ON ŁUKASIEWICZ GAMES

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**OVERVIEW**

- We introduce a compact representation of non-cooperative games based on finite-valued Łukasiewicz logics.
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- In Boolean games each individual player strives for the satisfaction of a goal, represented as a classical Boolean formula that encodes her payoff;

- The actions available to players correspond to valuations that can be made to variables under their control.

- The use of Łukasiewicz logics makes it possible to more naturally represent much richer payoff functions for players.
Łukasiewicz and Games

- Classic Game Theory:
  - Non-cooperative games:
    - Łukasiewicz Games on \( \mathcal{L}_k^c \) [M. & Wooldridge]
    - Constant Sum Łukasiewicz Games on \( \mathcal{L}_\infty \) [Kroupa & Majer]
  - Cooperative games: MV-coalitions [Kroupa]

- Game-Theoretic Semantics:
  - Dialogue games [Fermüller, Giles, …]
  - Evaluation games [Cintula & Majer]
  - Ulam games [Mundici]
DEFINITION I

A Łukasiewicz game $G$ on $Ł^c_k$ is a tuple

$$G = \langle P, V, \{V_i\}, \{S_i\}, \{\phi_i\} \rangle$$

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2. \( V = \{p_1, p_2, \ldots\} \) is a finite set of propositional variables;
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$$G = \langle P, V, \{V_i\}, \{S_i\}, \{\phi_i\} \rangle$$

where:
1. $P = \{P_1, \ldots, P_n\}$ is a set of players;
2. $V = \{p_1, p_2, \ldots \}$ is a finite set of propositional variables;
3. $V_i \subseteq V$ is the set of propositional variables under control of player $P_i$, so that the sets $V_i$ form a partition of $V$. 

**Definition II**

4. $S_i$ is the strategy set for player $i$ that includes all valuations $s_i : V_i \rightarrow L_k$ of the propositional variables in $V_i$, i.e.

$$S_i = \{ s_i \mid s_i : V_i \rightarrow L_k \}.$$


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\[
S_i = \{ s_i \mid s_i : V_i \rightarrow L_k \}.
\]

5. \( \phi_i(p_1, \ldots, p_t) \) is an \( \mathbb{L}^c_k \)-formula, built from variables in \( V \), whose associated function

\[
f_{\phi_i} : (L_k)^t \rightarrow L_k
\]

corresponds to the payoff function of \( P_i \), and whose value is determined by the valuations in \( \{ S_1, \ldots, S_n \} \).
EQUILIBRIA

- A tuple \((s_1, \ldots, s_n)\), with each \(s_i \in S_i\), is called a strategy combination.
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- \(s_{-i}\) the set of strategies \(\{s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n\}\) not including \(s_i\).
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- \(s_i\) the set of strategies \(\{s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n\}\) not including \(s_i\).

- The strategy \(s_i\) for \(P_i\) is called a *best response* whenever, fixing \(s_{-i}\), there exists no strategy \(s'_i\) such that

\[
f_{\phi_i}(s_i, s_{-i}) \leq f_{\phi_i}(s'_i, s_{-i}).
\]
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\]

▶ A strategy combination \((s^*_1, \ldots, s^*_n)\) is called a pure strategy Nash Equilibrium whenever \(s^*_i\) is a best response to \(s^*_{-i}\), for each \(1 \leq i \leq n\).
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Traveler’s Dilemma I [Basu 1994]

- Two travelers fly back home from a trip to a remote island where they bought exactly the same antiques.
Two travelers fly back home from a trip to a remote island where they bought exactly the same antiques.

Their luggage gets damaged and all the items acquired are broken.
Traveler’s Dilemma I [Basu 1994]

- Two travelers fly back home from a trip to a remote island where they bought exactly the same antiques.
- Their luggage gets damaged and all the items acquired are broken.
- The airline promises a refund for the inconvenience.
Two travelers fly back home from a trip to a remote island where they bought exactly the same antiques.

