Decidability for Gödel Modal Logics

George Metcalfe

Mathematical Institute
University of Bern, Switzerland

Joint work with Xavier Caicedo, Ricardo Rodríguez, and Jonas Rogger

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The Main Question

Are basic Gödel modal logics with crisp / fuzzy frames **decidable**?

(Yes, and so is the one-variable fragment of first-order Gödel logic.)
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Are basic Gödel modal logics with crisp / fuzzy frames **decidable**?

(Yes, and so is the one-variable fragment of first-order Gödel logic.)
A Side Question

What does it mean for a logic to have a finite model property?
Modal formulas in $\text{Fm}_{\square \Diamond}$ are built using connectives $
abla, \lor, \rightarrow, \bot, \top, \square, \text{ and } \Diamond$.

We also define $\neg \varphi = \text{def } \varphi \rightarrow \bot$ and the length of a formula $\varphi$ as $\ell(\varphi)$. 
For a non-empty set of worlds $W$, the ordered pair $\langle W, R \rangle$ is called

- a (crisp) Kripke frame if $R \subseteq W \times W$
- a fuzzy Kripke frame if $R : W \times W \rightarrow [0, 1]$. 
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Frames

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A **GK-model** $\langle W, R, V \rangle$ consists of a fuzzy Kripke frame $\langle W, R \rangle$ and a function $V : \text{Fm}_\Box \Diamond \times W \rightarrow [0, 1]$ satisfying

\[
V(\bot, x) = 0
\]

\[
V(\top, x) = 1
\]

\[
V(\varphi \land \psi, x) = \min(V(\varphi, x), V(\psi, x))
\]

\[
V(\varphi \lor \psi, x) = \max(V(\varphi, x), V(\psi, x))
\]

\[
V(\varphi \rightarrow \psi, x) = \begin{cases} 
1 & \text{if } V(\varphi, x) \leq V(\psi, x) \\
V(\psi, x) & \text{otherwise}
\end{cases}
\]
A **GK-model** $⟨W, R, V⟩$ consists of a fuzzy Kripke frame $⟨W, R⟩$ and a function $V : \text{Fm}_\square\diamond \times W \rightarrow [0, 1]$ satisfying

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V(\varphi \land \psi, x) &= \min(V(\varphi, x), V(\psi, x)) \\
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V(\varphi \rightarrow \psi, x) &= \begin{cases} 
1 & \text{if } V(\varphi, x) \leq V(\psi, x) \\
V(\psi, x) & \text{otherwise} 
\end{cases}
\end{align*}
\]
and for the modal connectives:

$$V(\Box \varphi, x) = \bigwedge_{y \in W} (R_{xy} \rightarrow_G V(\varphi, y))$$

$$V(\Diamond \varphi, x) = \bigvee_{y \in W} (\min(R_{xy}, V(\varphi, y))).$$
If \( \langle W, R \rangle \) is **crisp**, then \( \langle W, R, V \rangle \) is called a \( \text{GK}^C \)-**model** and

\[
V(\Box \varphi, x) = \bigwedge_{(x, y) \in R} V(\varphi, y)
\]

\[
V(\Diamond \varphi, x) = \bigvee_{(x, y) \in R} V(\varphi, y).
\]
Validity

A formula $\varphi \in Fm_{\Box\Diamond}$ is

- **valid** in a GK-model $\langle W, R, V \rangle$ if $V(\varphi, x) = 1$ for all $x \in W$;
- **GK-valid** if $\varphi$ is valid in all GK-models, written $\models_{GK} \varphi$;
- **GK$^C$-valid** if $\varphi$ is valid in all GK$^C$-models, written $\models_{GK^C} \varphi$.

