Sándor Jenei

University of Pécs, Hungary

NEW INVOLUTIVE FL_e-ALGEBRA CONSTRUCTIONS

Supported by the SROP-4.2.2.C-11/1/KONV-2012-0005 grant.

About the topic ...

- Connections between games and many-valued logic

ABOUT THE TOPIC ...

- My game is connectives for many-valued logic

Definitions

- *FL_e-algebra* = comm. RL + f, f is an arbitrary constant
- *involutive* = $x'' = x$, where $x' = x \to f$ (observe $f' = t$)
- *integral* = t is its greatest element
- *Group-like* = involutive + $f = t$

Conic Representation

- Conic representation: For any conic, involutive FL_e-algebra

$$x \circledast y = \begin{cases} x \otimes y & \text{if } x, y \in X_1 \\ x \oplus y & \text{if } x, y \in X_2 \\ (x \to_\oplus y')' & \text{if } x \in X_2,\ y \in X_1, \text{ and } x \leq y' \\ (y \to_\oplus x')' & \text{if } x \in X_1,\ y \in X_2, \text{ and } x \leq y' \\ (y \to_\otimes (x' \wedge t))' & \text{if } x \in X_2,\ y \in X_1, \text{ and } x \not\leq y' \\ (x \to_\otimes (y' \wedge t))' & \text{if } x \in X_1,\ y \in X_2, \text{ and } x \not\leq y' \end{cases}$$

- [S. Jenei, H. Ono, On Involutive FL_e-monoids, Archive for Mathematical Logic, 51 (7-8), 719-738 (2012)]

Twin Rotation

ASubL 4.pdf (page 5 of 19)

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Call $\circledast$ the twin-rotation of $\otimes$ and $\oplus$.

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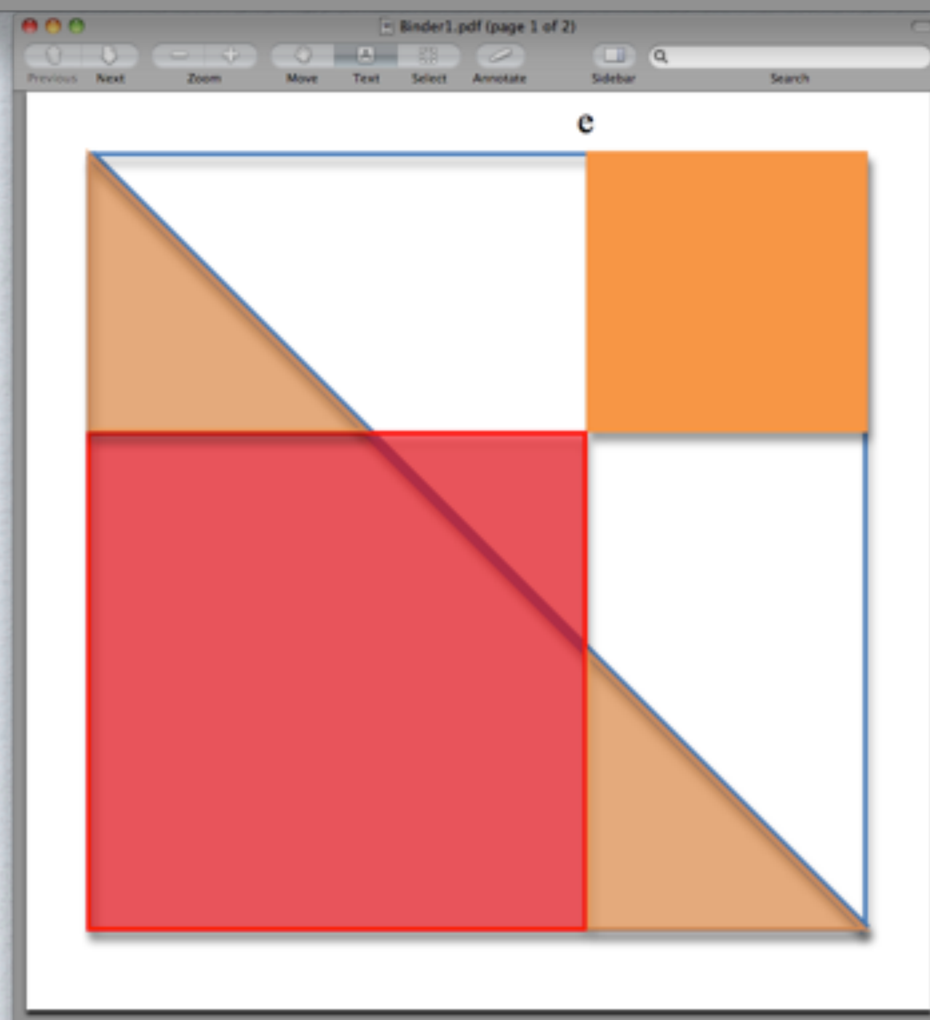

