ON ŁUKASIEWICZ GAMES

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OUTLINE

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- The actions available to players correspond to valuations that can be made to variables under their control.
- The use of Łukasiewicz logics makes it possible to more naturally represent much richer payoff functions for players.

ŁUKASIEWICZ AND GAMES

- Classic Game Theory:
  - Non-cooperative games:
    - Łukasiewicz Games on $Ł_k^c$ [M. & Wooldridge]
    - Constant Sum Łukasiewicz Games on $Ł_\infty$ [Kroupa & Majer]
  - Cooperative games: MV-coalitions [Kroupa]
- Game-Theoretic Semantics:
  - Dialogue games [Fermüller, Giles, . . .]
  - Evaluation games [Cintula & Majer]
  - Ulam games [Mundici]

DEFINITION I

A Łukasiewicz game $\mathcal{G}$ on $Ł_k^c$ is a tuple

$$\mathcal{G} = \langle \mathsf{P}, \mathsf{V}, \{\mathsf{V}_i\}, \{\mathsf{S}_i\}, \{\phi_i\} \rangle$$

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2. $\mathsf{V} = \{p_1, p_2, \dots\}$ is a finite set of propositional variables;
3. $\mathsf{V}_i \subseteq \mathsf{V}$ is the set of propositional variables under control of player P_i, so that the sets V_i form a partition of V.

DEFINITION II

4. S_i is the strategy set for player i that includes all valuations $s_i : \mathsf{V}_i \to L_k$ of the propositional variables in V_i, i.e.

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5. $\phi_i(p_1, \ldots, p_t)$ is an $Ł_k^c$-formula, built from variables in V, whose associated function

$$f_{\phi_i} : (L_k)^t \to L_k$$

corresponds to the *payoff function* of P_i, and whose value is determined by the valuations in $\{\mathsf{S}_1, \ldots, \mathsf{S}_n\}$.

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- The strategy s_i for P_i is called a *best response* whenever, fixing s_{-i}, there exists no strategy s_i' such that

$$f_{\phi_i}(s_i, s_{-i}) \leq f_{\phi_i}(s_i', s_{-i}).$$

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$$f_{\phi_i}(s_i, s_{-i}) \leq f_{\phi_i}(s'_i, s_{-i}).$$

- A strategy combination $(s^\star_1, \ldots, s^\star_n)$ is called a *pure strategy Nash Equilibrium* whenever $s^\star_i$ is a best response to $s^\star_{-i}$, for each $1 \leq i \leq n$.

OUTLINE

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- Two travelers fly back home from a trip to a remote island where they bought exactly the same antiques.

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- Their luggage gets damaged and all the items acquired are broken.
- The airline promises a refund for the inconvenience
- Both travelers must write on a piece of paper a number between 0 and 100 corresponding to the cost of the antiques.

TRAVELER’S DILEMMA II [BASU 1994]

- If they both write the same number x, they both receive $x - 1$.

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- The other player will receive $x - 2$.
- Travelers’ payoff is given by the functions:

$$f_1(x,y) = \begin{cases} \max(x-1,0) & x = y \\ \min(\min(x,y)+2,100) & x < y \\ \max(\min(x,y)-2,0) & y < x \end{cases} \quad ; \quad f_2(x,y) = \begin{cases} \max(x-1,0) & x = y \\ \min(\min(x,y)+2,100) & y < x \\ \max(\min(x,y)-2,0) & x < y \end{cases} .$$

