Propositional Dynamic Logic for Searching Games with Errors

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The Rényi – Ulam game is a searching game with errors

1. Alice chooses an element in $\{1, \ldots, M\}$.
2. Bob tries to guess this number by asking Yes/No questions.
3. Alice is allowed to lie $n - 1$ times in her answers.

Bob tries to guess Alice's number as fast as possible.

RÉNYI - ULAM game is used to illustrate $\mathcal{MV}_n$-algebras

Model of the game (MUNDICI)

1. Knowledge space $K = Ł_n^M$.
2. A state of knowledge (for BOB) $s \in Ł_n^M$: $s(m)$ is the seen as the distance between m and the set of elements of $\{1, \ldots, M\}$ that can be safely discarded.
3. A question Q is a subset of $\{1, \ldots, M\}$.
4. A way to compute states of knowledge from ALICE's answers (MV-algebra operations).

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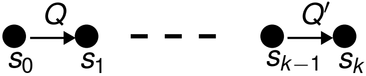

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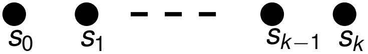

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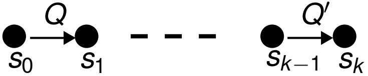
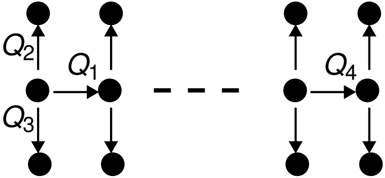


We want a language to talk about *all instances of any game*.



We use a language designed for stating many-valued program specifications

Programs $\alpha \in \Pi$ and formulas $\phi \in$ Form are mutually defined by

Formulas	$\phi ::= p \mid 0 \mid \phi \to \phi \mid \neg\phi \mid [\alpha]\phi$
Programs	$\alpha ::= a \mid \phi? \mid \alpha ; \alpha \mid \alpha \cup \alpha \mid \alpha^*$

where *p* is a propositional variable and *a* is an atomic program/question.

Word	Reading
$\alpha ; \beta$	α followed by β
$\alpha \cup \beta$	α or β
α^*	any number of execution of α
$\phi?$	test ϕ
$[\alpha]$	after any execution of α

We consider KRIPKE models in which worlds are many-valued

Definition

A (*dynamic* $n+1$*-valued*) KRIPKE *model* $\mathcal{M} = \langle W, R_{\cdot}, \mathrm{Val} \rangle$ where

- W is a non empty set,
- $R_{\cdot}$ maps any atomic program a to $R_a \subseteq W \times W$,
- Val assigns a truth value $\mathrm{Val}(u, p) \in \text{Ł}_n$ for any $u \in W$ and any propositional variable p.

$\mathrm{Val}(\cdot,\cdot)$ and $R_{\cdot}$ are extended to every formulas and programs

Val and $R_{\cdot}$ are extended by mutual induction :

- In a truth functional way for $\neg$ and $\rightarrow$,
- $\mathrm{Val}(u, [\alpha]\psi) := \bigwedge\{\mathrm{Val}(v, \psi) \mid (u, v) \in R_\alpha\}$,
- $R_{\alpha;\beta} := R_\alpha \circ R_\beta$,
- $R_{\alpha\cup\beta} := R_\alpha \cup R_\beta$,
- $R_{\phi?} = \{(u, u) \mid \mathrm{Val}(u, \phi) = 1\}$,
- $R_{\alpha^*} := (R_\alpha)^* = \bigcup_{k\in\omega} R_\alpha^k$.

Definition

We note $\mathcal{M}, u \models \phi$ if $\mathrm{Val}(u, \phi) = 1$ and $\mathcal{M} \models \phi$ if $\mathcal{M}, u \models \phi$ for every $u \in W$.

RÉNYI - ULAM game has a KRIPKE model

Language :

- a propositional variable p_m for any $m \in M$ that qualifies how m is far from the set of rejected elements.
- an atomic program m for any $\{m\} \subseteq \{1, \ldots, M\}$.

Model :

- $W = Ł_n^M$ is the knowledge space.
- $(s, t) \in R_{\{m\}}$ if t is a state of knowledge that can be obtained by updating s with an answer of ALICE to question $\{m\}$.
- $\mathrm{Val}(s, p_m) = s(m)$.

We want to axiomatize the theory of the KRIPKE models

Definition

$$T_n = \bigcap \{\{\phi \mid \mathcal{M} \models \phi\} \mid \mathcal{M} \text{ is a Kripke model}\}.$$

We aim to give an axiomatization of T_n.