Their luggage gets damaged and all the items acquired are broken.

The airline promises a refund for the inconvenience.

Both travelers must write on a piece of paper a number between 0 and 100 corresponding to the cost of the antiques.
Traveler’s Dilemma II [Basu 1994]

- If they both write the same number $x$, they both receive $x - 1$. 
Traveler’s Dilemma II [Basu 1994]

- If they both write the same number $x$, they both receive $x - 1$.
- If they write different numbers, say $x < y$, the one playing $x$ will receive $x + 2$. 
Traveler's Dilemma II [Basu 1994]

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- If they write different numbers, say $x < y$, the one playing $x$ will receive $x + 2$.
- The other player will receive $x - 2$. 

Travelers' payoff is given by the functions:

$$f_1(x, y) = \begin{cases} 
\max(x - 1, 0) & \text{if } x = y \\
\min(\min(x, y) + 2, 100) & \text{if } y < x \\
\max(\min(x, y) - 2, 0) & \text{if } x < y
\end{cases}$$

$$f_2(x, y) = \begin{cases} 
\max(x - 1, 0) & \text{if } x = y > 100 \\
\min(\min(x, y) + 2, 100) & \text{if } y < x \\
\max(\min(x, y) - 2, 0) & \text{if } x < y
\end{cases}$$
Traveler’s Dilemma II [Basu 1994]

- If they both write the same number $x$, they both receive $x - 1$.
- If they write different numbers, say $x < y$, the one playing $x$ will receive $x + 2$.
- The other player will receive $x - 2$.
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\min(\min(x, y) + 2, 100) & x < y \\
\max(\min(x, y) - 2, 0) & y < x 
\end{cases}$$

$$f_2(x, y) = \begin{cases} 
\max(x - 1, 0) & x = y \\
\min(\min(x, y) + 2, 100) & y < x \\
\max(\min(x, y) - 2, 0) & x < y 
\end{cases}$$
## Traveler’s Dilemma: Payoff Matrix

<table>
<thead>
<tr>
<th>T1</th>
<th>0, 0</th>
<th>2, 0</th>
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<th>...</th>
<th>2, 0</th>
<th>2, 0</th>
<th>2, 0</th>
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<td>2, 0</td>
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<td>2, 0</td>
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<td>0, 0</td>
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<td>...</td>
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<td>3, 0</td>
</tr>
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<td>1, 1</td>
<td>4, 0</td>
<td>...</td>
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<td>4, 0</td>
<td>4, 0</td>
<td>4, 0</td>
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<td>...</td>
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<td>...</td>
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<td>...</td>
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<td>0, 4</td>
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<td>0, 4</td>
<td>0, 5</td>
<td>...</td>
<td>95, 99</td>
<td>97, 97</td>
<td>100, 96</td>
<td>100, 96</td>
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<tr>
<td>99</td>
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<td>0, 3</td>
<td>0, 4</td>
<td>0, 5</td>
<td>...</td>
<td>95, 99</td>
<td>96, 100</td>
<td>98, 98</td>
<td>100, 97</td>
</tr>
<tr>
<td>100</td>
<td>0, 2</td>
<td>0, 3</td>
<td>0, 4</td>
<td>0, 5</td>
<td>...</td>
<td>95, 99</td>
<td>96, 100</td>
<td>97, 100</td>
<td>99, 99</td>
</tr>
</tbody>
</table>
**Traveler’s Dilemma as a Łukasiewicz Game over $L_{100}^c$**