TRAVELER'S DILEMMA: PAYOFF MATRIX

						T2				
		0	1	2	3	···	97	98	99	100
	0	0, 0	2, 0	2, 0	2, 0	···	2, 0	2, 0	2, 0	2, 0
	1	0, 2	0, 0	3, 0	3, 0	···	3, 0	3, 0	3, 0	3, 0
	2	0, 2	0, 3	1, 1	4, 0	···	4, 0	4, 0	4, 0	4, 0
	3	0, 2	0, 3	0, 4	2, 2	···	5, 0	4, 0	4, 0	4, 0
T1	⋮	⋮	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋮
	97	0, 2	0, 3	0, 4	0, 5	···	96, 96	99, 95	99, 95	99, 95
	98	0, 2	0, 3	0, 4	0, 5	···	95, 99	97, 97	100, 96	100, 96
	99	0, 2	0, 3	0, 4	0, 5	···	95, 99	96, 100	98, 98	100, 97
	100	0, 2	0, 3	0, 4	0, 5	···	95, 99	96, 100	97, 100	99, 99

TRAVELER'S DILEMMA AS A ŁUKASIEWICZ GAME OVER $Ł^c_{100}$

Let

$$\mathcal{G} = \langle \{\mathrm{T1}, \mathrm{T2}\}, \{p, q\}, \{p\}_1, \{q\}_2, \{\phi_1(p,q), \phi_2(p,q)\}\rangle,$$

where the payoff formulas are:

$$\begin{aligned}\phi_1(p,q) \quad := \quad & \left(\Delta\,(p \leftrightarrow q) \wedge \left(p \ominus \overline{\tfrac{1}{100}}\right)\right) \vee \left(\neg\Delta\,(q \to p) \wedge \left(p \ominus \overline{\tfrac{2}{100}}\right)\right) \vee \\ & \left(\neg\Delta\,(p \to q) \wedge \left(q \oplus \overline{\tfrac{2}{100}}\right)\right)\end{aligned},$$

$$\begin{aligned}\phi_2(p,q) \quad := \quad & \left(\Delta\,(p \leftrightarrow q) \wedge \left(p \ominus \overline{\tfrac{1}{100}}\right)\right) \vee \left(\neg\Delta\,(p \to q) \wedge \left(q \ominus \overline{\tfrac{2}{100}}\right)\right) \vee \\ & \left(\neg\Delta\,(q \to p) \wedge \left(p \oplus \overline{\tfrac{2}{100}}\right)\right)\end{aligned},$$

OTHER EXAMPLES

- Auctions.
- Coordination Games.
- Matching Pennies.
- Weak-Link Games.

OUTLINE

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2. $\bigcap_{i=1}^{n} \mathsf{B}_i \neq \emptyset$.
3. $\mathcal{E}_{\mathcal{G}}$ is satisfiable.
4. There exists a satisfiable normalized game $\mathcal{G}'$ equivalent to $\mathcal{G}$.

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NORMALIZED GAMES I

- Two games

$$\mathcal{G} = \langle \mathsf{P}, \mathsf{V}, \{\mathsf{V}_i\}, \{\mathsf{S}_i\}, \{\phi_i\} \rangle \quad \text{and} \quad \mathcal{G}' = \langle \mathsf{P}', \mathsf{V}', \{\mathsf{V}_i'\}, \{\mathsf{S}_i'\}, \{\phi_i'\} \rangle$$

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  are *equivalent* whenever:

  1. $\mathsf{P} = \mathsf{P}'$,
  2. $\mathsf{V} = \mathsf{V}'$,
  3. For each i, $\mathsf{V}_i = \mathsf{V}_i'$ and $\mathsf{S}_i = \mathsf{S}_i'$,
  4. $(s_1^\star, \ldots, s_n^\star)$ is a NE for $\mathcal{G}$ if and only if $(s_1^\star, \ldots, s_n^\star)$ is a NE for $\mathcal{G}'$.

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- A game $\mathcal{G}$ is *normalized* whenever each payoff formula $\phi_i(p_1, \ldots, p_m)$ contains all the variables from V.

NORMALIZED GAMES II

- An Ł$_k^c$-formula $\phi(p_1, \ldots, p_w)$ has an *equivalent extension* in $\{q_1, \ldots, q_v\}$ if there exists a formula

  $$\phi^\sharp(p_1, \ldots, p_w, q_1, \ldots, q_v)$$

  such that, for every $\{a_1, \ldots, a_w\} \in L_k$

  $$f_\phi(a_1, \ldots, a_w) = f_\phi^\sharp(a_1, \ldots, a_w, b_1, \ldots, b_v)$$

  for all $\{b_1, \ldots, b_v\} \in L_k$.

