &= 0, \quad \mathrm{mod}(l,4) = 3, \qquad \bar{x}_l = 1, \quad \mathrm{mod}(l,4) = 0.
\end{aligned}$$

Problem 25. Broyden tridiagonal function [22].

$$\begin{aligned}
F(x) &= \frac{1}{2} \sum_{k=1}^{m} f_k^2(x), \\
f_k(x) &= (3 - 2x_k)x_k - x_{k-1} - 2x_{k+1} + 1, \\
m &= n, \quad x_0 = x_{n+1} = 0\,, \\
\overline{x}_l &= -1, \quad l \geq 1.
\end{aligned}$$

Problem 26. Problem 201 in [25].

$$\begin{aligned}
F(x) &= \frac{1}{2}\sum_{k=1}^{n} f_k^2(x)\,, \\
f_k(x) &= x_k - 1 \qquad\qquad\qquad , \quad k = 1\,, \\
f_k(x) &= 10(k-1)(x_k - x_{k-1})^2, \quad 1 < k \le n\,, \\
\overline{x}_l &= -1.2, \quad 1 \le l < n, \quad x_l = -1, \quad l = n\,.
\end{aligned}$$

Problem 27. Generalized Broyden tridiagonal function [16].

$$\begin{aligned}
F(x) &= \frac{1}{2}\sum_{k=1}^{m} f_k^2(x), \\
f_k(x) &= (3 - 2x_k)\, x_k + 1 - x_{k-1} - x_{k+1}, \\
m &= n, \quad x_0 = x_{n+1} = 0, \\
\bar{x}_l &= -1, \quad l \ge 1.
\end{aligned}$$

Problem 28. Generalized Broyden banded function [16].

$$\begin{aligned}
F(x) &= \frac{1}{2}\sum_{k=1}^{m} f_k^2(x), \\
f_k(x) &= (2 + 5x_k^2)x_k + 1 + \sum_{j \in J_k} x_j(1 + x_j), \\
m &= n, \quad J_k = \{j : j \ne k,\ \max(1, k-5) \le j \le \min(n, k+1)\}, \\
\bar{x}_l &= -1, \quad l \ge 1.
\end{aligned}$$

Problem 29. Chained Freudenstein and Roth function [28].

$$\begin{aligned}
F(x) &= \frac{1}{2}\sum_{k=1}^{m} f_k^2(x), \\
f_k(x) &= x_i + x_{i+1}((5 - x_{i+1})x_{i+1} - 2) - 13\ , \quad \mathrm{mod}(k, 2) = 1, \\
f_k(x) &= x_i + x_{i+1}((1 + x_{i+1})x_{i+1} - 14) - 29, \quad \mathrm{mod}(k, 2) = 0, \\
m &= 2(n-1), \quad i = \mathrm{div}(k+1, 2), \\
\bar{x}_l &= 0.5, \quad l < n, \qquad \bar{x}_l = -2, \quad l = n.
\end{aligned}$$

Problem 30. Wright and Holt zero residual problem [29].

$$\begin{aligned}
F(x) &= \frac{1}{2}\sum_{k=1}^{m} f_k^2(x), \\
f_k(x) &= (x_i^a - x_j^b)^c, \\
a &= 1, \qquad k \le m/2, \qquad a = 2, \quad k > m/2, \\
b &= 5 - \mathrm{div}(k, m/4), \qquad c = \mathrm{mod}(k, 5) + 1, \\
m &= 5n, \quad i = \mathrm{mod}(k, n/2) + 1, \quad j = i + n/2, \\
\bar{x}_l &= \sin^2(l), \quad l \ge 1.
\end{aligned}$$

Problem 31. Toint quadratic merging problem [28].

$$
\begin{aligned}
F(x) &= \frac{1}{2}\sum_{k=1}^{m} f_k^2(x), \\
f_k(x) &= x_i + 3x_{i+1}(x_{i+2}-1) + x_{i+3}^2 - 1 &&, \quad \bmod(k,6)=1, \\
f_k(x) &= (x_i + x_{i+1})^2 + (x_{i+2}-1)^2 - x_{i+3} - 3 &&, \quad \bmod(k,6)=2, \\
f_k(x) &= x_i x_{i+1} - x_{i+2}x_{i+3} &&, \quad \bmod(k,6)=3, \\
f_k(x) &= 2x_i x_{i+2} + x_{i+1}x_{i+3} - 3 &&, \quad \bmod(k,6)=4, \\
f_k(x) &= (x_i + x_{i+1} + x_{i+2} + x_{i+3})^2 + (x_i - 1)^2 &&, \quad \bmod(k,6)=5, \\
f_k(x) &= x_i x_{i+1} x_{i+2} x_{i+3} + (x_{i+3}-1)^2 - 1 &&, \quad \bmod(k,6)=0, \\
m &= 3(n-2), \quad i = 2\,\mathrm{div}(k+5,6) - 1, \\
\bar{x}_l &= 5, \quad l \geq 1.
\end{aligned}
$$

Problem 32. Chained exponential problem [16].

$$
\begin{aligned}
F(x) &= \frac{1}{2}\sum_{k=1}^{m} f_k^2(x), \\
f_k(x) &= 4 - \exp(x_i) - \exp(x_{i+1}) &&, \quad \bmod(k,2)=1, \ \ i=1, \\
f_k(x) &= 8 - \exp(3x_{i-1}) - \exp(3x_i) \\
&+ 4 - \exp(x_i) - \exp(x_{i+1}) &&, \quad \bmod(k,2)=1, \ \ 1<i<n, \\
f_k(x) &= 8 - \exp(3x_{i-1}) - \exp(3x_i) &&, \quad \bmod(k,2)=1, \ \ i=n, \\
f_k(x) &= 6 - \exp(2x_i) - \exp(2x_{i+1}) &&, \quad \bmod(k,2)=0, \\
m &= 2n-1, \quad i = \mathrm{div}(k+1,2), \\
\bar{x}_l &= 0.2, \quad l \geq 1.
\end{aligned}
$$

Problem 33. Chained serpentine function [17].

