fMV-algebras and piecewise polynomial functions

Serafina Lapenta

Universitá degli Studi della Basilicata

joint work with Ioana Leuştean

Motivations

- f-algebras are a very well know and studied subject, with several analitics and functional results on them;
- fMV-algebras as common extention of the concept of PMV-algebras and Riesz MV-algebras;
- By means of adjuction fMV-algebras give a different point of view on Birkhoff-Pierce conjecture;
- for both PMV-algebras and fMV-algebras we are able to get a version of Hausdorff Moment Problem.

  It is a very central and important problem in statistic and probability.

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Section 1

Preliminary notions

MV-algebras.

In 1958, C.C. Chang introduced *MV*-algebras as algebraic counterpart of Łukasiewicz logic, and proved Completeness Theorem in the algebraic way.

- Chang, C.C., *Algebraic analysis of many valued logics*, Transactions American Mathematical Society, vol 88 (1958), pp. 467-490.
- Chang, C.C., *A new proof of the completeness of the Łukasiewicz axioms*, Transactions American Mathematical Society, vol 93 (1959), pp.74-80.

MV-algebras

Definition

An MV-algebra is an algebraic stucture A with two operation $\oplus$ and * and a distinguished element 0, that satisfied the following axioms:

for any $x, y, z \in A$,

- $x \oplus y = y \oplus x;$
- $x \oplus (y \oplus z) = (x \oplus y) \oplus z;$
- $x \oplus 0 = x;$
- $(x^*)^* = x;$
- $x \oplus 0^* = 0^*;$
- $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x.$

MV-algebras

A MV-algebra

$x,\ y \in A$

$x \odot y = (x^* \oplus y^*)^*, \qquad x \ominus y = x \odot y^*.$

Order on A:

$$x \leq y \text{ iff } x^* \oplus y = 1.$$

A is a lattice, with

$$x \vee y = (x \odot y^*) \oplus y, \qquad x \wedge y = (x^* \vee y^*)^* = x \odot (x^* \oplus y).$$

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Connection with Łukasiewicz logic

Let $\mathcal{L}$ be the Łukasiewicz propositional calculus.

Definition

Let $\varphi, \psi \in Form_L$, we say $\varphi \equiv \psi$ if and only if $\vdash \varphi \leftrightarrow \psi$.

We define

$$\mathcal{L} = (Form_L / \equiv, \oplus, \neg, 0),$$

where

- $[\varphi] \oplus [\psi] = [\neg\varphi \to \psi]$
- $[\varphi]^* = [\neg\varphi]$
- $0 = [\varphi]$ *where* $\vdash \neg\varphi$.

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$\mathcal{L}$ is a MV-algebra.

Lattice-ordered structures

$(G,+,0,\leq)$ ℓ-group	$(G,+,0)$ group, $(G,\leq)$ lattice, $x \leq y$ implies $x+z \leq y+z$
$(V,+,\{\mathbf{r} \mid r \in \mathbb{R}\},0,\leq)$ Riesz space	$(V,+,0,\leq)$ abelian ℓ-group $(V,+,\{\mathbf{r} \mid r \in \mathbb{R}\},0)$ real vector space $x \leq y$ implies $r \cdot x \leq r \cdot y$ for $r \geq 0$
$(R,+,\cdot,0,\leq)$ ℓ-ring	$(R,+,0,\leq)$ abelian ℓ-group, $(R,+,\cdot,0)$ ring $x \leq y$ implies $x \cdot z \leq y \cdot z$ and $z \cdot x \leq z \cdot y$ for $z \geq 0$
$(A,+,\cdot,\{\mathbf{r} \mid r \in \mathbb{R}\},0,\leq)$ ℓ-algebra	$(A,+,\cdot,0,\leq)$ ℓ-ring $(A,+,\{\mathbf{r} \mid r \in \mathbb{R}\},0,\leq)$ Riesz space $r(x \cdot y) = (rx) \cdot y = x \cdot (ry)$

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Lattice-ordered structures

f-ring (f-algebra) = subdirect product of chains

A f-ring (f-algebra).

for any $x, y \in A$, $z \in A^+$, if $x \wedge y = 0$ then $zx \wedge y = xz \wedge y = 0$.

Definition

A strong unit for an ℓ-group G is an element $u \geq 0$ such that, for each $x \in G$ there is an integer $n \geq 0$ with $|x| \leq nu$.

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Mundici's categorial equivalence

Theorem

The category of ℓ-groups with strong unit and the category of MV-algebras are equivalent.

