# Equilibrium semantics for languages with imperfect information

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IF logic and multivalued logic

- IF logic is an extension of FOL

- IF languages define games of imperfect information

- Imperfect information introduces indeterminacy

- To overcome indeterminacy we apply von Neumann's Minimax Theorem (Aitaj):

1. Equilibrium semantics under mixed strategies (Blass and Gurevich 1986, Sevenster, 2006)

2. Equilibrium semantics under behavior strategies (Galliani, 2008)

IF languages

- The sentence $\varphi_{inf}$

$$\forall x \exists y (\exists z/\{x\})(x = z \wedge w \neq c)$$

- Lewis sentence $\varphi_{sig}$

$$\forall x \exists z (\exists y/\{x\})\{S(x) \rightarrow (\Sigma(z) \wedge R(y) \wedge y = b(x))\}$$

- Monty Hall $\varphi_{MH}$

$$\forall x (\exists y/\{x\}) \forall z [x \neq z \wedge y \neq z \rightarrow (\exists t/\{x\}) x = t]$$

- Matching Pennies $\varphi_{MP}$

$$\forall x (\exists y/\ \{x\}) x = y$$

- Inverted Matching Pennies $\varphi_{IMP}$

$$\forall x (\exists y/\ \{x\}) x \neq y$$

Extensive IF games

- $G(\mathbb{M}, s, \varphi)$ where $\varphi$ is an IF sentence, $\mathbb{M}$ is a model, and $s$ is a partial assignment

- These are win-lose 2 player game of imperfect information

- The players are Eloise ($\exists$) and Abelard ($\forall$)

- An information set $\delta$ for player $i \in \{\exists, \forall\}$ is a set of partial plays (nonterminal histories)

- A strategy $s_i$ is a specification of what actions player $i$ should implement for each information set

Example: perfect information

- The game $G(\mathbb{M}, \varphi)$ where $\varphi$ is

$$\forall x \exists y x = y \quad \text{and} \quad \mathbb{M} = \{a, b\}$$

- Eloise has 2 inform. sets: $\delta_1 = \{a\}$ and $\delta_2 = \{b\}$

- A strategy $s_\exists$ for Eloise has the form

$$s_\exists = (s_\exists(\delta_1), s_\exists(\delta_2))$$

  where

$$s_\exists(\delta_1), s_\exists(\delta_2) \in \{a, b\}$$

- Abelard has 1 information set: $\gamma_1 = \varnothing$

- A strategy $s_\forall$ for Abelard has the form

$$s_\forall = (s_\forall(\gamma_1))$$

Example: imperfect information

- The game $G(\mathbb{M}, \varphi_{MP})$, where $\varphi_{MP}$ is
$$\forall x (\exists y / \{x\}) x = y$$
and
$$\mathbb{M} = \{a, b\}$$

- Eloise has one information set
$$\delta_1 = \{a, b\}$$
(which has 2 histories)

- Abelard has one information set
$$\gamma_1 = \varnothing$$

- A strategy for Eloise has the form
$$s_\exists = (s_\exists(\delta_1))$$

- A strategy for Abelard is as before.

Game-theoretical truth and falsity

- For $\varphi$ an IF-formula, $\mathbb{M}$ a model and $s$ an assignment in $M$, we stipulate:

- $\mathbb{M}, s \models^{+}_{GTS} \varphi$ iff there is a winning strategy for Eloise in $G(\mathbb{M}, s, \varphi)$

- $\mathbb{M}, s \models^{-}_{GTS} \varphi$ iff there is a winning strategy for Abelard in $G(\mathbb{M}, s, \varphi)$.

Expressive power

- Infinity. The sentence $\varphi_{inf}$

$$\forall x \exists y (\exists z/\{x\})(x = z \wedge y \neq c)$$

defines (Dedekind) infinity.

Model-theoretical properties

- We restrict the set of universes to those containing at least 2 objects.

- Compactness: An IF theory is satisable if every finite subtheory of it is satisable.

- Lowenheim-Skolem property.

- Separation property: any two contrary IF sentences can be separated by an elementary class.

- Interpolation property: Let $\varphi$ and $\psi$ be contrary IF $L$-sentences. Then there is an IF $L$-sentence $\chi$ such that

$$\varphi \equiv^{+} \chi \quad \text{and} \quad \psi \equiv^{+} \neg\chi$$

- Definability of truth.