In fact, we can consider just GK-tree-models of finite height.
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In fact, we can consider just GK-tree-models of finite height.
GK□ and GK◊, the box and diamond fragments of GK are axiomatized as extensions of Gödel logic with, respectively

\[ \Box (\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi) \quad \text{and} \quad \Diamond (\varphi \lor \psi) \rightarrow (\Diamond \varphi \lor \Diamond \psi) \]

\[ \neg \neg \Box \varphi \rightarrow \Box \neg \neg \varphi \]

\[ \varphi / \Box \varphi \]

\[ \varphi \lor / \Box \varphi \]

\[ \varphi 

GK□ and GK◊ coincide; GK□ is axiomatized by extending GK◊ with

\[ \chi \lor (\varphi \rightarrow \psi) / \Box \chi \lor (\Diamond \varphi \rightarrow \Diamond \psi). \]

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\[ \square(\varphi \rightarrow \psi) \rightarrow (\square\varphi \rightarrow \square\psi) \quad \text{and} \quad \diamond (\varphi \vee \psi) \rightarrow (\diamond\varphi \vee \diamond\psi) \]

\[ \neg
\neg\square\varphi \rightarrow \square\neg\neg\varphi \]

\[ \varphi / \square\varphi \]

\[ \varphi \rightarrow \psi / \diamond\varphi \rightarrow \diamond\psi. \]

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GK□ and GK♦, the box and diamond fragments of GK are axiomatized as extensions of Gödel logic with, respectively

\[ \Box (\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi) \quad \text{and} \quad \Diamond (\varphi \lor \psi) \rightarrow (\Diamond \varphi \lor \Diamond \psi) \]

\[ \neg \neg \Box \varphi \rightarrow \Box \neg \neg \varphi \]

\[ \varphi / \Box \varphi \]

\[ \Diamond \neg \neg \varphi \rightarrow \neg \neg \Diamond \psi \]

\[ \neg \Diamond \bot \]

\[ \varphi \rightarrow \psi / \Diamond \varphi \rightarrow \Diamond \psi. \]

GK^C and GK□ coincide; GK^C is axiomatized by extending GK♦ with

\[ \chi \lor (\varphi \rightarrow \psi) / \Diamond \chi \lor (\Diamond \varphi \rightarrow \Diamond \psi). \]

An axiomatization of the **full logic** GK is obtained by adding to the axiomatizations of the fragments, the Fischer Servi axioms

\[
\Diamond (\varphi \to \psi) \to (\Box \varphi \to \Diamond \psi) \\
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or by adding prelinearity to the intuitionistic modal logic IK.


No axiomatization has yet been found for the full logic GK^C.
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No axiomatization has yet been found for the full logic GK\(^C\).

But developing suitable systems for the full logics $\text{GK}$ and $\text{GK}_\Diamond^C$ seems to be more difficult...
Decidability and PSPACE-completeness has been established for the fragments $\text{GK}_\Box$, $\text{GK}_\Diamond$, and $\text{GK}_\Box^\Box$ using Gentzen-style proof systems in


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Failure of the Finite Model Property

The following formula is valid in all **finite** GK-models

\[ \Box \neg \neg p \rightarrow \neg \neg \Box p \]

but not in the **infinite** GK\(^C\)-model \( \langle \mathbb{N}, \mathbb{N}^2, V \rangle \) with

\[ V(p, x) = \frac{1}{x + 1} \quad (x \in \mathbb{N}). \]

Just note that:

\[
V(\Box \neg \neg p \rightarrow \neg \neg \Box p, 0) = (\bigwedge_{x \in \mathbb{N}} V(\neg \neg p, x)) \rightarrow_G (\neg \neg \bigwedge_{x \in \mathbb{N}} V(p, x))
\]

\[
= (\bigwedge_{x \in \mathbb{N}} 1) \rightarrow_G (\neg \neg \bigwedge_{x \in \mathbb{N}} \frac{1}{x + 1})
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= 1 \rightarrow_G 0 = 0.
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but not in the \textbf{infinite} \(GK^C\)-model \(⟨\mathbb{N}, \mathbb{N}^2, V⟩\) with
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V(□\neg\neg p → \neg\neg□p, 0) = (∏_{x∈\mathbb{N}} V(\neg\neg p, x)) →_G (\neg\neg ∏_{x∈\mathbb{N}} V(p, x))
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\[= 1 →_G 0 = 0.
\]
Consider fuzzy frames \( \langle W, R \rangle \) augmented with a function
\[
T : W \rightarrow \mathcal{P}_{<\omega}([0, 1])
\]
mapping worlds to \textbf{finite} subsets of \([0, 1]\) containing 0 and 1.
A GFK-model \( \langle W, R, T, V \rangle \) adds a valuation function \( V \) defined as before except that