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Product MV-algebras

Definition

- *Let A be an MV-algebra, for any $x, y \in A$*

  $x + y$ *is defined iff* $x \leq y^*$, *and* $x + y = x \oplus y$.

- *A admits product, if there is a binary operation $\cdot$ such that*

  (i) if $x + y$ is defined in A, also $z \cdot x + z \cdot y$ and $x \cdot z + y \cdot z$ are defined, and

  $$z \cdot (x + y) = z \cdot x + z \cdot y, \quad (x + y) \cdot z = x \cdot z + y \cdot z.$$

  (ii) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

  for any $x, y, z \in A$.

- *A PMV-algebra that has unit for $\cdot$, it is called unital.*

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- *A PMV-algebra that has unit for $\cdot$, it is called unital.*

Definition

A $PMVf$-algebra is a PMV-algebra such that

if $x \wedge y = 0$, then $x \cdot z \wedge y = z \cdot x \wedge y = 0$, for any x, y, z in the algebra.

Theorem

PMV-algebras are categorical equivalent to lattice-ordered rings with strong unit.

Di Nola A., Dvurecenskij A., *Product MV-algebras*, Multiple-Valued Logics 6 (2001), 193-215.

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Riesz MV-algebras

Definition

$(R, \star, \oplus, {}^*, 0)$ such that

$(R, \oplus, {}^*, 0)$ is a MV-algebra and $\star : [0,1] \times R \to R$ satisfies

- $r \star (x \odot y^*) = (r \star x) \odot (r \star y)^*$,
- $(r \odot q^*) \star x = (r \star x) \odot (q \star x)^*$,
- $r \star (q \star x) = (rq) \star x$,
- $1 \star x = x$.

for any $r, q \in [0,1]$ and any $x, y \in R$:

Riesz MV-algebras, equivalent definition

Definition

$[0,1]$ standard PMV-algebra, $(R, \star, \oplus, {}^*, 0)$ such that

(i) $x + y$ defined in R then $r \star x + r \star y$ defined and

$r \star (x + y) = r \star x + r \star y$,

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$x, y \in R$, $r, q \in [0,1]$.

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Theorem

Riesz MV-algebras with linear MV-algebra homomorphisms are categorical equivalent to Riesz Spaces with strong unit and linear ell-groups maps.

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Section 2

fMV-algebras

fMV-algebras: basic definitions

Definition

A PMV-algebra and Riesz MV-algebras. A is an fMV-algebra if

(f1) if $x \wedge y = 0$ then $x \wedge (z \cdot y) = x \wedge (y \cdot z) = 0$;

(f2) $\alpha(x \cdot y) = (\alpha x) \cdot y = x \cdot (\alpha y)$.

$\alpha \in [0,1]$ and any $x, y, z \in A$

The PMV-algebra reduct of an fMV-algebra is a PMVf-algebra.

fMV-algebras: basic definitions

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The PMV-algebra reduct of an fMV-algebra is a PMVf-algebra.

Equational characterization

Theorem

A is a fMV-algebra if and only if it satifies the following conditions:

$$(M1)\quad \alpha(x \odot y^*) = (\alpha x) \odot (\alpha y)^*$$

$$(M2)\quad (\alpha \odot \beta^*)x = (\alpha x) \odot (\beta) x)^*$$

$$(M3)\quad \alpha(\beta x) = (\alpha \cdot \beta)x$$

$$(M4)\quad 1x = x$$

$$(P1a)\quad z \cdot (x \odot (x \wedge y)^*) = (z \cdot x) \odot (z \cdot (x \wedge y))^*$$

$$(P1b)\quad (x \odot (x \wedge y)^*) \cdot z = (x \cdot z) \odot ((x \wedge y) \cdot z)^*$$

$$(P2)\quad x \cdot (y \cdot z) = (x \cdot y) \cdot z$$

$$(F1a)\quad (z \cdot (x \odot y^*)) \wedge (y \odot x^*) = 0$$

$$(F1b)\quad ((x \odot y^*) \cdot z) \wedge (y \odot x^*) = 0$$

$$(F2)\quad \alpha(x \cdot y) = (\alpha x) \cdot y = x \cdot (\alpha y).$$

Categorical equivalence

- **fMValg**, category whose objects are *fMV*-algebras and whose morphisms are *MV*-algebras homomorphisms that preserve both internal and external product.
- **falg**, category whose objects are *f*-algebras with strong unit u such that $u \cdot u \leq u$ and whose morphisms are linear ℓu-ring homomorphisms, that is ℓu-rings homomorphisms that preserve the external product.

We will call Γ_f the functor from **falg** to **fMValg** that extend Mundici's functor Γ.