Let

$$G = \langle \{T1, T2\}, \{p, q\}, \{p\}_1, \{q\}_2, \{\phi_1(p, q), \phi_2(p, q)\} \rangle,$$

where the payoff formulas are:

$$\phi_1(p, q) := \left( \Delta (p \leftrightarrow q) \land \left( p \ominus \frac{1}{100} \right) \right) \lor \left( \neg \Delta (q \rightarrow p) \land \left( p \ominus \frac{2}{100} \right) \right) \lor$$

$$\left( \neg \Delta (p \rightarrow q) \land \left( q \oplus \frac{2}{100} \right) \right),$$

$$\phi_2(p, q) := \left( \Delta (p \leftrightarrow q) \land \left( p \ominus \frac{1}{100} \right) \right) \lor \left( \neg \Delta (p \rightarrow q) \land \left( q \ominus \frac{2}{100} \right) \right) \lor$$

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OTHER EXAMPLES

- Auctions.
- Coordination Games.
- Matching Pennies.
- Weak-Link Games.
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Main Theorem

Let $G$ be any Łukasiewicz game on $\mathcal{L}_k^c$. Then there exists a formula $E_G$ of $\mathcal{L}_k^c$ so that the following statements are equivalent:
**Main Theorem**

Let $\mathcal{G}$ be any Łukasiewicz game on $\mathcal{L}_k^c$. Then there exists a formula $\mathcal{E}_\mathcal{G}$ of $\mathcal{L}_k^c$ so that the following statements are equivalent:

1. \( \mathcal{G} \) admits a pure strategy Nash Equilibrium
**Main Theorem**

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2. $\bigcap_{i=1}^n B_i \neq \emptyset$. 


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3. $E_G$ is satisfiable.
Let $G$ be any Łukasiewicz game on $\mathcal{L}_k^c$. Then there exists a formula $\mathcal{E}_G$ of $\mathcal{L}_k^c$ so that the following statements are equivalent:

1. $G$ admits a pure strategy Nash Equilibrium
2. $\bigcap_{i=1}^n B_i \neq \emptyset$.
3. $\mathcal{E}_G$ is satisfiable.
4. There exists a satisfiable normalized game $G'$ equivalent to $G$. 
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Two games

\[ \mathcal{G} = \langle P, V, \{V_i\}, \{S_i\}, \{\phi_i\} \rangle \quad \text{and} \quad \mathcal{G}' = \langle P', V', \{V'_i\}, \{S'_i\}, \{\phi'_i\} \rangle \]

are equivalent whenever:

1. \( P = P' \),
2. \( V = V' \),
3. For each \( i \), \( V_i = V'_i \) and \( S_i = S'_i \),
4. \( (s^{\star}_1, \ldots, s^{\star}_n) \) is a NE for \( \mathcal{G} \) if and only if \( (s^{\star}_1, \ldots, s^{\star}_n) \) is a NE for \( \mathcal{G}' \).

A game \( \mathcal{G} \) is normalized whenever each payoff formula \( \phi_i(p_1, \ldots, p_m) \) contains all the variables from \( V \).
NORMALIZED GAMES I

- Two games

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Normalized Games I

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- A game \( G \) is normalized whenever each payoff formula \( \phi_i(p_1, \ldots, p_m) \) contains all the variables from \( V \).
NORMALIZED GAMES II

- An \( L^c_k \)-formula \( \phi(p_1, \ldots, p_w) \) has an equivalent extension in \( \{q_1, \ldots, q_v\} \) if there exists a formula

\[
\phi^\#(p_1, \ldots, p_w, q_1, \ldots, q_v)
\]

such that, for every \( \{a_1, \ldots, a_w\} \in L_k \)

\[
f_\phi(a_1, \ldots, a_w) = f^{\#}_\phi(a_1, \ldots, a_w, b_1, \ldots, b_v)
\]

for all \( \{b_1, \ldots, b_v\} \in L_k \).
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  such that, for every \(\{a_1, \ldots, a_w\} \in L_k\)

  \[f_{\phi}(a_1, \ldots, a_w) = f_{\phi^\#}(a_1, \ldots, a_w, b_1, \ldots, b_v)\]

  for all \(\{b_1, \ldots, b_v\} \in L_k\).

