Image-finite first-order structures

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Introduction

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- **A**-structure: $\mathcal{M} = \langle M, \{P^{\mathcal{M}}\}\rangle$ with $P^{\mathcal{M}}\colon M^{ar(P)} \rightarrow \mathbf{A}$. An evaluation is a function $v\colon \mathcal{V} \rightarrow M$.

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- The interpretation of a formulae over an algebra with an structure $\mathcal{M}$ and an evaluation v:

$$
\begin{aligned}
\|P(x_1, \ldots, x_n)\|^{\mathbf{A}}_{\mathcal{M},v} &= P^{\mathcal{M}}(v(x_1), \ldots, v(x_n)) \\
\|\varphi \odot \psi\|^{\mathbf{A}}_{\mathcal{M},v} &= \|\varphi\|^{\mathbf{A}}_{\mathcal{M},v} \odot \|\psi\|^{\mathbf{A}}_{\mathcal{M},v} \\
\|\varphi \rightarrow \psi\|^{\mathbf{A}}_{\mathcal{M},v} &= \|\varphi\|^{\mathbf{A}}_{\mathcal{M},v} \rightarrow \|\psi\|^{\mathbf{A}}_{\mathcal{M},v} \\
\|\exists x \varphi\|^{\mathbf{A}}_{\mathcal{M},v} &= \sup_{m \in M} \{\|\psi\|^{\mathbf{A}}_{\mathcal{M},v[x \mapsto m]}\} \\
\|\forall x \varphi\|^{\mathbf{A}}_{\mathcal{M},v} &= \inf_{m \in M} \{\|\psi\|^{\mathbf{A}}_{\mathcal{M},v[x \mapsto m]}\}
\end{aligned}
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Introduction

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$$\sup_{m\in M}\{\|\psi\|^{\mathbf{A}}_{\mathcal{M},v}\} \in \mathbf{A}$$
$$\inf_{m\in M}\{\|\psi\|^{\mathbf{A}}_{\mathcal{M},v}\} \in \mathbf{A}$$

- An structure is *witnessed* whenever all infimum and supremum have a witness in the universe, i.e. for each formula ψ and variable x there exist m_0, m_1 such that

$$\sup_{m\in M}\{\|\psi\|^{\mathbf{A}}_{\mathcal{M},v[x\mapsto m]}\} = \|\psi\|^{\mathbf{A}}_{\mathcal{M},v[x\mapsto m_0]}$$
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- Standard semantics: $[0,1]$-valued structures.
- General semantics: **A**-valued (**A** chain) *safe* structures (product, Gödel, MV).

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First order tautologies over $[0,1]$, $[0,1] \cap \mathbb{Q}$ *and all* $Ł_n$ *coincide.*

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Theorem (I)

First order tautologies over $[0, 1]$, $[0, 1] \cap \mathbb{Q}$ *and all* $Ł_n$ *coincide.*

Theorem (II)

Standard and general semantics over MV-chains coincide for monadic 1-variable first order language.

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1. Generalize Rutledge's theorem (II) and present an alternative proof for it (until now, quite messy and only available in Rutledge's thesis).

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1. Generalize Rutledge's theorem (II) and present an alternative proof for it (until now, quite messy and only available in Rutledge's thesis).
2. Characterization of tautologies over a subalgebra of $[0,1]_{Ł}$ generated by one irrational.

Image-finite structures

Definition

$\mathcal{M}$ is an *image-finite* structure (over $\mathbf{A}$) if for each n-ary predicate symbol of the language, the sets $\{P(m_1, ..., m_n) : \langle m_1, .., m_n \rangle \in M^n\}$ are finite. $IFs(\mathbf{A})$ denotes all image-finite structures over $\mathbf{A}$.

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Lemma

If $\mathcal{M}$ is image-finite, then $\mathcal{M}$ is witnessed (and in particular, safe).