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Categorical equivalence

Theorem

The functor Γ_f *establish a categorical equivalence between the category* **falg** *whose objects are f-algebras with strong unit and whose morphisms are* ℓ*u-rings homomorphisms preserving the external product, and the category* **fMValg** *whose objects are fMV-algebra and whose morphisms are MV-homomorphisms preserving both products.*

Ideals and Representation Theorem

Definition

Let I be a subset of an fMV-algebra A. We will call I f-ideal if:

(I1) I is an MV-ideal;

(I2) for any $x \in A$, $y \in I$ we have $x \cdot y \in I$ and $y \cdot x \in I$;

(I3) for any $\alpha \in [0, 1]$ and any $x \in I$, $\alpha x \in I$.

Theorem

Any fMV-algebra A is subdirect product of totally ordered fMV-algebras.

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Theorem

Any fMV-algebra A is subdirect product of totally ordered fMV-algebras.

Special classes of *fMV*-algebras: Semiprime

Definition

(i) An f-algebra $\mathbf{V}$ is called *semiprime* if the only nilpotent element is 0. That is, if $x \cdot x = 0$, then $x = 0$ for any $x \in V$.

(ii) An fMV-algebra $\mathbf{A}$ is called *semiprime* if the only nilpotent element is 0.

They are related to Montagna's PMV^{+}.

Montagna F., *Subreducts of MV-algebras with product and product residuation*, Algebra Universalis 53 (2005) pp 109-137.

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Special classes of *fMV*-algebras: Semiprime

Definition

By fMV^+ we will denote the class of unital, commutative and semiprime fMV-algebras.

Proposition

A fMV-algebra A is semiprime if and only if the corresponding f-algebra V arising from the categorical equivalence is semiprime.

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The class of fMV^+-algebras is the quasi-variety generated by $[0,1]$.

Special classes of *fMV*-algebras: Semiprime

Proposition

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Theorem

The class of fMV^+*-algebras is the quasi-variety generated by* $[0, 1]$.

Special classes of fMV-algebras: Formally real

Definition

A fMV-algebra (PMV-algebra) is *formally real* if it belongs to HSP([0, 1]). We denote by $\mathbb{FR}$ the class of formally real fMV-algebras.

Theorem

For any formally real fMV-algebra A there exists an ultrapower of $^*[0,1]$ *of* $[0,1]$ *such that A embedds in* $(^*[0,1])^I$, *for some set I.*

Outline of the proof.

It is just an application of Theorem 4.2 of the paper

Flaminio T., Bianchi M., *A note for saturated models for many valued logic*, Mathematica Slovaca, submitted.

□

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Not any unital and commutative fMV-algebras is formally real

Example

It follows from Example 3.14 in

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Section 3

Piecewise polynomial functions and moment problem

Terms and term functions

-κ cardinal number;

-$\alpha < \kappa$, define $\pi_\alpha^\kappa : A^\kappa \mapsto A$, $\pi_\alpha^\kappa(a_1, \ldots, a_\alpha, \ldots) = a_\alpha$.

-S be a subring of $\mathbb{R}$

$\mathcal{L}_S$ is the alphabet $\{\oplus, ^*, \cdot, 0, \} \cup \{\delta_r \mid r \in [0,1] \cap S\}$, δ_r is a unary operation that is interpreted by $x \mapsto rx$ for any $r \in [0,1] \cap S$.

-a *term* over the set of variables $\{X_\alpha\}_{\alpha<\kappa}$ is a finite string of element over the alphabet $\mathcal{L}_S$.

$$Term_n^A(S) = \{\text{terms in the language } \mathcal{L}_S\}$$

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Definition

$t \in Term_n^A(S)$, and A a fMV-algebra. The *term function* $\tilde{t} : A^n \mapsto A$ of t is defined by

(i) For any $m \leq n$, $\widetilde{X_m} = \pi_m^n$;

(ii) $\widetilde{0}$ is the constant function equal to 0 on A^n;

(iii) $\widetilde{t^*} = (\widetilde{t})^*$;

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(v) $\widetilde{\delta_r t} = r\widetilde{t}$;

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$FT_n^A(S) = \{\widetilde{t} : A^n \mapsto A \mid t \in Term_n^A(S) \text{ and } \widetilde{t} \text{ is the term function of } t\}$

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$FR_n = \{\tilde{t} \mid t \in Term_n, \ \tilde{t} : [0,1]^n \to [0,1]$ *is the term function of* $t\}$;

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Piecewise polynomial functions

Definition

A *piecewise polynomial function in n variables with coefficients in S* ($\text{PWP}_n(S)$-*function*, shortly) is $f : \mathbb{R}^n \to \mathbb{R}$ such that:

there exists a finite number of polynomials f_1, ..., $f_k \in S[x_1, \ldots, x_n]$ such that

for any $(a_1, \ldots, a_n) \in [0,1]^n$ there is $i \in \{1, \ldots, k\}$ with $f(a_1, \ldots, a_n) = f_i(a_1, \ldots, a_n)$.