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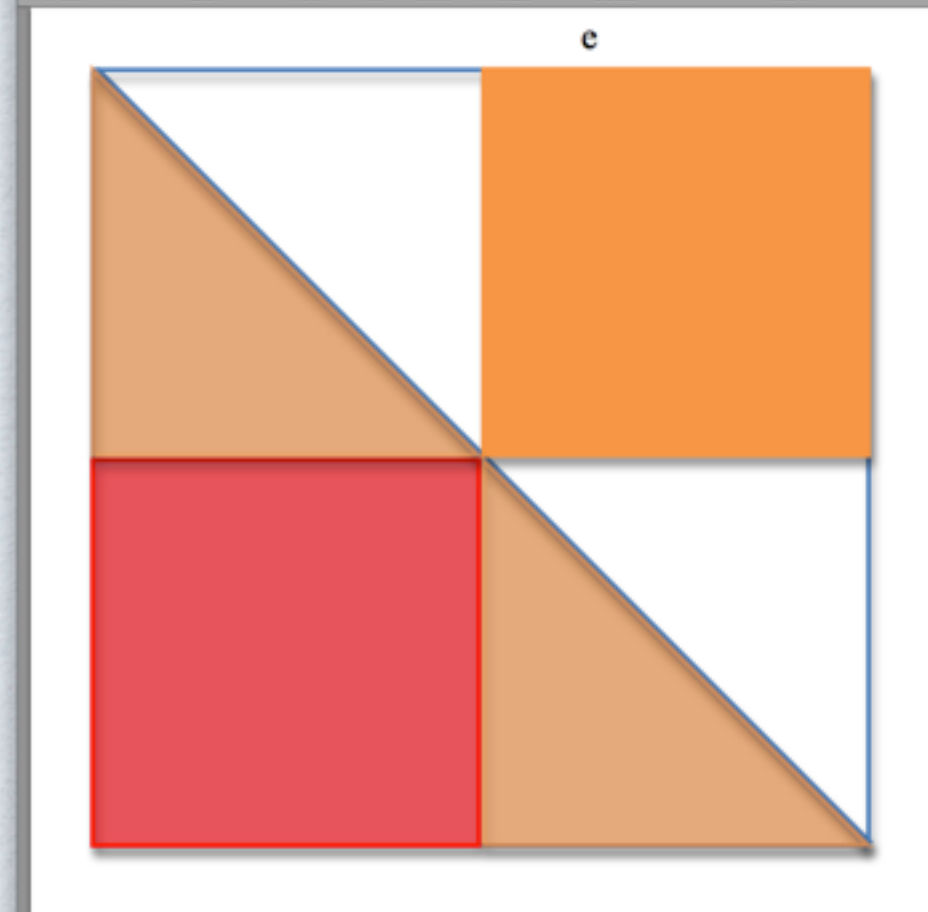