Indeterminacy

- Indeterminate sentences on finite models:

- $\varphi_{inf}$

$$\forall x \exists y (\exists z/\{x\})(x = z \wedge y \neq c)$$

- Lewis sentence $\varphi_{sig}$

$$\forall x \exists z (\exists y/\{x\})\{S(x) \rightarrow (\Sigma(z) \wedge R(y) \wedge y = b(x))\}$$

- Monty Hall $\varphi_{MH}$

$$\forall x (\exists y/\{x\}) \forall z [x \neq z \wedge y \neq z \rightarrow (\exists t/\{x\}) x = t]$$

- Matching Pennies $\varphi_{MP}$

$$\forall x (\exists y/\{x\}) x = y$$

- Inverted Matching Pennies $\varphi_{IMP}$

$$\forall x (\exists y/\{x\}) x \neq y$$

Strategic IF games: Definition

- Let $G(\mathbb{M}, \varphi)$ be an extensive IF game.

- $\Gamma(\mathbb{M}, \varphi) = (N, (S_i)_{i \in N}, u_{i \in N})$ is the strategic IF game where:

- $N = \{\exists, \forall\}$ is the set of players

- $S_i$ is the set of strategies of player $i$ in the extensive $G(\mathbb{M}, \varphi)$

- $u_i$ is the utility function of player $i$ such that $u_i(s, t) = 1$ if playing $s$ against $t$ in $G(\mathbb{M}, \varphi)$ yields a win for player $i$, and $u_i(s, t) = 0$, otherwise.

Example

- Let $\mathbb{M} = \{a, b\}$. The strategic game for $\forall x \exists y x = y$:

| | $a$ | $b$ |
|---|---|---|
| $(a,a)$ | $(1,0)$ | $(0,1)$ |
| $(a,b)$ | $(1,0)$ | $(1,0)$ |
| $(b,a)$ | $(0,1)$ | $(0,1)$ |
| $(b,b)$ | $(0,1)$ | $(1,0)$ |

- The strategic games for $\forall x (\exists y / \{x\}) x = y$ and $\forall x (\exists y / \{x\}) x \neq y$ :

| | $a$ | $b$ |
|---|---|---|
| $a$ | $(1,0)$ | $(0,1)$ |
| $b$ | $(0,1)$ | $(1,0)$ |

| | $a$ | $b$ |
|---|---|---|
| $a$ | $(0,1)$ | $(1,0)$ |
| $b$ | $(1,0)$ | $(0,1)$ |

- There are no equilibria in the last two games

Multivalues

- Mixed strategy equilibria in strategic IF games
- Behavior strategy equilibria in extensive IF games (Galliani)

Mixed strategies in strategic IF games

- Fix a strategic IF game $\Gamma(\mathbb{M}, \varphi) = (N, (S_i)_{i\in N}, u_{i\in N}$

- A mixed str. $\sigma_i$ for player $i$, $\sigma_i : S_i \to [0,1]$ such that
$$\sum_{s\in S_i} \sigma_i(s) = 1$$

- $\sigma_i$ is uniform over $S_i' \subseteq S_i$ if it assigns equal probability to all the strategies in $S_i'$

- Let $\sigma$ be a mixed str. for $\exists$ and $\tau$ a mixed str. for $\forall$.

- The expected utility for player $i$ for the strategy profile $(\sigma, \tau)$:
$$U_i(\sigma, \tau) = \sum_{s\in S_\exists} \sum_{t\in S_\forall} \sigma(s)\tau(t)u_i(s,t)$$

Behavior strategies

- Fix an extensive IF game $G(\mathbb{M}, \varphi)$

- Let $\delta_1, ..., \delta_n$ be the information sets of player $\exists$

- A pure strategy for pl. $\exists$ has the form
$$s_\exists = (s_\exists(\delta_1), ..., s_\exists(\delta_n))$$
where each $s_\exists(\delta_i) \in A(\delta_i)$.

- A behavior strategy for pl. $\exists$ has the form
$$b_\exists = (p_1(\delta_1), ..., p_n(\delta_2))$$
where each $p_i(\delta_i)$ is a probability distribution over $A(\delta_i)$

- Let $a \in A(\delta_i)$ for some $i$. Let $p_i(a/\delta)$ denote $(p_i(\delta_i))(a)$

- We must have:

$$\textstyle\sum_{a \in A(\delta)} p_i(a/\delta) \;=\; 1$$

Example

- The extensive game $G(\mathbb{M}, \varphi)$ where $\varphi$ is $\forall x(\exists y/\{x\})x = y$ and $\mathbb{M}$ is $\{a, b\}$

- Pl. $\exists$ has one information set $\delta_1 = \{a, b\}$

- A behavior strategy for $\exists$:

$$b_\exists \;=\; (1/2a \oplus 1/2b\;)$$

- Pl. $\forall$ has one information set $\gamma_1 = \{\varnothing\}$

- A behavior strategy for $\forall$:

$$b_\forall \;=\; (1/2a \oplus 1/2b\;)$$

Example continued: expected utility

- When the strategy profile $(b_\exists, b_\forall)$ is played, each terminal history will receive a probability.