- Every Ł\(_c\)-formula \(\phi(p_1, \ldots, p_w)\) has an equivalent extension in \(\{q_1, \ldots, q_v\}\) by taking

  \[\phi(p_1, \ldots, p_w) \oplus \bigoplus_{j=1}^{v} (q_j \circ \neg q_j)\].
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\]

- Every game is equivalent to a normalized game.
Best Response Sets

- We assume that every game is normalized.
**BEST RESPONSE SETS**

- We assume that every game is normalized.

- For each $i$, let $\vec{x}_i$ be tuple of variables controlled by $i$. 
Best Response Sets

- We assume that every game is normalized.
- For each $i$, let $\vec{x}_i$ be tuple of variables controlled by $i$.
- The slice of $f_{\phi_i}$ at $s_{-i}$, denoted as
  \[
  \sigma_{s-1}(f_{\phi_i}),
  \]
  is the function
  \[
  f_{\phi_i}(\vec{x}_i, s_{-1}).
  \]
**Best Response Sets**

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- For each $i$, let $\vec{x}_i$ be tuple of variables controlled by $i$.
- The slice of $f_{\phi_i}$ at $s_{-i}$, denoted as
  \[ \sigma_{s-1}(f_{\phi_i}), \]
  is the function
  \[ f_{\phi_i}(\vec{x}_i, s_{-1}). \]
- The set
  \[ B_i = \left\{ (s_i, s_{-i}) \mid \text{argmax} (\sigma_{s_{-i}}(f_{\phi_i})) = s_i \right\}, \]
  is called the *best response set* for $i$. 

**EXAMPLE**

Take the game

\[ G = \langle \{A1, A2\}, \{p, q\}, \{p\}_1, \{q\}_2, \{\phi_1(p, q), \phi_2(p, q)\} \rangle, \]

where

\[ \phi_1(p, q) := (p \rightarrow q), \quad \phi_2(p, q) := (q \rightarrow p), \]

and their associated functions are

\[ f_{\phi_1}(x, y) = \min(1 - x + y, 1) \quad f_{\phi_2}(x, y) = \min(1 - y + x, 1). \]
**Example: Payoff Matrix**

<table>
<thead>
<tr>
<th>T1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>...</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
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<td>...</td>
<td>8,10</td>
<td>9,10</td>
<td>10,10</td>
</tr>
</tbody>
</table>
EXAMPLE: $f_{\phi_1}$

The diagram shows the function $f_{\phi_2}(x, y)$ in a 3D space with $x$ and $y$ axes ranging from 0 to 1 and $f_{\phi_2}(x, y)$ ranging from 0 to 1. The data points are distributed across the space with a visible pattern.
**Example:** The slice of $f_{\phi_1}$ at 0
**Example: The slice of $f_{\varphi_1}$ at 0.1**
**Example: The slice of** $f_{\phi_1}$ **at 0.2**
**Example:** The slice of $f_{\phi_1}$ at 0.3

![3D graph showing the slice of $f_{\phi_1}$ at 0.3]
**EXAMPLE: DEFINING** $B_i$

$$f_{\phi_1}(x, y) = \{(0,0)\}$$
**EXAMPLE: DEFINING** $B_i$

$$f_{\phi_1}(x, y) \{ (0, 0), (0, 0.1), (0.1, 0.1) \}$$
**Example: Defining** $B_i$

$\{(0,0), (0,0.1), (0.1,0.1), (0,0.2), (0.1,0.2), (0.2,0.2)\}$
**Example: Defining $B_i$**

$$f_{\phi_1}(x,y) = \{(0, 0), (0, 0.1), (0.1, 0.1), (0, 0.2), (0.1, 0.2), (0.2, 0.2), (0, 0.3), (0.1, 0.3), (0.2, 0.3), (0.3, 0.3)\}$$
EXAMPLE: INTERSECTION OF BEST RESPONSE SETS
Best Response Sets and Equilibria