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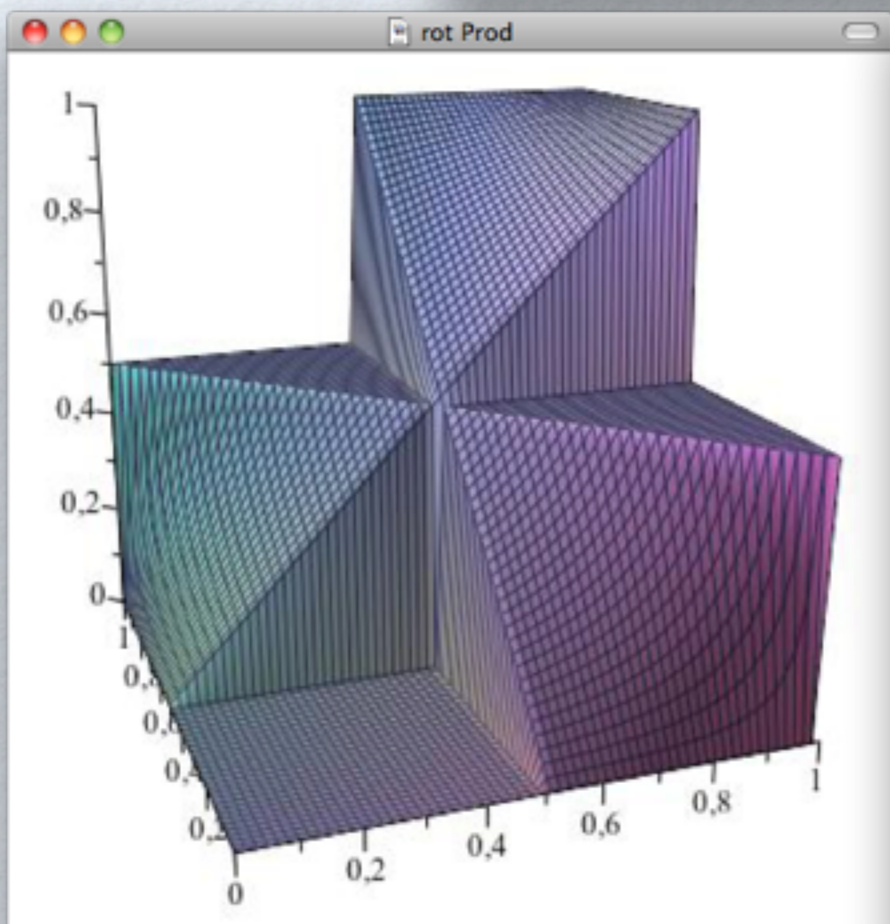