$$
\begin{aligned}
F(x) &= \frac{1}{2}\sum_{k=1}^{m} f_k^2(x), \\
f_k(x) &= 10(2x_i/(1+x_i^2) - x_{i+1}) &&, \quad \bmod(k,2)=1, \\
f_k(x) &= x_i - 1 &&, \quad \bmod(k,2)=0, \\
m &= 2(n-1), \quad i = \mathrm{div}(k+1,2), \\
\bar{x}_l &= -0.8, \quad l \geq 1.
\end{aligned}
$$

Problem 34. Chained and modified problem HS47 [17].

$$
\begin{aligned}
F(x) &= \frac{1}{2}\sum_{k=1}^{m} f_k^2(x), \\
f_k(x) &= 10(x_i^2 - x_{i+1}) &&, \quad \bmod(k,6)=1, \\
f_k(x) &= x_{i+2} - 1 &&, \quad \bmod(k,6)=2, \\
f_k(x) &= (x_{i+3} - 1)^2 &&, \quad \bmod(k,6)=3, \\
f_k(x) &= (x_{i+4} - 1)^3 &&, \quad \bmod(k,6)=4,
\end{aligned}
$$

$$
\begin{aligned}
f_k(x) &= x_i^2 x_{i+3} + \sin(x_{i+3} - x_{i+4}) - 10 \ , \quad \mathrm{mod}(k,6) = 5, \\
f_k(x) &= x_{i+1} + x_{i+2}^4 x_{i+3}^2 - 20 \qquad\qquad , \quad \mathrm{mod}(k,6) = 0, \\
m &= 6(\mathrm{div}(n-5,3)+1), \quad i = 3\ \mathrm{div}(k+5,6) - 2, \\
\bar{x}_l &= -1, \quad l \geq 1.
\end{aligned}
$$

Problem 35. Chained and modified problem HS48 [17].

$$
\begin{aligned}
F(x) &= \frac{1}{2} \sum_{k=1}^{m} f_k^2(x), \\
f_k(x) &= 10(x_i^2 - x_{i+1}) \qquad , \quad \mathrm{mod}(k,7) = 1, \\
f_k(x) &= 10(x_{i+1}^2 - x_{i+2}) \qquad , \quad \mathrm{mod}(k,7) = 2, \\
f_k(x) &= (x_{i+2} - x_{i+3})^2 \qquad , \quad \mathrm{mod}(k,7) = 3, \\
f_k(x) &= (x_{i+3} - x_{i+4})^2 \qquad , \quad \mathrm{mod}(k,7) = 4, \\
f_k(x) &= x_i + x_{i+1}^2 + x_{i+2} - 30 \quad , \quad \mathrm{mod}(k,7) = 5, \\
f_k(x) &= x_{i+1} - x_{i+2}^2 + x_{i+3} - 10 \ , \quad \mathrm{mod}(k,7) = 6, \\
f_k(x) &= x_i x_{i+4} - 10 \qquad , \quad \mathrm{mod}(k,7) = 0, \\
m &= 7(\mathrm{div}(n-5,3)+1), \quad i = 3\ \mathrm{div}(k+6,7) - 2, \\
\bar{x}_l &= -1, \quad l \geq 1.
\end{aligned}
$$

Problem 36. Sparse signomial function [17].

$$
\begin{aligned}
F(x) &= \frac{1}{2} \sum_{k=1}^{m} f_k^2(x), \\
f_k(x) &= y_j - \sum_{p=1}^{3} (p^2/j) \prod_{q=1}^{4} \mathrm{sign}(x_{i+q}) |x_{i+q}|^{q/(pj)}, \\
m &= 4(\mathrm{div}(n-4,2)+1), \quad i = 2\ \mathrm{div}(k+3,4) - 2, \quad j \ = \ \mathrm{mod}(k-1,4)+1, \\
\bar{x}_l &= -0.8 \quad , \quad \mathrm{mod}(l,4) = 1, \quad y_1 = 14.4, \\
\bar{x}_l &= 1.2 \quad , \quad \mathrm{mod}(l,4) = 2, \quad y_2 = 6.8, \\
\bar{x}_l &= -1.2 \quad , \quad \mathrm{mod}(l,4) = 3, \quad y_3 = 4.2, \\
\bar{x}_l &= 0.8 \quad , \quad \mathrm{mod}(l,4) = 0, \quad y_4 = 3.2.
\end{aligned}
$$

Problem 37. Sparse exponential function [17].

$$
\begin{aligned}
F(x) &= \frac{1}{2} \sum_{k=1}^{m} f_k^2(x), \\
f_k(x) &= y_j - \sum_{p=1}^{3} (p^2/j) \exp\left(\sum_{q=1}^{4} x_{i+q} q/(pj) \right), \\
m &= 4(\mathrm{div}(n-4,2)+1), \quad i = 2\ \mathrm{div}(k+3,4) - 2, \quad j \ = \ \mathrm{mod}(k-1,4)+1, \\
\bar{x}_l &= -0.8 \quad , \quad \mathrm{mod}(l,4) = 1, \quad y_1 = 35.8, \\
\bar{x}_l &= 1.2 \quad , \quad \mathrm{mod}(l,4) = 2, \quad y_2 = 11.2, \\
\bar{x}_l &= -1.2 \quad , \quad \mathrm{mod}(l,4) = 3, \quad y_3 = 6.2, \\
\bar{x}_l &= 0.8 \quad , \quad \mathrm{mod}(l,4) = 0, \quad y_4 = 4.4.
\end{aligned}
$$

Problem 38. Sparse trigonometric function [17].

$$
\begin{aligned}
F(x) &= \frac{1}{2}\sum_{k=1}^{m} f_k^2(x), \\
f_k(x) &= y_j - \sum_{q=1}^{4}[(-1)^q j q^2 \sin(x_{i+q}) + j^2 q \cos(x_{i+q})], \\
m &= 4(\mathrm{div}(n-4,2)+1), \quad i = 2\,\mathrm{div}(k+3,4)-2, \quad j = \mathrm{mod}(k-1,4)+1, \\
\bar{x}_i &= -0.8 \quad , \quad \mathrm{mod}(l,4)=1, \quad y_1 = 30.6, \\
\bar{x}_i &= 1.2 \quad , \quad \mathrm{mod}(l,4)=2, \quad y_2 = 72.2, \\
\bar{x}_i &= -1.2 \quad , \quad \mathrm{mod}(l,4)=3, \quad y_3 = 124.4, \\
\bar{x}_i &= 0.8 \quad , \quad \mathrm{mod}(l,4)=0, \quad y_4 = 187.4.
\end{aligned}
$$