- This probability is the product of the probabilities of the actions which compose the history.

- In the example, each terminal history has probability 1/4.

- The expected utility $U_i(b_\exists, b_\forall)$: we sum up the probability of each terminal history with the payoff of player $i$.

Example: mixed strategies $\Rightarrow$ behavior strategies

- Let $\varphi$ be $\exists x(\exists y/\{x\})x = y$ and $\mathbb{M} = \{a, b\}$

- In the game $G(\mathbb{M}, \varphi)$, $\exists$ has 2 information sets

$$\delta_1 = \{\varnothing\} \quad \text{and} \quad \delta_2 = \{a, b\}$$

- $\exists$ has 4 pure strategies:

$$(a, a), (a, b), (b, a), (b, b)$$

- Let $\sigma$ be the mixed strategy

$$\sigma(a, a) = \sigma(b, b) = 1/2$$

- The behavior strategy induced by $\sigma$

$$P(a/\delta_1) = P(b/\delta_1) = 1/2$$

and

$$P(a/\delta_2) = P(b/\delta_2) = 1/2$$

- However this induces a different probability (1/4) on terminal histories than $\sigma$.

Example continued

- The mixed str. $\sigma$ allows $\exists$ to create a different probability distribution at each of the nodes of the same information set.

- At the left node she chooses $a$ with probability 1; at the right node she chooses $a$ with probability 0.

- A conditional probability on the other side will impose the same probability distribution on both nodes.

Mixed strategy equilibria

- Let $N = \{\exists, \forall\}$ and $\Gamma = ((S_i)_{i \in N}, (u_i)_{i \in N})$ be a constant sum, strategic game

- Let $(\sigma_\exists, \sigma_\forall)$ be a pair of mixed strategies in $\Gamma$. $(\sigma_\exists, \sigma_\forall)$ is an equilibrium if

- for every mixed strategy $\sigma$ of Eloise: $U_\exists(\sigma_\exists, \sigma_\forall) \geq U_\exists(\sigma, \sigma_\forall)$

- for every mixed strategy $\sigma$ of Abelard: $U_\forall(\sigma_\exists, \sigma_\forall) \geq U_\forall(\sigma_\exists, \sigma)$

Von Neumann's Minimax Theorem: equilibrium semantics

- Every finite, constant sum, two-player game has an equilibrium in mixed strategies

- Every two such equilibria have the same expected utility

- We can talk about the probabilistic value of an IF sentence on a finite model $\mathbb{M}$.

- The satisfaction relation $\models_\varepsilon$ between IF sentences $\varphi$ and models $\mathbb{M}$, with $\varepsilon$ such that $0 \leq \varepsilon \leq 1$ defined by:

  $\mathbb{M} \models_\varepsilon \varphi$ iff the value of the strategic game $\Gamma(\mathbb{M}, \varphi)$ is $\varepsilon$.

Equilibrium semantics: A conservative extension of classical GTS

- Conservativity:

**(i)** $\mathbb{M} \models^{+}_{GTS} \varphi$ iff $\mathbb{M} \models_1 \varphi$

**(ii)** $\mathbb{M} \models^{-}_{GTS} \psi$ iff $\mathbb{M} \models_0 \varphi$.

Example

- Recall the strategic games $\Gamma(\mathbb{M}, \varphi_{MP})$ and $\Gamma(\mathbb{M}, \varphi_{IMP})$, where $\mathbb{M} = \{a, b, c\}$:

| | $a$ | $b$ | $c$ |
|---|---|---|---|
| $a$ | $(1,0)$ | $(0,1)$ | $(0,1)$ |
| $b$ | $(0,1)$ | $(1,0)$ | $(0,1)$ |
| $c$ | $(0,1)$ | $(1,0)$ | $(1,0)$ |

| | $a$ | $b$ | $c$ |
|---|---|---|---|
| $a$ | $(0,1)$ | $(1,0)$ | $(1,0)$ |
| $b$ | $(1,0)$ | $(0,1)$ | $(1,0)$ |
| $c$ | $(1,0)$ | $(1,0)$ | $(0,1)$ |

- Let $\sigma$ and $\tau$ be uniform probability distributions over $\{a, b, c\}$.