Image-finite structures

It is well known that at a propositional level

$$Var(\mathrm{K}_1) = Var(\mathrm{K}_2) \;\Leftrightarrow\; Th(\mathrm{K}_1) = Th(\mathrm{K}_2).$$

At a predicate level we can get:

Lemma

Let K_1 and K_2 be two classes of MTL-chains that generate the same variety. Then, image-finite structures over chains in K_1 and image-finite structures over chains in K_2 share the same 1-valid sentences, i.e.

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Proof (sketch):

φ s.t. $\|\varphi\|^{\mathbf{A}}_{\mathcal{M},v} < 1$, $\mathbf{A} \in \mathsf{K}_1$

- Add $|M|$ FO variables evaluated to its correspondent element of M. Change quantified subformulae of φ by (finite) disjunctions or conjunctions over these new variables. Resulting (still FO, non-quantified) formula evaluates to the same value.

Image-finite structures

Proof (cont.)

- Transform the non-quantified formula into a propositional one $\overline{\varphi}$, and the structure defines a (propositional) evaluation e such that $e(\overline{\varphi}) = \|\varphi\|^{\mathbf{A}}_{\mathcal{M},v} < 1$.
- Since K_1 and K_2 generate the same variety, there exists e' over $\mathbf{B} \in \mathrm{K}_2$ s.t. $e'(\overline{\varphi}) < 1$.
- An image-finite structure $\mathcal{M}'$ over $\mathbf{B}$ can be defined from e' over the same universe M such that $\|\varphi\|^{\mathbf{B}}_{\mathcal{M}',v} = e'(\overline{\varphi}) < 1$.

Image-finite structures

From Rutledge's theorem (I) we have

Lemma

A formula is a $[0,1]_{Ł}$ *tautology iff it is 1-valid for each image-finite structure over* $[0,1]_{Ł}$.

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Applying the continuity idea from Rutledge's proof, we also get

Lemma

A formula is a tautology of some subalgebra of the standard MV-*chain generated by one irrational iff it is 1-valid for each image-finite structure over it.*

Image-finite structures

Using previous results we get a list of structures that share 1-valid sentences

Theorem (Łukasiewicz Standard Semantics.)

The following families of first-order structures have the same set of 1-valid sentences.

- *Image-finite structures over the class of all* MV*-chains;*
- *(Image-finite) structures over the standard* MV*-chain;*
- *(Image-finite) structures over the rational standard* MV*-chain;*
- *(Image-finite) first-order structures over some subalgebra of the standard* MV*-chain generated by one irrational.*

Monadic Restriction

Lemma

Let φ be a monadic formula with only one variable x, $\mathcal{M}$ a witnessed structure over **A** *and v an $\mathcal{M}$-evaluation. There exists $M_\varphi \subseteq M$ finite such that*

$$\|\varphi\|^{\mathbf{A}}_{\mathcal{M},v} = \|\varphi\|^{\mathbf{A}}_{\mathcal{M}_\varphi,v}.$$

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Proof (sketch)

Let $\|\forall x\psi(x)\|^{\mathbf{A}}_{\mathcal{M}} = \|\psi(x)\|^{\mathbf{A}}_{\mathcal{M},[x\mapsto m_{\forall\psi}]}$ (anag. $\exists$ witness)

- *With only one variable, quantified formulae are always sentences.*
- $M_\varphi = \{v(x)\} \cup \{m_{\forall\psi} : \forall x\psi$ *is a subformula of* $\varphi\} \cup \{m_{\exists\psi} : \exists x\psi$ *is a subformula of* $\varphi\}$

Monadic restriction

As a corollary (using completeness wrt witnessed models of Ł∀, [Háj07]) we get Rutledge's theorem (II) ([Rut59]):

Theorem

1-valid sentences with one variable over the class of all MV*-chains coincide with those over* $[0,1]_Ł$.

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As a corollary (using completeness wrt witnessed models of $Ł\forall$, [Háj07]) we get Rutledge's theorem (II) ([Rut59]):

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Note that Bou has (still unpublished) some results showing this is not true in general (with three variables it fails).

Conclusions and Future work

- Rutledge's result (II) gets an alternative (nicer?) proof.
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  - Monadic product logic with one variable? (do standard and general semantics coincide?)
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Thank you!

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