Problem 39. Countercurrent reactors problem 1 [6] (modified).

$$
\begin{aligned}
F(x) &= \frac{1}{2}\sum_{k=1}^{n} f_k^2(x), \\
f_k(x) &= \alpha - (1-\alpha)x_{k+2} - x_k(1+4x_{k+1}) \quad , \ k=1, \\
f_k(x) &= -(2-\alpha)x_{k+2} - x_k(1+4x_{k-1}) \quad , \ k=2, \\
f_k(x) &= \alpha x_{k-2} - (1-\alpha)x_{k+2} - x_k(1+4x_{k+1}) \quad , \ \mathrm{mod}\ (k,2)=1\ , \ 2<k<n-1, \\
f_k(x) &= \alpha x_{k-2} - (2-\alpha)x_{k+2} - x_k(1+4x_{k-1}) \quad , \ \mathrm{mod}\ (k,2)=0\ , \ 2<k<n-1, \\
f_k(x) &= \alpha x_{k-2} - x_k(1+4x_{k+1}) \quad , \ k=n-1, \\
f_k(x) &= \alpha x_{k-2} - (2-\alpha) - x_k(1+4x_{k-1}) \quad , \ k=n, \\
\alpha &= 1/2, \\
\overline{x}_l &= 0.1 \quad , \ \mathrm{mod}\ (l,8)=1, \quad \overline{x}_l = 0.2 \quad , \ \mathrm{mod}\ (l,8)=2, \\
\overline{x}_l &= 0.3 \quad , \ \mathrm{mod}\ (l,8)=3, \quad \overline{x}_l = 0.4 \quad , \ \mathrm{mod}\ (l,8)=4, \\
\overline{x}_l &= 0.5 \quad , \ \mathrm{mod}\ (l,8)=5, \quad \overline{x}_l = 0.4 \quad , \ \mathrm{mod}\ (l,8)=6, \\
\overline{x}_l &= 0.3 \quad , \ \mathrm{mod}\ (l,8)=7, \quad \overline{x}_l = 0.2 \quad , \ \mathrm{mod}\ (l,8)=0.
\end{aligned}
$$

Problem 40. Tridiagonal system [15].

$$
\begin{aligned}
F(x) &= \frac{1}{2}\sum_{k=1}^{n} f_k^2(x), \\
f_k(x) &= 4(x_k - x_{k+1}^2) \quad , \ k=1, \\
f_k(x) &= 8x_k(x_k^2 - x_{k-1}) - 2(1-x_k) + 4(x_k - x_{k+1}^2) \quad , \ 1<k<n, \\
f_k(x) &= 8x_k(x_k^2 - x_{k-1}) - 2(1-x_k) \quad , \ k=n, \\
\overline{x}_l &= 12, \quad l \geq 1.
\end{aligned}
$$

Problem 41. Structured Jacobian problem [11].

$$
F(x) = \frac{1}{2}\sum_{k=1}^{n} f_k^2(x),
$$

$$
\begin{aligned}
f_k(x) &= -2x_k^2 + 3x_k - 2x_{k+1} + 3x_{n-4} - x_{n-3} \\
&- \; x_{n-2} + 0.5x_{n-1} - x_n + 1 \qquad , \; k = 1, \\
f_k(x) &= -2x_k^2 + 3x_k - x_{k-1} - 2x_{k+1} + 3x_{n-4} - x_{n-3} \\
&- \; x_{n-2} + 0.5x_{n-1} - x_n + 1 \qquad , \; 1 < k < n, \\
f_k(x) &= -2x_k^2 + 3x_k - x_{k-1} + 3x_{n-4} - x_{n-3} \\
&- \; x_{n-2} + 0.5x_{n-1} - x_n + 1 \qquad , \; k = n, \\
\overline{x}_l &= -1, \quad l \geq 1.
\end{aligned}
$$

Problem 42. Modified discrete boundary value problem [17].

$$
\begin{aligned}
F(x) &= \frac{1}{2} \sum_{k=1}^{n} f_k^2(x), \\
f_k(x) &= 2x_k + (1/2)h^2(x_k + hk + 1)^3 - x_{k-1} - x_{k+1} + 1, \\
h &= 1/(n+1), \quad x_0 = x_{n+1} = 0, \\
\bar{x}_l &= lh(lh - 1), \quad l \geq 1.
\end{aligned}
$$

Problem 43. Chained and modified problem HS48 [17].

$$
\begin{aligned}
F(x) &= \frac{1}{2} \sum_{k=1}^{m} f_k^2(x), \\
f_k(x) &= 10(x_i^2 - x_{i+1}) \quad , \quad \mathrm{mod}(k, 7) = 1, \\
f_k(x) &= x_{i+1} + x_{i+2} - 2 \quad , \quad \mathrm{mod}(k, 7) = 2, \\
f_k(x) &= x_{i+3} - 1 \quad , \quad \mathrm{mod}(k, 7) = 3, \\
f_k(x) &= x_{i+4} - 1 \quad , \quad \mathrm{mod}(k, 7) = 4, \\
f_k(x) &= x_i + 3x_{i+1} \quad , \quad \mathrm{mod}(k, 7) = 5, \\
f_k(x) &= x_{i+2} + x_{i+3} - 2x_{i+4} \quad , \quad \mathrm{mod}(k, 7) = 6, \\
f_k(x) &= 10(x_{i+1}^2 - x_{i+4}) \quad , \quad \mathrm{mod}(k, 7) = 0, \\
m &= 7(\mathrm{div}(n-5, 3) + 1) \quad , \quad i = 3\,\mathrm{div}(k+6, 7) - 2, \\
\bar{x}_l &= -1, \quad l \geq 1.
\end{aligned}
$$

Problem 44. Attracting-Repelling problem [17].