- The pair $(\sigma, \tau)$ is an equilibrium in both games.

- The value of $\varphi_{MP}$ on $\mathbb{M}$ is $1/3$ and that of $\varphi_{IMP}$ is $2/3$.

- As the size of $\mathbb{M}$ increases, the value of $\varphi_{MP}$ on $\mathbb{M}$ asymptotically approaches 0 and that of $\varphi_{IMP}$ asymptotically approaches 1.

Example (Galliani): the value of the game is different in the two semantics

- Let $\varphi$ be

  $$\exists x(\exists y/\{x\})(\forall z/\{x,y\})(x = y \wedge x \neq z)$$

  and $\mathbb{M} = \{a, b\}$

- The strategic IF game:

| | $a$ | $b$ |
|---|---|---|
| $(a,a)$ | $(0,1)$ | $(1,0)$ |
| $(a,b)$ | $(0,1)$ | $(0,1)$ |
| $(b,a)$ | $(0,1)$ | $(0,1)$ |
| $(b,b)$ | $(1,0)$ | $(0,1)$ |

- The strategies $(a,b)$ and $(b,a)$ are weakly dominated by $(a,a)$

- The game is equivalent to the Matching Pennies game

| | $a$ | $b$ |
|---|---|---|
| $(a, a)$ | $(0, 1)$ | $(1, 0)$ |
| $(b, b)$ | $(1, 0)$ | $(0, 1)$ |

- The value of the game under Nash equilibrium semantics is $1/2$.

Example continued: behavior semantics

- The pair of behavior strategies

$$b_\exists = ({}^1\!/_2 a \oplus {}^1\!/_2 b, {}^1\!/_2 a \oplus {}^1\!/_2 b)$$
$$b_\forall = ({}^1\!/_2 a \oplus {}^1\!/_2 b)$$

  is an equilibrium.

- Each terminal history has probability $1/8$.

- The value of the game under behavior strategies is $1/4$.

Imperfect recall

- IF extensive games are game of imperfect recall

- In the sentence

$$\forall x \exists y (\exists z / \{x\}) x = z$$

Eloise does not have knowledge memory.

- In the sentence

$$\exists x (\exists y / \{x\}) (\forall z / \{x, y\}) (x = y \wedge x \neq z)$$

Eloise does not have action recall.

- By Kuhn's Theorem, on formulas with perfect recall, the two semantics coincide.

- Theorem (Sevenster 2006; Mann, Sandu,Sevenster 2011) Every regular IF sentence for which Eloise (Abelard) has perfect recall is truth (falsity) equivalent to a first-order sentence

Example: Infinity

- Recall the sentence $\varphi_{inf}$

$$\forall x \exists y (\exists z/\{x\})(x = z \wedge c \neq y)$$

- When $\mathbb{M}$ contains $n$ elements, the value of $\varphi_{inf}$ on $\mathbb{M}$ is ${n-1}/{n}$.

- Thus as the size of $\mathbb{M}$ increases, the value of $\varphi_{inf}$ on $\mathbb{M}$ approaches 1.

Expressing the rationals (Sevenster and Sandu, Galliani)

- Let $0 \leq m < n$ be integers and $q = {}^{m}\!/_{n}$.

- There exists an IF sentence that has value $q$ on every structure with at least two objects.

The game: informal description

- Let $M$ be a set of at least $n$ objects and $C \subseteq M$, $\mid C \mid = n$

- We formulate a two-step game:

**S1** $\forall$ chooses $m$ distinct objects, $b_1, ..., b_m \in M$.

**S2** $\exists$ chooses one object $c \in M$ not knowing the objects chosen in S1.

Payoffs

- $\exists$ gets payoff 1 iff at least one of the following conditions is met for at least some distinct $i, j \leq m$:

1. $b_i = b_j$ ($\forall$ chooses the same object)

2. $b_i \notin C$ ($\forall$ chooses outside $C$)

3. $b_i = c$ ($\exists$ chooses one of the objects chosen by $\forall$)

Expressing the game in IF logic

- Let $\mathbb{M}$ be a model which interprets the constants $c_1, ..., c_n$ in such a way that
$$C = \left\{c_1^M, ..., c_n^M\right\}$$

- The following IF sentence defines the rational game:
$$\forall x_1 ... \forall x_m (\exists y / \{x_1, ..., x_m\})(\beta_1 \vee \beta_2 \vee \beta_3)$$
where $\beta_1$ is
$$\bigvee_{i \in \{1,...,m\}} \bigvee_{j \in \{i,...,m\} - \{i\}} x_i = x_j$$
$\beta_2$ is
$$\bigvee_{i \in \{1,...,m\}} \bigwedge_{j \in \{1,...,n\}} x_i \neq c_j$$
and $\beta_3$ is
$$\bigvee_{i \in \{1,...,m\}} x_i = y$$

- The value of the game is $\frac{m}{n}$.