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  for all $\{b_1, \ldots, b_v\} \in L_k$.
- Every $Ł_k^c$-formula $\phi(p_1, \ldots, p_w)$ has an equivalent extension in $\{q_1, \ldots, q_v\}$ by taking

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- Every game is equivalent to a normalized game.

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  is the function

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- The set

  $$\mathsf{B}_i = \left\{ (s_i, s_{-i}) \mid \operatorname*{argmax}_{s_i' \in \mathsf{S}_i}(\sigma_{s_{-i}}(f_{\phi_i})) = s_i \right\},$$

  is called the *best response set* for i.

EXAMPLE

Take the game

$$\mathcal{G} = \langle \{\mathrm{A1}, \mathrm{A2}\}, \{p, q\}, \{p\}_1, \{q\}_2, \{\phi_1(p,q), \phi_2(p,q)\} \rangle,$$

where

$$\phi_1(p,q) \quad := \quad (p \to q), \qquad \phi_2(p,q) \quad := \quad (q \to p),$$

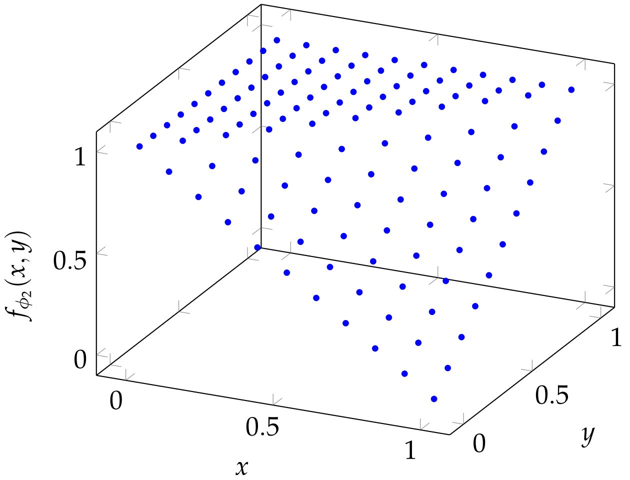
and their associated functions are

$$f_{\phi_1}(x,y) = \min(1 - x + y, 1) \qquad f_{\phi_2}(x,y) = \min(1 - y + x, 1).$$

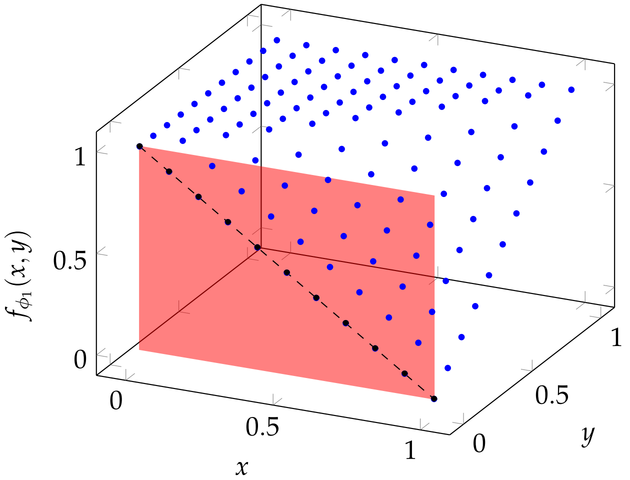
EXAMPLE: PAYOFF MATRIX

		T2							
		0	1	2	3	···	8	9	10
T1	0	10, 10	10, 9	10, 8	10, 7	···	10, 2	10, 1	10, 0
	1	9, 10	10, 10	10, 9	10, 8	···	10, 3	10, 2	10, 1
	2	8, 10	9, 10	10, 10	10, 9	···	10, 4	10, 3	10, 2
	3	7, 10	8, 10	9, 10	10, 10	···	10, 5	10, 4	10, 3
	⋮	⋮	⋮	⋮	⋮	⋱	⋮	⋮	⋮
	8	2, 10	3, 10	4, 10	5, 10	···	10, 10	10, 9	10, 8
	9	1, 10	2, 10	3, 10	4, 10	···	9, 10	10, 10	10, 9
	10	0, 10	1, 10	2, 10	3, 10	···	8, 10	9, 10	10, 10