$$
\begin{aligned}
F(x) &= \frac{1}{2} \sum_{k=1}^{m} f_k^2(x), \\
f_k(x) &= x_1 - 1 \quad , \quad k = 1, \\
f_k(x) &= 10(x_i^2 - x_{i+1}), \quad , \quad k > 1, \quad \mathrm{mod}(k, 2) = 0, \\
f_k(x) &= 2\exp(-(x_i - x_{i+1})^2) + \exp(-2(x_{i+1} - x_{i+2})^2) \; , \quad k > 1, \quad \mathrm{mod}(k, 2) = 1, \\
m &= 2(n-1), \quad i = \mathrm{div}(k, 2), \\
\bar{x}_l &= -1.2 \;\; , \;\; \mathrm{mod}(l, 2) = 1, \qquad \bar{x}_l = 1.0 \;\; , \;\; \mathrm{mod}(l, 2) = 0.
\end{aligned}
$$

Problem 45. Countercurrent reactors problem 2 [6] (modified).

$$
F(x) = \frac{1}{2} \sum_{k=1}^{n} f_k^2(x),
$$

$$
\begin{aligned}
f_k(x) &= x_1 - (1-x_1)x_{k+2} - \alpha(1+4x_{k+1}) && ,\ k=1,\\
f_k(x) &= -(1-x_1)x_{k+2} - \alpha(1+4x_k) && ,\ k=2,\\
f_k(x) &= \alpha x_1 - (1-x_1)x_{k+2} - x_k(1+4x_{k-1}) && ,\ k=3,\\
f_k(x) &= x_1x_{k-2} - (1-x_1)x_{k+2} - x_k(1+4x_{k-1}) && ,\ 3<k<n-1,\\
f_k(x) &= x_1x_{k-2} - x_k(1+4x_{k-1}) && ,\ k=n-1,\\
f_k(x) &= x_1x_{k-2} - (1-x_1) - x_k(1+4x_{k-1}) && ,\ k=n,\\
\alpha &= 0.414214,\\
\overline{x}_i &= 0.1\ ,\ \mathrm{mod}\ (i,8)=1, \qquad \overline{x}_i = 0.2\ ,\ \mathrm{mod}\ (i,8)=2,\\
\overline{x}_i &= 0.3\ ,\ \mathrm{mod}\ (i,8)=3, \qquad \overline{x}_i = 0.4\ ,\ \mathrm{mod}\ (i,8)=4,\\
\overline{x}_i &= 0.5\ ,\ \mathrm{mod}\ (i,8)=5, \qquad \overline{x}_i = 0.4\ ,\ \mathrm{mod}\ (i,8)=6,\\
\overline{x}_i &= 0.3\ ,\ \mathrm{mod}\ (i,8)=7, \qquad \overline{x}_i = 0.2\ ,\ \mathrm{mod}\ (i,8)=0.
\end{aligned}
$$

Problem 46.

$$
\begin{aligned}
F(x) &= \frac{1}{2}\sum_{k=1}^{n} f_k^2(x),\\
f_k(x) &= 5 - (i+1)(1-\cos x_k) - \sin x_k - \sum_{j=5i+1}^{5i+5} \cos x_j,\\
i &= \mathrm{div}(k-1,5), \qquad \bar{x}_l = 1/n, \qquad l \ge 1.
\end{aligned}
$$

Problem 47. Trigonometric - exponential system (trigexp 1) [27].

$$
\begin{aligned}
F(x) &= \frac{1}{2}\sum_{k=1}^{n} f_k^2(x),\\
f_k(x) &= 3x_k^3 + 2x_{k+1} - 5 + \sin(x_k - x_{k+1})\sin(x_k + x_{k+1})\ , && k=1,\\
f_k(x) &= 3x_k^3 + 2x_{k+1} - 5 + \sin(x_k - x_{k+1})\sin(x_k + x_{k+1})\\
&+ 4x_k - x_{k-1}\exp(x_{k-1} - x_k) - 3 && ,\quad 1<k<n,\\
f_k(x) &= 4x_k - x_{k-1}\exp(x_{k-1} - x_k) - 3 && ,\quad k=n,\\
\overline{x}_i &= 0, \quad i \ge 1.
\end{aligned}
$$

Problem 48. Trigonometric - exponential system (trigexp 2) [27].

$$
\begin{aligned}
F(x) &= \frac{1}{2}\sum_{k=1}^{n} f_k^2(x),\\
f_k(x) &= 3(x_k - x_{k+2})^3 - 5 + 2x_{k+1},\\
&+ \sin(x_k - x_{k+1} - x_{k+2})\sin(x_k + x_{k+1} - x_{k+2}) && ,\ \mathrm{mod}\ (k,2)=1,\ k=1,\\
f_k(x) &= -6(x_{k-2} - x_k)^3 + 10 - 4x_{k-1}\\
&- 2\sin(x_{k-2} - x_{k-1} - x_k)\sin(x_{k-2} + x_{k-1} - x_k)\\
&+ 3(x_k - x_{k+2})^3 - 5 + 2x_{k+1}\\
&+ \sin(x_k - x_{k+1} - x_{k+2})\sin(x_k + x_{k+1} - x_{k+2}) && ,\ \mathrm{mod}\ (k,2)=1,\ 1<k<n,\\
f_k(x) &= -6(x_{k-2} - x_k)^3 + 10 - 4x_{k-1}
\end{aligned}
$$

$$
\begin{aligned}
&- && 2\sin(x_{k-2}-x_{k-1}-x_k)\sin(x_{k-2}+x_{k-1}-x_k) && , \bmod (k,2)=1,\ k=n,\\
f_k(x) &= && 4x_k-(x_{k-1}-x_{k+1})\exp(x_{k-1}-x_k-x_{k+1})-3 && , \bmod (k,2)=0,\\
\overline{x}_i &= && 1, \quad i\ge 1.
\end{aligned}
$$

Problem 49. Singular Broyden problem [11].

$$
\begin{aligned}
F(x) &= \frac{1}{2}\sum_{k=1}^{n} f_k^2(x),\\
f_k(x) &= ((3-2x_k)x_k-2x_{k+1}+1)^2 && , k=1,\\
f_k(x) &= ((3-2x_k)x_k-x_{k-1}-2x_{k+1}+1)^2 && , 1<k<n,\\
f_k(x) &= ((3-2x_k)x_k-x_{k-1}+1)^2 && , k=n,\\
\overline{x}_i &= -1, \quad i\ge 1.
\end{aligned}
$$