\[
V(\Box \varphi, x) = \max\{r \in T(x) : r \leq \bigwedge_{y \in W} (Rxy \rightarrow^G V(\varphi, y))\}
\]

\[
V(\Diamond \varphi, x) = \min\{r \in T(x) : r \geq \bigvee_{y \in W} \min(Rxy, V(\varphi, y))\}.
\]

\( \langle W, R, T, V \rangle \) is called a GFK\(^C\)-model if \( \langle W, R \rangle \) is crisp.
A GFK-model $\langle W, R, T, V \rangle$ adds a valuation function $V$ defined as before except that

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$\langle W, R, T, V \rangle$ is called a GFK$^C$-model if $\langle W, R \rangle$ is crisp.
The formula $\Box \neg \neg p \rightarrow \neg \neg \Box p$ is not valid in the finite $\text{GFK}^C$-model

$$\langle \{a\}, \{(a, a)\}, T, V \rangle$$

where $V(p, a) = \frac{1}{2}$ and $T(a) = \{0, 1\}$.

Just observe that:

$$V(\Box \neg \neg p, a) = \max\{r \in T(a) : r \leq V(\neg \neg p, a)\} = 1$$

$$V(\neg \neg \Box p, a) = \neg \neg \max\{r \in T(a) : r \leq V(p, a)\} = 0$$

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V(\Box \neg \neg p \rightarrow \neg \neg \Box p, a) & = 1 \rightarrow_{G} 0 = 0.
\end{align*}
\]
The formula $\Box \neg \neg p \rightarrow \neg \neg \Box p$ is not valid in the finite GFK$^C$-model

\[ \langle \{a\}, \{(a, a)\}, T, V \rangle \quad \text{where} \quad V(p, a) = \frac{1}{2} \quad \text{and} \quad T(a) = \{0, 1\}. \]

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\end{align*}
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The formula $\square \neg \neg p \rightarrow \neg \neg \square p$ is not valid in the finite GFK$^C$-model

$\langle \{a\}, \{(a, a)\}, T, V \rangle$ where $V(p, a) = \frac{1}{2}$ and $T(a) = \{0, 1\}$.

Just observe that:

$V(\square \neg \neg p, a) = \max\{r \in T(a) : r \leq V(\neg \neg p, a)\} = 1$

$V(\neg \neg \square p, a) = \neg \neg \max\{r \in T(a) : r \leq V(p, a)\} = 0$

$V(\square \neg \neg p \rightarrow \neg \neg \square p, a) = 1 \rightarrow_G 0 = 0$. 
Lemma

Let $\langle W, R, T, V \rangle$ be a GFK-tree-model with root $x_0$. Then there exists a GK-tree-model $\langle \hat{W}, \hat{R}, \hat{V} \rangle$ with root $\hat{x}_0$ satisfying for each $\varphi \in \text{Fm}_{\Box \Diamond}$:

$$\hat{V}(\varphi, \hat{x}_0) = V(\varphi, x_0).$$

Moreover, if $\langle W, R \rangle$ is crisp, then so is $\langle \hat{W}, \hat{R} \rangle$. 
A Second Tricky Lemma

Lemma

Let $\langle W, R, V \rangle$ be a GK-tree-model with root $x_0$ and let $\varphi \in Fm_{\Box \Diamond}$. Then there exists a GFK-tree-model $\langle \hat{W}, \hat{R}, \hat{T}, \hat{V} \rangle$ satisfying

- $\langle \hat{W}, \hat{R} \rangle \subseteq \langle W, R \rangle$ and $x_0 \in \hat{W}$
- $\hat{V}(\varphi, x_0) = V(\varphi, x_0)$
- $|\hat{W}| \leq \ell(\varphi)^{\ell(\varphi)}$. 
The Main Theorems

Theorem

The following are equivalent:

1. \( \models_{\text{GK}} \varphi \)
2. \( \varphi \) is valid in all \( \text{GFK}\)-models \( \langle W, R, T, V \rangle \) with \( |W| \leq \ell(\varphi)\ell(\varphi) \).