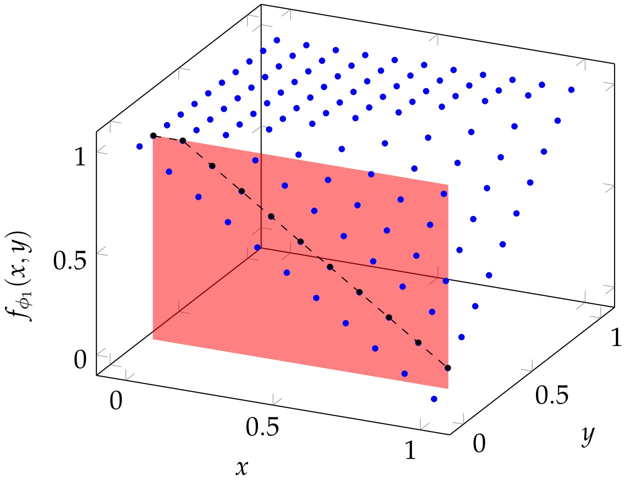
EXAMPLE: f_{ϕ_1}



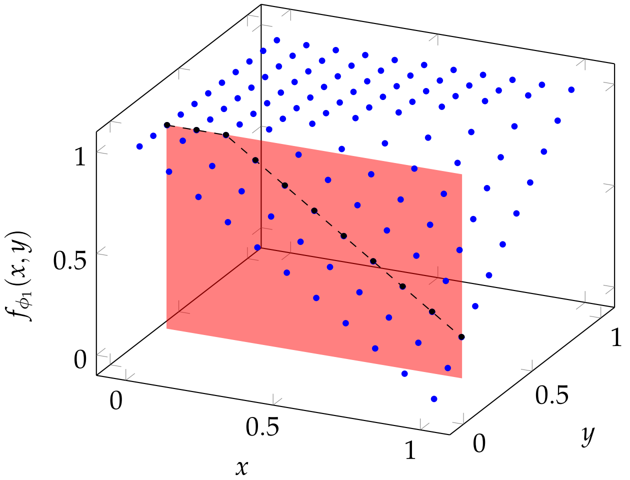
EXAMPLE: THE SLICE OF f_{ϕ_1} AT 0



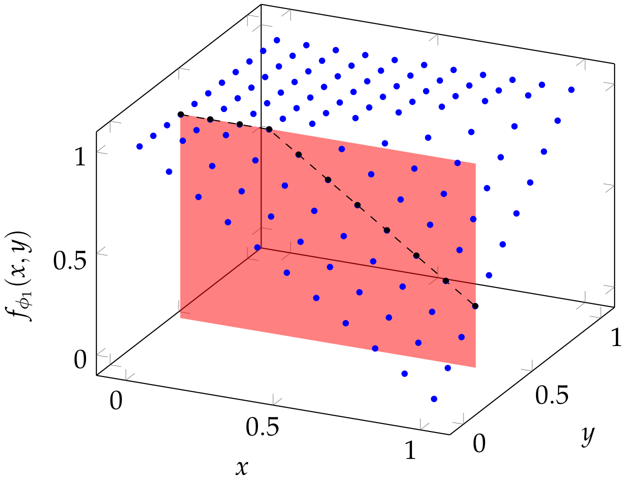
EXAMPLE: THE SLICE OF f_{ϕ_1} AT 0.1



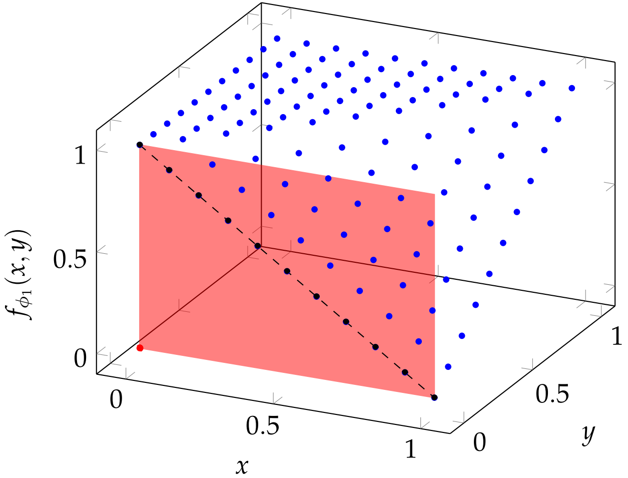
EXAMPLE: THE SLICE OF f_{ϕ_1} AT 0.2



EXAMPLE: THE SLICE OF f_{ϕ_1} AT 0.3

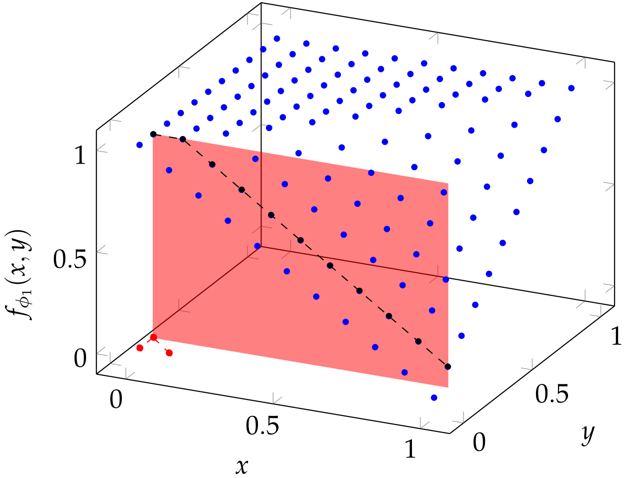


EXAMPLE: DEFINING B_i



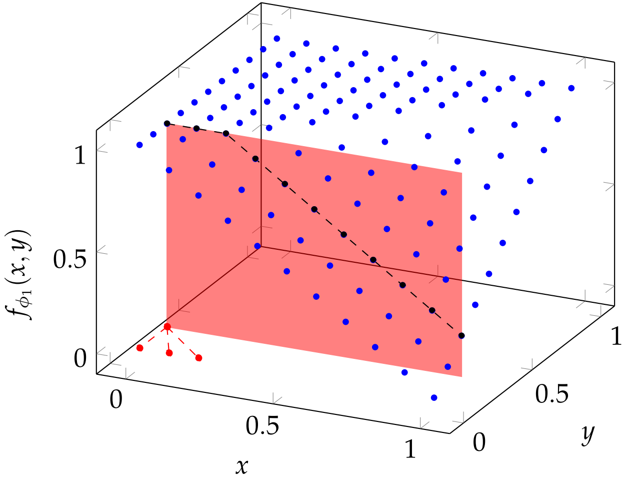
$\{\color{red}(0,0)\}$

EXAMPLE: DEFINING B_i



$\{(0, 0), \color{red}{(0, 0.1), (0.1, 0.1)}\}$

EXAMPLE: DEFINING B_i



$\{(0,0),(0,0.1),(0.1,0.1),{\color{red}(0,0.2),(0.1,0.2),(0.2,0.2)}\}$

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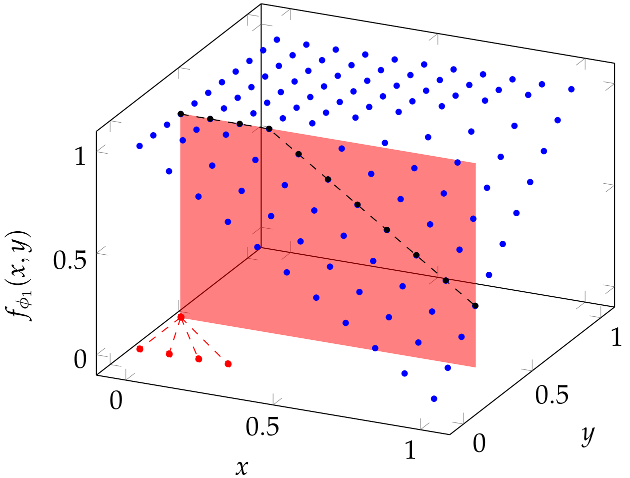
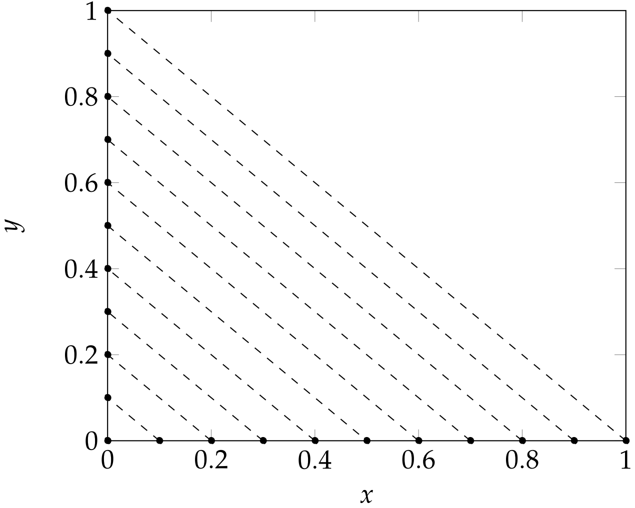