Problem 50. Five-diagonal system [15].

$$
\begin{aligned}
F(x) &= \frac{1}{2}\sum_{k=1}^{n} f_k^2(x),\\
f_k(x) &= 4(x_k-x_{k+1}^2)+x_{k+1}-x_{k+2}^2 && , k=1,\\
f_k(x) &= 8x_k(x_k^2-x_{k-1})-2(1-x_k)\\
&+ 4(x_k-x_{k+1}^2)+x_{k+1}-x_{k+2}^2 && , k=2,\\
f_k(x) &= 8x_k(x_k^2-x_{k-1})-2(1-x_k)\\
&+ 4(x_k-x_{k+1}^2)+x_{k-1}^2-x_{k-2}+x_{k+1}-x_{k+2}^2 && , 2<k<n-1,\\
f_k(x) &= 8x_k(x_k^2-x_{k-1})-2(1-x_k)\\
&+ 4(x_k-x_{k+1}^2)+x_{k-1}^2-x_{k-2} && , k=n-1,\\
f_k(x) &= 8x_k(x_k^2-x_{k-1})-2(1-x_k)+x_{k-1}^2-x_{k-2} && , k=n,\\
\overline{x}_i &= -2, \quad i\ge 1.
\end{aligned}
$$

Problem 51. Seven-diagonal system [15].

$$
\begin{aligned}
F(x) &= \frac{1}{2}\sum_{k=1}^{n} f_k^2(x),\\
f_k(x) &= 4(x_k-x_{k+1}^2)+x_{k+1}-x_{k+2}^2+x_{k+2}-x_{k+3}^2 && , k=1,\\
f_k(x) &= 8x_k(x_k^2-x_{k-1})-2(1-x_k)\\
&+ 4(x_k-x_{k+1}^2)+x_{k-1}^2+x_{k+1}-x_{k+2}^2+x_{k+2}-x_{k+3}^2 && , k=2,\\
f_k(x) &= 8x_k(x_k^2-x_{k-1})-2(1-x_k)\\
&+ 4(x_k-x_{k+1}^2)+x_{k-1}^2-x_{k-2}+x_{k+1}-x_{k+2}^2\\
&+ x_{k-2}^2+x_{k+2}-x_{k+3}^2 && , k=3,\\
f_k(x) &= 8x_k(x_k^2-x_{k-1})-2(1-x_k)\\
&+ 4(x_k-x_{k+1}^2)+x_{k-1}^2-x_{k-2}+x_{k+1}-x_{k+2}^2\\
&+ x_{k-2}^2+x_{k+2}-x_{k-3}-x_{k+3}^2 && , 3<k<n-2,\\
f_k(x) &= 8x_k(x_k^2-x_{k-1})-2(1-x_k)
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad 4(x_k - x_{k+1}^2) + x_{k-1}^2 - x_{k-2} + x_{k+1} - x_{k+2}^2 \\
&+ \quad x_{k-2}^2 + x_{k+2} - x_{k-3} && , \; k = n-2, \\
f_k(x) &= 8x_k(x_k^2 - x_{k-1}) - 2(1 - x_k) \\
&+ \quad 4(x_k - x_{k+1}^2) + x_{k-1}^2 - x_{k-2} + x_{k+1} \\
&+ \quad x_{k-2}^2 - x_{k-3} && , \; k = n-1, \\
f_k(x) &= 8x_k(x_k^2 - x_{k-1}) - 2(1 - x_k) + x_{k-1}^2 - x_{k-2} \\
&+ \quad x_{k-2}^2 - x_{k-3} && , \; k = n, \\
\overline{x}_i &= -3, \quad i \geq 1.
\end{aligned}
$$

Problem 52. Extended Freudenstein and Roth function [5].

$$
\begin{aligned}
F(x) &= \frac{1}{2}\sum_{k=1}^{n} f_k^2(x), \\
f_k &= x_k + ((5 - x_{k+1})x_{k+1} - 2)x_{k+1} - 13 \;, \quad \mathrm{mod}(k,2) = 1, \\
f_k &= x_{k-1} + ((x_k + 1)x_k - 14)x_k - 29 \;, \quad \mathrm{mod}(k,2) = 0, \\
\overline{x}_i &= 90 \;, \quad \mathrm{mod}(i,2) = 1, \qquad \overline{x}_i = 60 \;, \quad \mathrm{mod}(i,2) = 0.
\end{aligned}
$$

Problem 53. Extended Cragg and Levy problem [22].

$$
\begin{aligned}
F(x) &= \frac{1}{2}\sum_{k=1}^{n} f_k^2(x), \\
f_k(x) &= (\exp(x_k) - x_{k+1})^2 \;, \quad \mathrm{mod}(k,4) = 1, \\
f_k(x) &= 10(x_k - x_{k+1})^3 \;, \quad \mathrm{mod}(k,4) = 2, \\
f_k(x) &= \tan^2(x_k - x_{k+1}) \;, \quad \mathrm{mod}(k,4) = 3, \\
f_k(x) &= x_k - 1 \;, \quad \mathrm{mod}(k,4) = 0, \\
\overline{x}_i &= 1 \;, \quad \mathrm{mod}(i,4) = 1, \qquad \overline{x}_i = 2 \;, \quad \mathrm{mod}(i,4) \neq 1.
\end{aligned}
$$

Problem 54. Broyden tridiagonal problem [22].

$$
\begin{aligned}
F(x) &= \frac{1}{2}\sum_{k=1}^{n} f_k^2(x), \\
f_k(x) &= x_k(0.5x_k - 3) + 2x_{k+1} - 1 \;, \quad k = 1, \\
f_k(x) &= x_k(0.5x_k - 3) + x_{k-1} + 2x_{k+1} - 1 \;, \quad 1 < k < n, \\
f_k(x) &= x_k(0.5x_k - 3) - 1 + x_{k-1} \;, \quad k = n, \\
\overline{x}_i &= -1, \quad i \geq 1.
\end{aligned}
$$

Problem 55. Extended Powell badly scaled function [22].