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Notations

$PF_n(S) =$
$\{f : [0,1]^n \to [0,1] \mid f$ *is a cont.* $PWP_n(S)$*-function def. on the n-cube*$\}$

$PF_n(S)_r = \{f|_{[0,1]^n} | f : \mathbb{R}^n \to [0,1]$ *is a continuous* $PWP_n(S)$*-function*$\}$

$SID_n(S) = \{g : [0,1]^n \to [0,1] \mid g \in PF_n(S),\ g = \bigvee_{i\in I} \bigwedge_{j\in J} g_{ij}\}$ *where* g_{ij} *are polynomials in* $S[x_1, \ldots, x_n]$.

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A different description for FR_n

Proposition

The elements of $FT_n(S)$ are continuous piecewise polynomial functions defined on the n-cube, i.e. $FT_n(S) \subseteq PF_n(S)$.

Definition

$\varrho : \mathbb{R} \mapsto [0,1]$, $\varrho(x) = x \wedge 1 \vee 0$, for any $x \in \mathbb{R}$.

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Let S be a subring of $\mathbb{R}$.
(a)For any polynomial function $p : [0,1]^n \to \mathbb{R}$ with coefficients in S, there exists a term $t \in Term_n(S)$ such that $\varrho \circ p = \widetilde{t}$ and $\widetilde{t} \in FT_n(S)$.
(b)For any continuous function $g \in SID_n(S)$ there exists a term $t \in Term_n(S)$ such that $g = \widetilde{t}$.

A different description for FR_n

Corollary

(1) $SID_n(S) \subseteq FT_n(S) \subseteq PF_n(S)$

(2) $SID_n(S) \subseteq PF_n(S)_r \subseteq PF_n(S)$.

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Theorem

For $n \leq 2$, $PF_n(\mathbb{R})_r = PF_n(\mathbb{R}) = FR_n = SID_n(\mathbb{R})$.
In consequence, the fMV-algebra FR_n is the set of all continuous piecewise polynomial functions defined on the n-cube, i.e any continuous piecewise polynomial functions defined on the n-cube is a term function from FR_n.

- Birkhoff-Pierce conjecture is proved for $n < 3$ in
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Moment Problem

Given a interval $I \subseteq \mathbb{R}$, the n^{th}-moment of a probability measure μ on I is defined as $\int_I x^n d\mu$. Let $\{m_k\}_{k\geq 0}$ be a sequence of real numbers, the Moment Problems on I consistes on finding out the condition on $\{m_k\}_{k\geq 0}$ for which there exists a probability measure μ on I such that m_k is the k^{th} moment of μ.

When $I = [0,1]$ we get the Hausdorff moment problem

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A *state* for a MV-algebra A is a map $s : A \to [0,1]$ such that for any $x, y \in A$ with $x \odot y = 0$, $s(x \oplus y) = s(x) + s(y)$ and $s(1) = 1$.

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$\{m_k | k \geq 0\} \subseteq [0,1]$.
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The sequence $\{m_k\}_k$ *satisfies the* Hausdorff moment condition *if* $m_0 = 1$ *and* $(-1)^r \Delta^r m_k \geq 0$ *for any* $r, k \geq 0$.

$C([0,1]) = \Gamma(C([0,1], \mathbb{R}), 1)$, C *be any semisimple* PMV*-subalgebra (unital and commutative) of* $C([0,1])$ *such that* $p_1 \in C$.

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There exists a state $s : C \to [0,1]$ *such that* $s(p_k) = m_k$ *if and only if the sequence* $\{m_k\}$ *satisfies the* Hausdorff moment condition.

Outline of the proof.

By Kroupa-Panti rapresentation for states we get $s(f) = \int_0^1 f d\mu$. Then it follows by calculations.

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Corollary

There exists a state $s : FR \to [0,1]$ *such that* $s(p_k) = m_k$ *if and only if the sequence* $\{m_k\}$ *satisfies the* Hausdorff moment condition.

Conclusions

- definition and categorical equivalence for fMV-algebras;
- description of special classes of fMV-algebras;
- for $n \leq 2$ a different description of the free formally real fMV-algebra with two generators, relying on Birkhoff-Pierce conjecture;
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THANK YOU FOR YOUR ATTENTION