$\{(0, 0), (0, 0.1), (0.1, 0.1), (0, 0.2), (0.1, 0.2), (0.2, 0.2), (0, 0.3), (0.1, 0.3), (0.2, 0.3), (0.3, 0.3)\}$

EXAMPLE: INTERSECTION OF BEST RESPONSE SETS



BEST RESPONSE SETS AND EQUILIBRIA

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OUTLINE

Łukasiewicz Games
    Basic Definitions

Examples
    Traveler's Dilemma

Results
    Theorem
    Best Response Sets
    Equilibrium Formula
    Satisfiable Games

THE FORMULA $\mathcal{E}_{\mathcal{G}}$ (I)

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- We want to define an $Ł_k^c$-formula $\mathcal{E}_{\mathcal{G}}$ whose satisfiability encodes the existence of equilibria.
- $\mathcal{E}_{\mathcal{G}}$ should not require additional constants (apart from the payoff formulas).
- For every variable p and every valuation $v : \{p\} \to L_k$ there exists a formula $\psi(p)$ such that

$$v(p) = \frac{i}{k} \qquad \text{IFF} \qquad v(\psi(p)) = 1.$$

THE FORMULA $\mathcal{E}_\mathcal{G}$ (I)

- We want to define an $Ł_k^c$-formula $\mathcal{E}_\mathcal{G}$ whose satisfiability encodes the existence of equilibria.
- $\mathcal{E}_\mathcal{G}$ should not require additional constants (apart from the payoff formulas).
- For every variable p and every valuation $v : \{p\} \to L_k$ there exists a formula $\psi(p)$ such that

$$v(p) = \frac{i}{k} \qquad \text{IFF} \qquad v(\psi(p)) = 1.$$

- This means that every strategy combination $(s_1, \ldots, s_n)$ can be encoded by a formula $\psi(\vec{p}_1, \ldots, \vec{p}_n)$ so that

$$f_\psi(s'_1, \ldots, s'_n) = 1 \qquad \text{IFF} \qquad s_i = s'_i$$

for all i.