- Notice that the sentence depends on the model.

Expressing the rationals (Barbero and Sandu)

- The Lewis sentence $\varphi_{sig}$

  $\forall x \exists z (\exists y/\{x\})\{S(x) \rightarrow (\Sigma(z) \wedge R(y) \wedge y = x)\}$

  and models of the form

  $$\mathbb{M} = (M, S^M, \Sigma^M, R^M)$$

  where

  $$M = \{s_1, ..., s_n, t_1, ..., t_m\}$$
  $$S^M = R^M = \{s_1, ..., s_n\}$$
  $$\Sigma^M = \{t_1, ..., t_m\}$$

- When $0 \leq m < n$, the value of the game is $m/n$.

- Notice that here the sentence does not depend on the model.

- The sentence is a monadic sentence with identity

Remark

- Compare

  $\varphi_{sig} = \forall x \exists z (\exists y/\{x\})\{(S(x) \to (\Sigma(z) \wedge R(y) \wedge y = x))$

  and

  $$\varphi_{inf} = \forall x \exists y (\exists z/\{x\})(x = z \wedge c \neq y)$$

- $\varphi_{sig}$ put a constraint on the available signals: they are restricted to a set $\Sigma^M$.

- $\varphi_{inf}$ put a constraint on the available signals: they must be different from $c$.

- If the structure is infinite, then all the objects may be signalled.

- If the structure has fixed cardinality $n$, then at most $n-1$ objects may be signalled.

Numerical impact of the relation of independence

- We are given a prefix $\overrightarrow{Q}$ of IF quantifiers

- We attach $\overrightarrow{Q}$ infront of some IF formula $\psi$ to obtain an IF sentence $\varphi = \overrightarrow{Q}\psi$.

- We evaluate $\varphi$ on same (finite) structure $\mathbb{M}$ : the value of $\varphi$ is some rational number $p$.

- We remove some of the independence relations in $\overrightarrow{Q}$, e.g.

$$(\exists y / \{u, v, x, z\}) \quad \rightsquigarrow \quad (\exists y / \{u, v, z\})$$

- In this way we turn $\overrightarrow{Q}$ into a new quantifier prefix $\overrightarrow{Q}^{y \leftarrow x}$: the dependence of $y$ on $x$ has been restored.

- We form the IF sentence $\varphi^{y \leftarrow x} = \overrightarrow{Q}^{y \leftarrow x} \psi$.

- The probabilistic value $q$ of $\varphi^{y \leftarrow x}$ on $\mathbb{M}$ is such that $q \geq p$.

Numerical impact of the relation of independence

- (Barbero and Sandu, forthcoming) Let $\overrightarrow{Q}$ a quantifier prefix containing a relevant relation of independence (of $y$ from $x$). Then there is an IF sentence $\varphi = \overrightarrow{Q}\psi$ such that for each $0 < p, q \leq 1$ with $q/p \in \mathbb{N}$, we may associate a structure $\mathbb{M}$ such that

$$\mathbb{M} \vDash_p \varphi \text{ and } \mathbb{M} \vDash_q \varphi^{y \leftarrow x}$$

Game-theoretical probabilities

- We extend the object language to include identities of the form $NE(\varphi) = r$.

- $\mathbb{M} \vDash NE(\varphi) = r$ if and only if the value of $\varphi$ in $\mathbb{M}$ is $r$.

- Properties of the equilibrium semantics (Mann, Sandu, and Sevenster)

P1 $\quad NE(\varphi \vee \psi) = max(NE(\varphi), NE(\psi))$

P2 $\quad NE(\varphi \wedge \psi) = min(NE(\varphi), NE(\psi))$

P3 $\quad NE(\neg\varphi) = 1 - NE(\varphi)$.

- It follows that:

| | |
|---|---|
| Ax1 | $NE(\varphi) \geq 0$ |
| Ax2 | $NE(\varphi) + NE(\neg\varphi) = 1$ |
| Ax3 | $NE(\varphi) + NE(\psi) \geq NE(\varphi \vee \psi)$ |
| Ax4 | $NE(\varphi \wedge \psi) = 0 \rightarrow NE(\varphi) + NE(\psi) = NE(\varphi \vee \psi)$ |