Let $\mathcal{G}$ be any Łukasiewicz game on $\mathcal{L}_k^c$. Then there exists a formula $\mathcal{E}_\mathcal{G}$ of $\mathcal{L}_k^c$ so that the following statements are equivalent:

1. $\mathcal{G}$ admits a pure strategy Nash Equilibrium.
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**The Formula $\mathcal{E}_G$ (I)**

- We want to define an $L^c_k$-formula $\mathcal{E}_G$ whose satisfiability encodes the existence of equilibria.
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- $\mathcal{E}_G$ should not require additional constants (apart from the payoff formulas).

- For every variable $p$ and every valuation $\nu : \{p\} \to L_k$ there exists a formula $\nu(p)$ such that

\[ \nu(p) = \frac{i}{k} \quad \text{IFF} \quad \nu(\psi(p)) = 1. \]
THE FORMULA $\mathcal{E}_G$ (I)

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- $\mathcal{E}_G$ should not require additional constants (apart from the payoff formulas).

- For every variable $p$ and every valuation $v : \{p\} \rightarrow L_k$ there exists a formula $\psi(p)$ such that

\[ v(p) = \frac{i}{k} \quad \text{IFF} \quad v(\psi(p)) = 1. \]

- This means that every strategy combination $(s_1, \ldots, s_n)$ can be encoded by a formula $\psi(\vec{p}_1, \ldots, \vec{p}_n)$ so that

\[ f_\psi(s'_1, \ldots, s'_n) = 1 \quad \text{IFF} \quad s_i = s'_i \]

for all $i$. 
THE FORMULA $\mathcal{E}_G$ (II)

$$\mathcal{E}_G := \bigvee_{\tilde{s} \in (L_k)^n} \sum_{i=1}^{m_i} \left[ \bigwedge_{i=1}^{n} \left( \psi_{\alpha_{1i}}(x_{1i}) \land \cdots \land \psi_{\alpha_{mi}}(x_{mi}) \right) \land \left( \phi_i(x_{11}, \ldots, x_{m1}, \ldots, x_{1i-1}, \ldots, x_{mi-1}, \ldots, y_{1i}^{\beta_{1i}}, \ldots, y_{mi}^{\beta_{mi}}, \ldots, x_{1i+1}, \ldots, x_{mi+1}, \ldots, x_{1n}, \ldots, x_{mn}) \rightarrow \right) \phi_i(x_{11}, \ldots, x_{m1}, \ldots, x_{1i-1}, \ldots, x_{mi-1}, \ldots, x_{1i}, \ldots, x_{mi}, \ldots, x_{1i+1}, \ldots, x_{mi+1}, \ldots, x_{1n}, \ldots, x_{mn}) \right] \right]\]$$
The Formula $\mathcal{E}_G$ (II)

\[
\mathcal{E}_G := \bigvee_{\vec{s} \in (L_k)^*} \sum_{i=1}^{m_i} \left[ \bigwedge_{i=1}^{n} \left( \psi_{\alpha_1} (x_1) \land \cdots \land \psi_{\alpha_{m_i}} (x_{m_i}) \right) \right] \land \\
\bigwedge_{i=1}^{n} \left[ \bigwedge_{s_i \in (L_k)^{m_i}} \left( \psi_{\beta_1} (y_1) \land \cdots \land \psi_{\beta_{m_i}} (y_{m_i}) \right) \right] \land \\
(\phi_i (x_1, \ldots, x_{m_1}, \ldots, x_{1_i-1}, \ldots, x_{m_i-1}, \ldots, y_{1_i}^\beta, \ldots, y_{m_i}^\beta, \ldots) \rightarrow \\
\phi_i (x_1, \ldots, x_{m_1}, \ldots, x_{1_i-1}, \ldots, x_{m_i-1}, \ldots, x_{1_i}^\phi, \ldots, x_{m_i}^\phi, \ldots) \\
x_{1_{i+1}}, \ldots, x_{m_{i+1}}, \ldots, x_{1_n}, \ldots, x_{m_n})
\]
The Formula $\mathcal{E}_G$ (II)