THE FORMULA $\mathcal{E}_{\mathcal{G}}$ (II)

$$
\begin{aligned}
\mathcal{E}_{\mathcal{G}} := \quad & \bigvee_{\vec{s}\in (L_k)^{\sum_{i=1}^{n} m_i}} \Bigg[\bigwedge_{i=1}^{n} \Big(\psi_{\alpha_{1_i}}\left(x_{1_i}\right) \wedge \cdots \wedge \psi_{\alpha_{m_i}}\left(x_{m_i}\right) \Big) \wedge \\
& \qquad \bigwedge_{i=1}^{n} \Bigg[\bigwedge_{s_i \in (L_k)^{m_i}} \Bigg[\psi_{\beta_{1_i}}\left(y_{1_i}^{\beta_{1_i}}\right) \wedge \cdots \wedge \psi_{\beta_{m_i}}\left(y_{m_i}^{\beta_{m_i}}\right) \wedge \\
& \qquad\qquad (\phi_i(x_{1_1}, \ldots, x_{m_1}, \ldots, x_{1_{i-1}}, \ldots, x_{m_{i-1}}, \ldots, y_{1_i}^{\beta_{1_i}}, \ldots, y_{m_i}^{\beta_{m_i}}, \ldots \\
& \qquad\qquad\quad x_{1_{i+1}}, \ldots, x_{m_{i+1}}, \ldots x_{1_n}, \ldots, x_{m_n}) \rightarrow \\
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SATISFIABILITY AND EQUILIBRIA

Let $\mathcal{G}$ be any Łukasiewicz game on $Ł_k^c$. Then there exists a formula $\mathcal{E}_{\mathcal{G}}$ of $Ł_k^c$ so that the following statements are equivalent:

1. $\mathcal{G}$ admits a pure strategy Nash Equilibrium.
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OUTLINE

SATISFIABLE GAMES (I)

- A game $\mathcal{G}$ is called *satisfiable* if there exists a strategy combination

  $$(s_1, \dots .s_n)$$

  such that for every i, ϕ_i is satisfied under $(s_1, \dots .s_n)$.

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- Every satisfiable game admits a NE.
- Every ϕ_i is satisfiable under $(s_1, \ldots .s_n)$, so no player can unilaterally improve her payoff.

SATISFIABLE GAMES (II)

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- Define the formula $E_{\mathcal{G}}$:

$$\exists \vec{x}_1, \dots, \vec{x}_n \forall \vec{y}_1, \dots, \vec{y}_n \quad \textstyle\bigsqcap_{i=1}^{n} \quad \big(\phi_i(\vec{x}_1, \dots, \vec{x}_{i-1}, \vec{y}_i, \vec{x}_{i+1}, \dots, \vec{x}_n) \leq \phi_i(\vec{x}_1, \dots, \vec{x}_{i-1}, \vec{x}_i, \vec{x}_{i+1}, \dots, \vec{x}_n)\big)$$

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- A game $\mathcal{G}$ admits a NE iff $E_{\mathcal{G}}$ holds in $\mathsf{Th}(L_k)$.

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- $E_{\mathcal{G}}$ holds in $\mathsf{Th}(L_k)$ iff the set defined by $E'_{\mathcal{G}}$

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- There exists a quantifier-free $E_{\mathcal{G}}^{free}$ logically equivalent to $E'_{\mathcal{G}}$ that defines the same set as $E'_{\mathcal{G}}$.
- There exists an $Ł_k^c$-formula $\epsilon_{\mathcal{G}}$ that is satisfiable off so is $E_{\mathcal{G}}^{free}$.

SATISFIABLE GAMES (IV)

- Given a game
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WORK IN PROGRESS

- Games with costs and efficiency.
- Classes of games.
- Complexity and tractable games.
- Games with external influence.
- Games with mixed strategies.
- And more...

THANKS!