$$
\begin{aligned}
F(x) &= \frac{1}{2}\sum_{k=1}^{n} f_k^2(x), \\
f_k(x) &= 10000\, x_k\, x_{k+1} - 1 \;, \quad \mathrm{mod}(k,2) = 1, \\
f_k(x) &= \exp(-x_{k-1}) + \exp(-x_k) - 1.0001 \;, \quad \mathrm{mod}(k,2) = 0, \\
\overline{x}_i &= 0 \;, \quad \mathrm{mod}(i,2) = 1, \qquad \overline{x}_i = 1 \;, \quad \mathrm{mod}(i,2) = 0.
\end{aligned}
$$

Problem 56. Extended Wood problem [12].

$$\begin{aligned}
F(x) &= \frac{1}{2}\sum_{k=1}^{n} f_k^2(x), \\
f_k(x) &= -200x_k(x_{k+1} - x_k^2) - (1 - x_k) \quad , \quad \mathrm{mod}(k,4) = 1, \\
f_k(x) &= 200(x_k - x_{k-1}^2) + 20.2(x_k - 1) + 19.8(x_{k+2} - 1) \quad , \quad \mathrm{mod}(k,4) = 2, \\
f_k(x) &= -180x_k(x_{k+1} - x_k^2) - (1 - x_k) \quad , \quad \mathrm{mod}(k,4) = 3, \\
f_k(x) &= 180(x_k - x_{k-1}^2) + 20.2(x_k - 1) + 19.8(x_{k-2} - 1) \quad , \quad \mathrm{mod}(k,4) = 0, \\
\overline{x}_i &= -3, \quad \mathrm{mod}(i,2) = 1, \quad \overline{x}_i = -1, \quad \mathrm{mod}(i,2) = 0.
\end{aligned}$$

Problem 57. Tridiagonal exponential problem [5].

$$\begin{aligned}
F(x) &= \frac{1}{2}\sum_{k=1}^{n} f_k^2(x), \\
f_k(x) &= x_k - \exp(\cos(k(x_k + x_{k+1}))) \quad , \ k = 1, \\
f_k(x) &= x_k - \exp(\cos(k(x_{k-1} + x_k + x_{k+1}))) \quad , \ 1 < k < n, \\
f_k(x) &= x_k - \exp(\cos(k(x_{k-1} + x_k))) \quad , \ k = n, \\
\overline{x}_i &= 1.5, \quad i \ge 1.
\end{aligned}$$

Problem 58. Brent problem [4].

$$\begin{aligned}
F(x) &= \frac{1}{2}\sum_{k=1}^{n} f_k^2(x), \\
f_k(x) &= 3x_k(x_{k+1} - 2x_k) + x_{k+1}^2/4 \quad , \ k = 1, \\
f_k(x) &= 3x_k(x_{k+1} - 2x_k + x_{k-1}) + (x_{k+1} - x_{k-1})^2/4 \quad , \ 1 < k < n, \\
f_k(x) &= 3x_k(20 - 2x_k + x_{k-1}) + (20 - x_{k-1})^2/4 \quad , \ k = n, \\
\overline{x}_i &= 10, \quad i \ge 1.
\end{aligned}$$

Problem 59. Troesch problem [24].

$$\begin{aligned}
F(x) &= \frac{1}{2}\sum_{k=1}^{n} f_k^2(x), \\
f_k(x) &= 2x_k + \rho h^2 \sinh(\rho x_k) - x_{k+1} \quad , \ k = 1, \\
f_k(x) &= 2x_k + \rho h^2 \sinh(\rho x_k) - x_{k-1} - x_{k+1} \quad , \ 1 < k < n, \\
f_k(x) &= 2x_k + \rho h^2 \sinh(\rho x_k) - x_{k-1} - 1 \quad , \ k = n, \\
\rho &= 10, \quad h = 1/(n+1), \\
\overline{x}_i &= 1, \quad i \ge 1.
\end{aligned}$$

Problem 60. Flow in a channel [3].

$$F(x) = \frac{1}{2} f^T(x) f(x),$$

where equation $f(x) = 0$ is a finite difference analogue of the following nonlinear ordinary differential equation

$$u'''' = R(u'u'' - uu'''), \quad R = 500$$

over unit interval Ω with boundary conditions $u(0) = 0$, $u'(0) = 0$, $u(1) = 1$, $u'(1) = 0$. We use standard 5-point finite differences on an uniform grid having 5000 internal nodes. The initial approximate solution is a discretization of $u_0(x) = (x - 1/2)^2$.

Problem 61. Swirling flow [3].

$$F(x) = \frac{1}{2} f^T(x) f(x),$$

where equation $f(x) = 0$ is a finite difference analogue of the following system of two nonlinear ordinary differential equations

$$\begin{aligned} u'''' + R(uu''' + vv') &= 0 \\ v'' + R(uv' + u'v) &= 0, \quad R = 500 \end{aligned}$$

over unit interval Ω with boundary conditions $u(0) = u'(0) = u(1) = u'(1) = 0$, $v(0) = -1$, $v(1) = 1$. We use standard 5-point finite differences on an uniform grid having 2500 internal nodes. The initial approximate solution is a discretization of $u_0(x) = (x - 1/2)^2$ and $v_0(x) = x - 1/2$.

Problem 62. Bratu problem [13].

$$F(x) = \frac{1}{2} f^T(x) f(x),$$

where equation $f(x) = 0$ is a finite difference analogue of the following nonlinear partial differential equation

$$\Delta u + R \exp(u) = 0, \quad R = 6.8$$

over unit square Ω with Dirichlet boundary conditions $u = 0$ on $\partial\Omega$. We use standard 5-point finite differences on a uniform grid having 70×70 internal nodes. The initial approximate solution is a discretization of $u_0(x, y) = 0$.

Problem 63. Poisson problem 1 [11].

$$F(x) = \frac{1}{2} f^T(x) f(x),$$

where equation $f(x) = 0$ is a finite difference analogue of the following nonlinear partial differential equation

$$\Delta u = \frac{u^3}{1 + x^2 + y^2}$$

over unit square Ω with Dirichlet boundary conditions $u(0, y) = 1$, $u(1, y) = 2 - \exp(y)$, $u(x, 0) = 1$, $u(x, 1) = 2 - \exp(x)$. We use standard 5-point finite differences on a uniform grid having 70×70 internal nodes. The initial approximate solution is a discretization of $u_0(x, y) = -1$.