The same holds also for \( \text{GK}^C \)-validity and \( \text{GFK}^C \)-models.

Theorem

\( \text{GK}\)-validity and \( \text{GK}^C \)-validity are decidable.
The Main Theorems

**Theorem**

The following are equivalent:

1. $\models_{\text{GK}} \varphi$
2. $\varphi$ is valid in all GFK-models $\langle W, R, T, V \rangle$ with $|W| \leq \ell(\varphi) \ell(\varphi)$.

The same holds also for GK$^C$-validity and GFK$^C$-models.

**Theorem**

GK-validity and GK$^C$-validity are decidable.
A *GS5*-model is a *GK*-model where $R$ is an equivalence relation.

In fact we can restrict to universal *GS5*-models $\langle W, V \rangle$ with

$$V(\Box \varphi, x) = \bigwedge_{y \in W} V(\varphi, y) \quad \text{and} \quad V(\Diamond \varphi, x) = \bigvee_{y \in W} V(\varphi, y).$$

Moreover, *GS5* can be interpreted as the one-variable fragment of first-order Gödel logic.
A Gödel S5 Logic

A $\text{GS}^{5C}$-model is a $\text{GK}^{C}$-model where $R$ is an equivalence relation.

In fact we can restrict to universal $\text{GS}^{5C}$-models $\langle W, V \rangle$ with

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Moreover, $\text{GS}^{5C}$ can be interpreted as the one-variable fragment of first-order Gödel logic.
A GS5\(^C\)-model is a GK\(^C\)-model where \( R \) is an equivalence relation.

In fact we can restrict to universal GS5\(^C\)-models \( \langle W, V \rangle \) with

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V(\Box \varphi, x) = \bigwedge_{y \in W} V(\varphi, y) \quad \text{and} \quad V(\Diamond \varphi, x) = \bigvee_{y \in W} V(\varphi, y).
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Moreover, GS5\(^C\) can be interpreted as the one-variable fragment of first-order Gödel logic.
The New Semantics

A universal $\text{GFS5}^C$-model $\langle W, T, V \rangle$ is a universal $\text{GS5}^C$-model $\langle W, V \rangle$ with a finite set $T$ satisfying $\{0, 1\} \subseteq T \subseteq [0, 1]$, and

$$V(\Box \varphi, x) = \max \{ r \in T : r \leq \bigwedge_{y \in W} V(\varphi, y) \}$$

$$V(\Diamond \varphi, x) = \min \{ r \in T : r \geq \bigvee_{y \in W} V(\varphi, y) \}.$$
The following are equivalent:

1. $\models_{\text{GS5}^C} \varphi$
2. $\varphi$ is valid in all universal $\text{GFS5}^C$-models $\langle W, T, V \rangle$ with $|W| \leq \ell(\varphi)$.

GS5$^C$-validity and validity in the one-variable fragment of first-order Gödel logic are decidable and indeed co-NP-complete.
Finite Model Property and Decidability

Theorem

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Theorem

GS5\(^C\)-validity and validity in the one-variable fragment of first-order Gödel logic are decidable and indeed co-NP-complete.
Final Questions

We have established decidability for GK and GK\textsuperscript{C}, and also co-NP completeness for the one-variable fragment of first-order G\ddot{o}del logic.

However, intriguing questions remain:

- What is the complexity of GK and GK\textsuperscript{C}? Proof systems?
- Do these techniques extend to other modal G\ddot{o}del logics?
- Which first-order logics have a decidable one-variable fragment?
- Is the two-variable fragment of first-order G\ddot{o}del logic decidable?
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