Decidability for Gödel Modal Logics

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The Main Question

Are basic Gödel modal logics with crisp / fuzzy frames **decidable**?

(Yes, and so is the one-variable fragment of first-order Gödel logic.)

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A Side Question

What does it mean for a logic to have a **finite model property**?

A Language

Modal formulas in $\mathrm{Fm}_{\Box\Diamond}$ are built using connectives

$$\wedge,\ \vee,\ \to,\ \bot,\ \top,\ \Box,\ \text{and}\ \ \Diamond.$$

We also define $\neg\varphi =_{\mathrm{def}} \varphi \to \bot$ and the **length** of a formula φ as $\ell(\varphi)$.

Frames

For a non-empty set of worlds W, the ordered pair $\langle W, R \rangle$ is called

- a **(crisp) Kripke frame** if $R \subseteq W \times W$
- a **fuzzy Kripke frame** if $R \colon W \times W \to [0, 1]$.

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GK-models

A GK-**model** $\langle W, R, V\rangle$ consists of a fuzzy Kripke frame $\langle W, R\rangle$ and a function $V\colon \mathrm{Fm}_{\Box\Diamond} \times W \to [0,1]$ satisfying

$$
\begin{aligned}
V(\bot, x) &= 0 \\
V(\top, x) &= 1 \\
V(\varphi \wedge \psi, x) &= \min(V(\varphi, x), V(\psi, x)) \\
V(\varphi \vee \psi, x) &= \max(V(\varphi, x), V(\psi, x)) \\
V(\varphi \to \psi, x) &= \begin{cases} 1 & \text{if } V(\varphi, x) \le V(\psi, x) \\ V(\psi, x) & \text{otherwise} \end{cases}
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GK-models

and for the modal connectives:

$$V(\Box\varphi, x) = \bigwedge_{y \in W} (Rxy \to_{\mathrm{G}} V(\varphi, y))$$

$$V(\Diamond\varphi, x) = \bigvee_{y \in W} (\min(Rxy, V(\varphi, y))).$$

$\mathrm{GK^C}$-models

If $\langle W, R \rangle$ is **crisp**, then $\langle W, R, V \rangle$ is called a $\mathrm{GK^C}$**-model** and

$$
\begin{aligned}
V(\Box\varphi, x) &= \bigwedge_{(x,y)\in R} V(\varphi, y) \\
V(\Diamond\varphi, x) &= \bigvee_{(x,y)\in R} V(\varphi, y).
\end{aligned}
$$

Validity

A formula $\varphi \in \mathrm{Fm}_{\Box\Diamond}$ is

- **valid** in a GK-model $\langle W, R, V \rangle$ if $V(\varphi, x) = 1$ for all $x \in W$
- GK-**valid** if φ is valid in all GK-models, written $\models_{\mathrm{GK}} \varphi$
- GK^{C}-**valid** if φ is valid in all GK^{C}-models, written $\models_{\mathrm{GK}^{\mathrm{C}}} \varphi$.

In fact, we can consider just GK-tree-models of finite height.

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Fragments

$\mathrm{GK}_\Box$ and $\mathrm{GK}_\Diamond$, the **box** and **diamond** fragments of GK are axiomatized as extensions of Gödel logic with, respectively

$$\begin{array}{lcl} \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) & \text{and} & \Diamond(\varphi \lor \psi) \to (\Diamond\varphi \lor \Diamond\psi) \\ \neg\neg\Box\varphi \to \Box\neg\neg\varphi & & \Diamond\neg\neg\varphi \to \neg\neg\Diamond\psi \\ \varphi \;/\; \Box\varphi & & \neg\Diamond\bot \\ & & \varphi \to \psi \;/\; \Diamond\varphi \to \Diamond\psi. \end{array}$$

$\mathrm{GK}^C_\Box$ and $\mathrm{GK}_\Box$ coincide; $\mathrm{GK}^C_\Diamond$ is axiomatized by extending $\mathrm{GK}_\Diamond$ with

$$\chi \lor (\varphi \to \psi) \;/\; \Diamond\chi \lor (\Diamond\varphi \to \Diamond\psi).$$

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Axiomatizing GK

An axiomatization of the **full logic** GK is obtained by adding to the axiomatizations of the fragments, the Fischer Servi axioms

$$\Diamond(\varphi \to \psi) \to (\Box\varphi \to \Diamond\psi)$$
$$(\Diamond\varphi \to \Box\psi) \to \Box(\varphi \to \psi),$$

or by adding prelinearity to the intuitionistic modal logic IK.

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Decidability?

Decidability and PSPACE-completeness has been established for the fragments $\mathrm{GK}_{\Box}$, $\mathrm{GK}_{\Diamond}$, and $\mathrm{GK}^{\mathrm{C}}_{\Diamond}$ using Gentzen-style proof systems in

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Failure of the Finite Model Property

The following formula is valid in all **finite** GK-models

$$\Box\neg\neg p \to \neg\neg\Box p$$

but not in the **infinite** GK^C-model $\langle \mathbb{N}, \mathbb{N}^2, V \rangle$ with

$$V(p, x) = \frac{1}{x+1} \qquad (x \in \mathbb{N}).$$

Just note that:

$$\begin{aligned} V(\Box\neg\neg p \to \neg\neg\Box p, 0) &= (\bigwedge_{x\in\mathbb{N}} V(\neg\neg p, x)) \to_G (\neg\neg \bigwedge_{x\in\mathbb{N}} V(p, x)) \\ &= (\bigwedge_{x\in\mathbb{N}} 1) \to_G (\neg\neg \bigwedge_{x\in\mathbb{N}} \frac{1}{x+1}) \\ &= 1 \to_G 0 = 0. \end{aligned}$$

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Fuzzy Frames Revisited

Consider fuzzy frames $\langle W, R \rangle$ augmented with a function

$$T\colon W \to \mathcal{P}_{<\omega}([0,1])$$

mapping worlds to **finite** subsets of $[0,1]$ containing 0 and 1.