Problem 64. Poisson problem 2 [21].

$$F(x) = \frac{1}{2} f^T(x) f(x),$$

where equation $f(x) = 0$ is a finite difference analogue of the following nonlinear partial differential equation

$$\Delta u + \sin(2\pi u) + \sin\left(2\pi \frac{\partial u}{\partial x}\right) + \sin\left(2\pi \frac{\partial u}{\partial y}\right) + f(x, y) = 0,$$

where $f(x, y) = 1000((x-1/4)^2 + (y-3/4)^2)$, over unit square Ω with Dirichlet boundary conditions $u = 0$ on $\partial\Omega$. We use standard 5-point finite differences on a uniform grid having 70×70 internal nodes. The initial approximate solution is a discretization of $u_0(x, y) = 0$.

Problem 65. Porous medium problem [8].

$$F(x) = \frac{1}{2} f^T(x) f(x),$$

where equation $f(x) = 0$ is a finite difference analogue of the following nonlinear partial differential equation

$$\Delta u^2 + R\left(\frac{\partial u^3}{\partial x} + f(x, y),\right) = 0, \quad R = 50,$$

where $f(1/71, 1/71) = 1$ and $f(x, y) = 0$ for $(x, y) \neq (1/71, 1/71)$, over unit square Ω with Dirichlet boundary conditions $u(0, y) = 1$, $u(1, y) = 0$, $u(x, 0) = 1$, $u(x, 1) = 0$. We use standard 5-point finite differences on a uniform grid having 70×70 internal nodes. The initial approximate solution is a discretization of $u_0(x, y) = 1 - xy$.

Problem 66. Convection-difussion problem [14].

$$F(x) = \frac{1}{2} f^T(x) f(x),$$

where equation $f(x) = 0$ is a finite difference analogue of the following nonlinear partial differential equation

$$\Delta u - Ru\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right) + f(x, y) = 0, \quad R = 20,$$

where $f(x, y) = 2000x(1 - x)y(1 - y)$, over unit square Ω with Dirichlet boundary conditions $u = 0$ on $\partial\Omega$. We use standard 5-point finite differences on a uniform grid having 70×70 internal nodes. The initial approximate solution is a discretization of $u_0(x, y) = 0$.

Problem 67. Nonlinear biharmonic problem [18].

$$F(x) = \frac{1}{2} f^T(x) f(x),$$

where equation $f(x) = 0$ is a finite difference analogue of the following nonlinear partial differential equation

$$\Delta\Delta u + R\,(\max(0,u) + \mathrm{sign}(x - 1/2)) = 0, \quad R = 500$$

over unit square Ω with the boundary conditions $u = 0$ on $\partial\Omega$ and $\partial u(0,y)/\partial x = 0$, $\partial u(1,y)/\partial x = 0$, $\partial u(x,0)/\partial y = 0$, $\partial u(x,1)/\partial y = 0$. We use standard 13-point finite differences on a shifted uniform grid having 50×50 internal nodes [13]. The initial approximate solution is a discretization of $u_0(x,y) = 0$.

Problem 68. Driven cavity problem [13].

$$F(x) = \frac{1}{2} f^T(x) f(x),$$

where equation $f(x) = 0$ is a finite difference analogue of the following nonlinear partial differential equation

$$\Delta\Delta u + R\left(\frac{\partial u}{\partial y}\frac{\partial \Delta u}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial \Delta u}{\partial y}\right) = 0, \quad R = 500$$

over unit square Ω with the boundary conditions $u = 0$ on $\partial\Omega$ and $\partial u(0,y)/\partial x = 0$, $\partial u(1,y)/\partial x = 0$, $\partial u(x,0)/\partial y = 0$, $\partial u(x,1)/\partial y = 1$. We use standard 13-point finite differences on a shifted uniform grid having 50×50 internal nodes [13]. The initial approximate solution is a discretization of $u_0(x,y) = 0$.

Problem 69.

$$\begin{aligned}
F(x) &= \frac{1}{2}\sum_{k=1}^{n} f_k^2(x)\,, \\
f_k(x) &= 2x_k - x_{k+1} - x_{k-1} \\
&\quad + h^2\left(x_k^3 + 2.10^{-4}(2.10^{-4}a_2 - 1)x_k - 10^9\exp(-3.10^4 a_2)\right)\,, \\
h &= 1/(n+1), \quad a_1 = h\,k, \quad a_2 = (a_1 - 1/2)^2\,, \\
x_0 &= x_{n+1} = 0, \quad \overline{x}_l = 5\min(lh, 1 - lh), \quad l \geq 1\,.
\end{aligned}$$

Problem 70.

$$\begin{aligned}
F(x) &= \frac{1}{2}\sum_{k=1}^{n} f_k^2(x)\,, \\
f_k(x) &= 2x_k - x_{k+1} - x_{k-1} \\
&\quad + h^2\left(x_k^3\exp(x_k) + 5.10^8\exp(-10^4 a_2)\sqrt{|a_1 - 1/2|}(x_{k+1} - x_{k-1}) + a_3\right)\,, \\
h &= 1/(n+1), \quad a_1 = h\,k, \quad a_2 = (a_1 - 1/2)^2, \quad a_3 = 10^6\mathrm{sign}(a_1 - 1/2)\,, \\
x_0 &= x_{n+1} = 0, \quad \overline{x}_l = 5\min(lh, 1 - lh), \quad l \geq 1\,.
\end{aligned}$$

Problem 71. Problem 202 in [25].

$$F(x) = \frac{1}{2}\sum_{k=1}^{n} f_k^2(x)\,,$$

$$
\begin{aligned}
f_k(x) &= x_k - \frac{x_{k+1}^2}{10}, \quad 1 \le k < n\,, \\
f_k(x) &= x_k - \frac{x_1^2}{10}, \quad k = n\,, \\
\overline{x}_l &= 2, \quad l \ge 1\,.
\end{aligned}
$$

Problem 72. Problem 206 in [25].

$$
\begin{aligned}
F(x) &= \frac{1}{2}\sum_{k=1}^{n} f_k^2(x)\,, \\
f_k(x) &= x_{k-1} - 2x_k + x_{k+1} - h^2 \exp(x_k), \quad 1 \le k \le n\,, \\
h &= 1/(n+1), \quad x_0 = x_{n+1} = 0\,, \\
\overline{x}_l &= 1, \quad l \ge 1\,.
\end{aligned}
$$