$$\mathcal{E}_G := \bigvee_{\bar{s} \in (L_k) \sum_{i=1}^{m_i} s_i \in (L_k)} \left[ \prod_{i=1}^{n} \left( \psi_{\alpha_1 i} (x_1) \land \cdots \land \psi_{\alpha_m i} (x_m) \right) \land \right.$$ 

$$\prod_{i=1}^{n} \left[ \prod_{s_i \in (L_k)^{m_i}} \left( \psi_{\beta_1 i} (y_{1_i}) \land \cdots \land \psi_{\beta_{m_i}} (y_{m_i}) \right) \right] \land$$

$$(\phi_i (x_{11}, \ldots, x_{m1}, \ldots, x_{1i-1}, \ldots, x_{mi-1}, \ldots, y_{1i}, \ldots, y_{mi}, \ldots$$

$$x_{1i+1}, \ldots, x_{mi+1}, \ldots, x_{1n}, \ldots, x_{mn}) \rightarrow$$

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The Formula $\mathcal{E}_G$ (II)

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\bigwedge_{i=1}^n \left[ \bigwedge_{s_i \in (L_k)^{m_i}} \left( \psi_{\beta_{1i}} (y_{1i}^{\beta_1 i}) \land \cdots \land \psi_{\beta_{m_i} i} (y_{mi}^{\beta_{m_i} i}) \right) \land \\
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**THE FORMULA** $\mathcal{E}_G$ (II)

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\prod_{i=1}^{m_i} \left[ \prod_{j=1}^{n} \left( \psi_{\beta_1} (y_{1_i}) \land \cdots \land \psi_{\beta_{m_i}} (y_{m_i}) \right) \land \\
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x_{1_{i+1}}, \ldots, x_{m_{i+1}}, \ldots, x_{1_n}, \ldots, x_{m_n}) \right] \right] \right]
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**THE FORMULA \( \mathcal{E}_G \) (II)**

\[ \mathcal{E}_G := \bigvee_{\bar{s} \in (L_k)^{\sum_{i=1}^{n} m_i}} \left[ \bigwedge_{i=1}^{n} \left( \bigwedge_{s_i \in (L_k)^{m_i}} \left[ \psi_{\alpha_1} (x_{1i}) \wedge \cdots \wedge \psi_{\alpha_{mi}} (x_{mi}) \right] \wedge \right. \right. \]

\[ \left. \left. \bigwedge_{i=1}^{n} \left[ \bigwedge_{s_i \in (L_k)^{m_i}} \left[ \psi_{\beta_1} (y_{1i}) \wedge \cdots \wedge \psi_{\beta_{mi}} (y_{mi}) \right] \wedge \right. \right. \]

\[ \left. \left. (\phi_i(x_{1i}, \ldots, x_{mi}, \ldots, x_{1i-1}, \ldots, x_{mi-1}, \ldots, y_{1i}, \ldots, y_{mi}, \ldots, \right. \right. \]

\[ \left. \left. x_{1i+1}, \ldots, x_{mi+1}, \ldots, x_{1n}, \ldots, x_{mn} \right) \rightarrow \phi_i(x_{1i}, \ldots, x_{mi}, \ldots, x_{1i-1}, \ldots, x_{mi-1}, \ldots, x_{1i}, \ldots, x_{mi}, \ldots \right. \right. \]