There are three ingredients in the axiomatization

Definition

An $n+1$-*valued propositional dynamic logic* is a set of formulas that contains formulas in Ax_1, Ax_2, Ax_3 and closed for the rules in Ru_1, Ru_2.

Łukasiewicz $n+1$-valued logic	
Ax_1	Axiomatization
Ru_1	MP, uniform substitution

Crisp modal $n+1$-valued logic	
Ax_2	$[\alpha](p \to q) \to ([\alpha]p \to [\alpha]q)$, $[\alpha](p \oplus p) \leftrightarrow [\alpha]p \oplus [\alpha]p$, $[\alpha](p \odot p) \leftrightarrow [\alpha]p \odot [\alpha]p$,
Ru_2	$\phi \,/\, [\alpha]\phi$

Program constructions	
Ax_3	$[\alpha \cup \beta]p \leftrightarrow [\alpha]p \wedge [\beta]p$ $[\alpha;\beta]p \leftrightarrow [\alpha][\beta]p$, $[q?]p \leftrightarrow (\neg q^n \vee p)$ $[\alpha^*]p \leftrightarrow (p \wedge [\alpha][\alpha^*]p)$, $[\alpha^*]p \rightarrow [\alpha^*][\alpha^*]p$, $(p \wedge [\alpha^*](p \rightarrow [\alpha]p)^n) \rightarrow [\alpha^*]p$.

The last axiom means

> 'if after an undetermined number of executions of α the truth value of p cannot decrease after a new execution of α, then the truth value of p cannot decrease after any undetermined number of executions of α'.

Our main result is a completeness theorem

Definition

We denote by PDL_n the smallest $n+1$-valued propositional dynamic logic.

Theorem

$$T_n = \mathrm{PDL}_n$$

Sketch of the proof.

1. Construction of the canonical model of PDL_n.
2. Truth lemma.
3. Filtration of the canonical model.

$\square$

We construct a model in which truth formulas are precisely the elements of PDL_n

The MV-reduct of the LINDENBAUM - TARSKI algebra $\mathcal{F}_n$ of PDL_n is a member of $\mathbb{ISP}(Ł_n)$.

Definition

The *canonical model* of PDL_n is $\mathcal{M}^c = \langle W^c, R^c, \mathrm{Val}^c \rangle$ where

1. $W^c = \mathcal{MV}(\mathcal{F}_n, Ł_n)$;
2. For any program α,

$$R^c_\alpha := \{(u, v) \mid \forall \phi \in \mathcal{F}_n \; (u([\alpha]\phi) = 1 \Rightarrow v(\phi) = 1)\};$$

3. For any formula ϕ,

$$\mathrm{Val}^c(u, \phi) = u(\phi).$$

We use filtration to overcome the fact that the canonical model is not a KRIPKE model

$R^c_{\alpha^*}$ may be a proper extension of $(R^c_\alpha)^*$.

Definition

$\mathrm{FL}(\phi)$ is the finite set of formulas that are a subexpression of ϕ.

Definition

Fix a formula ϕ. Let $\equiv_\phi$ be the equivalence defined on W^c by

$$u \equiv_\phi v \quad \text{if} \quad \forall\, \psi \in \mathrm{FL}(\phi)\; u(\psi) = v(\psi).$$

Theorem (Filtration)

$W^c / \equiv_\phi$ can be equipped with a Kripke model structure $[\mathcal{M}^c]_\phi$ that satisfies

$$\mathcal{M}^c \models \psi \;\Leftrightarrow\; [\mathcal{M}^c]_\phi \models \psi, \quad \psi \in \mathrm{FL}(\phi).$$

We can finalize the proof of the completeness theorem

Theorem

$$T_n = \mathsf{PDL}_n$$

Sketch of the proof.

1. ✓ Construction of the canonical model of PLD_n.
2. ✓ Truth lemma.
3. ✓ Filtration of the canonical model.

If ϕ is a tautology then $[\mathcal{M}^c]_\phi \models \phi$. Hence $\mathcal{M}^c \models \phi$, which means that $\phi \in \mathsf{PDL}_n$. □

If $n = 1$, everything boils down to PDL (introduced by FISCHER and LADNER in 1979).

There is room for future work

1. Shows that PDL_n can actually help in stating many-valued program specifications.
2. There is an epistemic interpretation of PDL. Can it be generalized to the $n+1$-valued realm ?
3. What happens if KRIPKE models are not crisp.
4. Can coalgebras explain why PDL and PDL_n works are so related ?