GFK-models

A GFK-**model** $\langle W, R, T, V\rangle$ adds a valuation function V defined as before except that

$$V(\Box\varphi, x) = \max\{r \in T(x) : r \leq \bigwedge_{y \in W} (Rxy \to_{\mathrm{G}} V(\varphi, y))\}$$

$$V(\Diamond\varphi, x) = \min\{r \in T(x) : r \geq \bigvee_{y \in W} \min(Rxy, V(\varphi, y))\}.$$

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A Finite Counter Model

The formula $\Box\neg\neg p \to \neg\neg\Box p$ is not valid in the **finite** GFK^{C}-model

$$\langle\{a\},\{(a,a)\},T,V\rangle \qquad \text{where} \;\; V(p,a)=\tfrac{1}{2} \;\; \text{and} \;\; T(a)=\{0,1\}.$$

Just observe that:

$$\begin{aligned}
V(\Box\neg\neg p,a) &= \max\{r\in T(a): r\leq V(\neg\neg p,a)\} = 1\\
V(\neg\neg\Box p,a) &= \neg\neg\max\{r\in T(a): r\leq V(p,a)\} = 0\\
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A First Tricky Lemma

Lemma

Let $\langle W, R, T, V\rangle$ *be a* GFK*-tree-model with root* x_0*. Then there exists a* GK*-tree-model* $\langle \widehat{W}, \widehat{R}, \widehat{V}\rangle$ *with root* $\widehat{x}_0$ *satisfying for each* $\varphi \in \mathrm{Fm}_{\Box\Diamond}$*:*

$$\widehat{V}(\varphi, \widehat{x}_0) = V(\varphi, x_0).$$

Moreover, if $\langle W, R\rangle$ *is crisp, then so is* $\langle \widehat{W}, \widehat{R}\rangle$*.*

A Second Tricky Lemma

Lemma

Let $\langle W, R, V \rangle$ *be a* GK*-tree-model with root* x_0 *and let* $\varphi \in \mathrm{Fm}_{\Box\Diamond}$*.*
Then there exists a GFK*-tree-model* $\langle \widehat{W}, \widehat{R}, \widehat{T}, \widehat{V} \rangle$ *satisfying*

- $\langle \widehat{W}, \widehat{R} \rangle \subseteq \langle W, R \rangle$ *and* $x_0 \in \widehat{W}$
- $\widehat{V}(\varphi, x_0) = V(\varphi, x_0)$
- $|\widehat{W}| \leq \ell(\varphi)^{\ell(\varphi)}$.

The Main Theorems

Theorem

The following are equivalent:

(1) $\models_{\mathrm{GK}} \varphi$

(2) φ *is valid in all* GFK*-models* $\langle W, R, T, V \rangle$ *with* $|W| \leq \ell(\varphi)^{\ell(\varphi)}$.

The same holds also for $\mathrm{GK^C}$*-validity and* $\mathrm{GFK^C}$*-models.*

Theorem

GK*-validity and* $\mathrm{GK^C}$*-validity are decidable.*

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Theorem

GK*-validity and* GK^{C}*-validity are decidable.*

A $GS5^C$-**model** is a GK^C-model where R is an equivalence relation.

In fact we can restrict to **universal** $GS5^C$-**models** $\langle W, V \rangle$ with

$$V(\Box\varphi, x) = \bigwedge_{y \in W} V(\varphi, y) \quad \text{and} \quad V(\Diamond\varphi, x) = \bigvee_{y \in W} V(\varphi, y).$$

Moreover, $GS5^C$ can be interpreted as the **one-variable fragment of first-order Gödel logic**.

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Moreover, GS5^{C} can be interpreted as the **one-variable fragment of first-order Gödel logic**.

A **universal** GFS5$^{\mathrm{C}}$**-model** $\langle W, T, V\rangle$ is a universal GS5$^{\mathrm{C}}$-model $\langle W, V\rangle$ with a finite set T satisfying $\{0,1\} \subseteq T \subseteq [0,1]$, and

$$V(\Box\varphi, x) \;=\; \max\{r \in T : r \leq \bigwedge_{y \in W} V(\varphi, y)\}$$

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Finite Model Property and Decidability

Theorem

The following are equivalent:

(1) $\models_{\mathrm{GS5^C}} \varphi$

(2) φ *is valid in all universal* $\mathrm{GFS5^C}$*-models* $\langle W, T, V\rangle$ *with* $|W| \leq \ell(\varphi)$.

Theorem

$\mathrm{GS5^C}$*-validity and validity in the one-variable fragment of first-order Gödel logic are decidable and indeed co-NP-complete.*

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Final Questions

We have established decidability for GK and GK^C, and also co-NP completeness for the one-variable fragment of first-order Gödel logic.

However, intriguing questions remain:

- What is the complexity of GK and GK^C? Proof systems?
- Do these techniques extend to other modal Gödel logics?
- Which first-order logics have a decidable one-variable fragment?
- Is the two-variable fragment of first-order Gödel logic decidable?

Final Questions

We have established decidability for GK and GK^C, and also co-NP completeness for the one-variable fragment of first-order Gödel logic.

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