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$$\left( \bigwedge_{i=1}^{n} \left( \left( \bigwedge_{i=1}^{n} \left( \psi_{\beta_1} (y_1) \land \cdots \land \psi_{\beta_{m_i}} (y_{m_i}) \right) \land \right) \right) \land \right) \land$$

$$(\phi_i(x_1, \ldots, x_{m_1}, \ldots, x_{1_{i-1}}, \ldots, x_{m_{i-1}}, \ldots, x_1, \ldots, x_{1_n}, \ldots, x_{m_n}) \rightarrow \phi_i(x_1, \ldots, x_{m_1}, \ldots, x_{1_{i-1}}, \ldots, x_{m_{i-1}}, \ldots, x_1, \ldots, x_{1_n}, \ldots, x_{1_{i+1}}, \ldots, x_{m_{i+1}}, \ldots, x_{1_n}, \ldots, x_{m_n})) \right) \right) \right)$$
Satisfiability and Equilibria

Let $\mathcal{G}$ be any Łukasiewicz game on $\mathcal{L}_k^c$. Then there exists a formula $\mathcal{E}_\mathcal{G}$ of $\mathcal{L}_k^c$ so that the following statements are equivalent:

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Satisfiable Games (I)

- A game $G$ is called *satisfiable* if there exists a strategy combination $(s_1, \ldots, s_n)$ such that for every $i$, $\phi_i$ is satisfied under $(s_1, \ldots, s_n)$. 

 Every satisfiable game admits a NE. Every $\phi_i$ is satisfiable under $(s_1, \ldots, s_n)$, so no player can unilaterally improve her payoff.
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- Take the first-order theory $\text{Th}(L_k)$ of the finite MV-chain $L_k$ in the language of MV-algebras.
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- We want to show that there exists a sentence $E_G$ that encodes the existence of equilibria.
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- Define the formula $E_G$:

$$\exists \vec{x}_1, \ldots, \vec{x}_n \forall \vec{y}_1, \ldots, \vec{y}_n \prod_{i=1}^{n} \left( \phi_i(\vec{x}_1, \ldots, \vec{x}_{i-1}, \vec{y}_i, \vec{x}_{i+1}, \ldots, \vec{x}_n) \leq \phi_i(\vec{x}_1, \ldots, \vec{x}_{i-1}, \vec{x}_i, \vec{x}_{i+1}, \ldots, \vec{x}_n) \right)$$

where each $\vec{x}_i, \vec{y}_i$ refers to the tuple of variables assigned to player $i$. 
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\]

where each $\bar{x}_i, \bar{y}_i$ refers to the tuple of variables assigned to player $i$.

- A game $\mathcal{G}$ admits a NE iff $E_G$ holds in $\text{Th}(L_k)$.
Satisfiable Games (III)

- $E_G$ holds in $\text{Th}(L_k)$ iff the set defined by $E'_G$

\[
\forall \vec{y}_1, \ldots, \vec{y}_n \prod_{i=1}^{n} \left( \phi_i(\vec{x}_1, \ldots, \vec{x}_{i-1}, \vec{y}_i, \vec{x}_{i+1}, \ldots, \vec{x}_n) \leq \phi_i(\vec{x}_1, \ldots, \vec{x}_{i-1}, \vec{x}_i, \vec{x}_{i+1}, \ldots, \vec{x}_n) \right)
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- There exists a quantifier-free $E_{G}^{\text{free}}$ logically equivalent to $E'_G$ that defines the same set as $E'_G$.

- There exists an $\mathcal{L}_k^c$-formula $\epsilon_G$ that is satisfiable off so is $E_{G}^{\text{free}}$. 
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- Given a game

\[ G = \langle P, V, \{V_i\}, \{S_i\}, \{\phi_i\} \rangle \]
**SATISFIABLE GAMES (IV)**

- Given a game
  
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- Define a new game

  \[ G' = \langle P', V', \{V'_i\}, \{S'_i\}, \{\phi'_i\} \rangle \]

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**Work in Progress**

- Games with costs and efficiency.
- Classes of games.
- Complexity and tractable games.
- Games with external influence.
- Games with mixed strategies.
- And more...
THANKS!