Problem 73. Problem 207 in [25].

$$
\begin{aligned}
F(x) &= \frac{1}{2}\sum_{k=1}^{n} f_k^2(x)\,, \\
f_k(x) &= (3 - x_k/10)x_k + 1 - x_{k-1} - 2x_{k+1}, \quad 1 \le k \le n\,, \\
x_0 &= x_{n+1} = 0\,, \\
\overline{x}_l &= -1, \quad l \ge 1\,.
\end{aligned}
$$

Problem 74. Problem 208 in [25].

$$
\begin{aligned}
F(x) &= \frac{1}{2}\sum_{k=1}^{n} f_k^2(x)\,, \\
f_k(x) &= (1 + x_k^2)x_k + 1 - \sum_{i \in I_k}(x_i + x_i^2), \quad 1 \le k \le n\,, \\
I_k &= \{i : i \ne k, \max(1, k-3) \le i \le \min(n, k+3)\}\,, \\
\overline{x}_l &= -1, \quad l \ge 1\,.
\end{aligned}
$$

Problem 75. Problem 212 in [25].

$$
\begin{aligned}
F(x) &= \frac{1}{2}\sum_{k=1}^{n} f_k^2(x)\,, \\
f_k(x) &= x_k \qquad\qquad\qquad\qquad , \quad k = 1\,, \\
f_k(x) &= \cos(x_{k-1}) + x_k - 1), \quad 1 < k \le n\,, \\
\overline{x}_l &= 1/2, \quad l \ge 1\,.
\end{aligned}
$$

Problem 76. Problem 213 in [25].

$$
\begin{aligned}
F(x) &= \frac{1}{2}\sum_{k=1}^{n} f_k^2(x)\,, \\
f_k(x) &= 2x_k + h^2(x_k + \sin(x_k)) - x_{k-1} - x_{k+1}, \quad 1 \le k \le n\,, \\
h &= 1/(n+1), \quad x_0 = 0, \quad x_{n+1} = 1\,, \\
\overline{x}_l &= 1, \quad l \ge 1\,.
\end{aligned}
$$

Problem 77. Problem 214 in [25].

$$
\begin{aligned}
F(x) &= \frac{1}{2}\sum_{k=1}^{n} f_k(x)\,, \\
f_k(x) &= x_k(2+5x_k^2)+1-\sum_{i\in I_k} x_i(1+x_i), \quad 1\le k\le n\,, \\
I_k &= \{i : i\ne k, \max(1,k-5)\le i\le \min(n,k+1)\}\,, \\
\overline{x}_l &= -1, \quad l\ge 0\,.
\end{aligned}
$$

Problem 78. Ascher and Russel boundary value problem [2].

$$
\begin{aligned}
F(x) &= \frac{1}{2}\sum_{k=1}^{n} f_k(x)\,, \\
f_k(x) &= 2x_k-2h^2\left(x_k^2+\frac{x_{k+1}-x_{k-1}}{2h}\right)-x_{k-1}-x_{k+1}, \quad 1\le k\le n\,, \\
h &= 1/(n+1), \quad x_0=0, \quad x_{n+1}=1/2\,, \\
\overline{x}_l &= 1, \quad l\ge 1\,.
\end{aligned}
$$

Problem 79. Allgower and Georg boundary value problem [1].

$$
\begin{aligned}
F(x) &= \frac{1}{2}\sum_{k=1}^{n} f_k(x)\,, \\
f_k(x) &= 2x_k+0.3h^2\left[\exp(20(x_k+25(kh-1)))-\exp(-20(x_k+25kh))-t_k\right] \\
&\quad -x_{k-1}-x_{k+1}\,, \\
t_k &= \mathrm{sign}(kh-0.009), \quad k\ge 1\,, \\
h &= 0.01/(n+1), \quad x_0=0, \quad x_{n+1}=25\,, \\
\overline{x}_l &= 1, \quad l\ge 1\,.
\end{aligned}
$$

Problem 80. Potra and Rheinboldt boundary value problem [23].

$$
\begin{aligned}
F(x) &= \frac{1}{2}\sum_{k=1}^{n} f_k(x)\,, \\
f_k(x) &= 2x_k-x_{k-1}-x_{k+1}+h^2(x_k^2+x_k+0.1x_{k+n/2}-1.2)\;, \quad 1\le k\le n/2\,, \\
f_k(x) &= 2x_k-x_{k-1}-x_{k+1}+h^2(0.2x_{k-n/2}^2+x_k^2+2x_k-0.6), \quad n/2<k\le n\,, \\
h &= 1/(n/2+1), \quad x_0=x_{n+1}=0\,, \\
\overline{x}_l &= lh(1-lh), \quad \overline{x}_{l+n/2}=\overline{x}_l, \quad 1\le l\le n/2\,.
\end{aligned}
$$

Problem 81.

$$
\begin{aligned}
F(x) &= \frac{1}{2}\sum_{k=1}^{n} f_k(x)\,, \\
f_k(x) &= 4x_k-x_{k-1}-x_{k+1}-x_{k-\sqrt{n}}-x_{k+\sqrt{n}}+h^2\exp(x_k), \quad 1\le k\le n\,, \\
h &= 1/(\sqrt{n}+1), \quad x_l=0 \quad \text{for} \quad l<1 \quad \text{or} \quad l>n, \\
\overline{x}_l &= 1, \quad 1\le l\le n\,.
\end{aligned}
$$

Problem 82.

$$\begin{aligned}
F(x) &= \frac{1}{2}\sum_{k=1}^{n} f_k(x)\,, \\
f_k(x) &= 4x_k - x_{k-1} - x_{k+1} - x_{k-\sqrt{n}} - x_{k+\sqrt{n}} + h^2 x_k^2 - y_k, \quad 1 \le k \le n\,, \\
h &= 1/(\sqrt{n}+1), \quad x_l = 0 \quad \text{for} \quad l < 1 \quad \text{or} \quad l > n, \\
\overline{x}_l &= 1, \quad 1 \le l \le n\,.
\end{aligned}$$

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