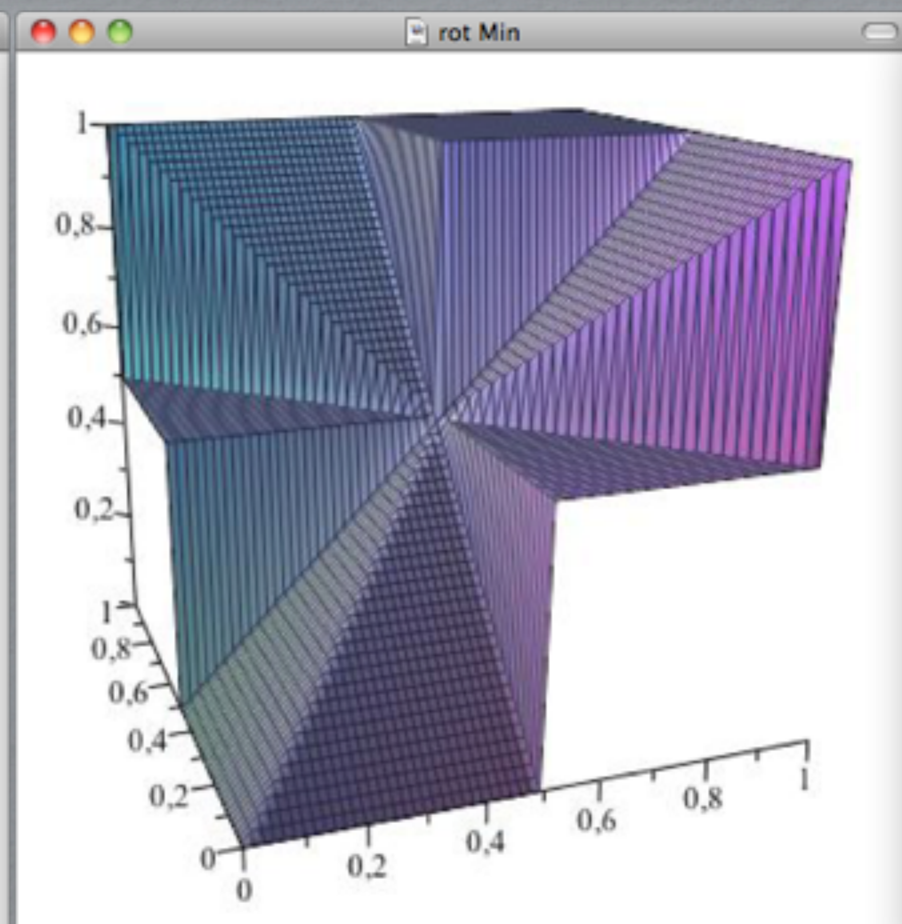

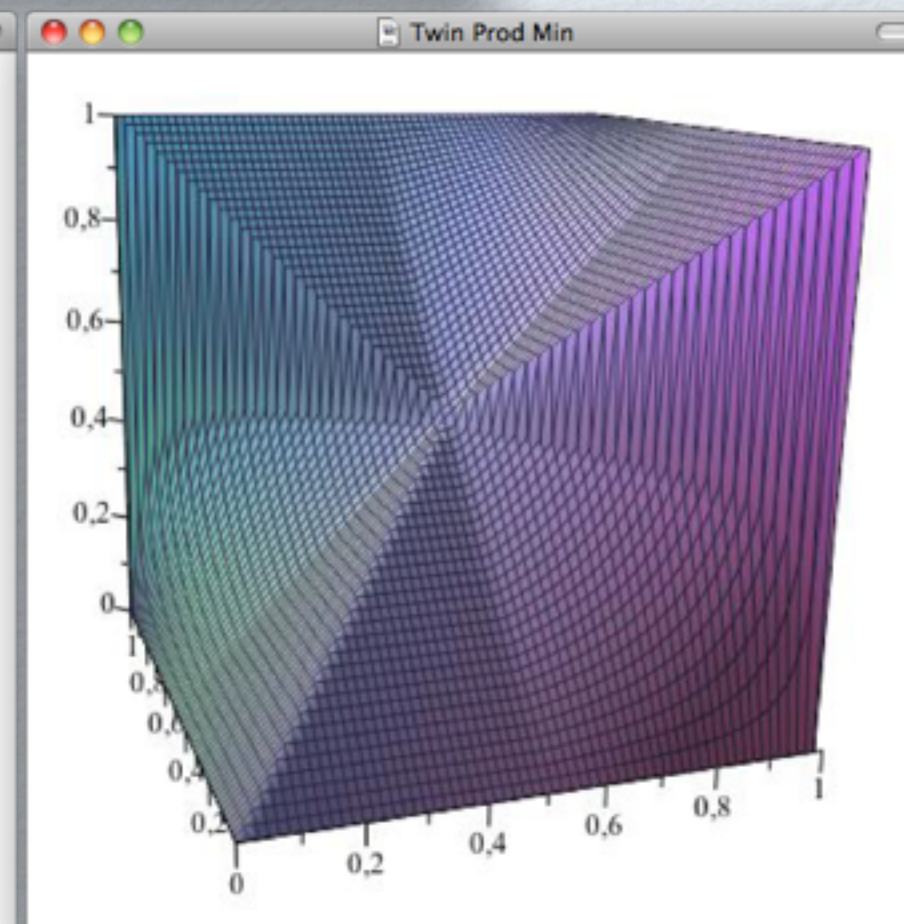

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\end{cases}
$$

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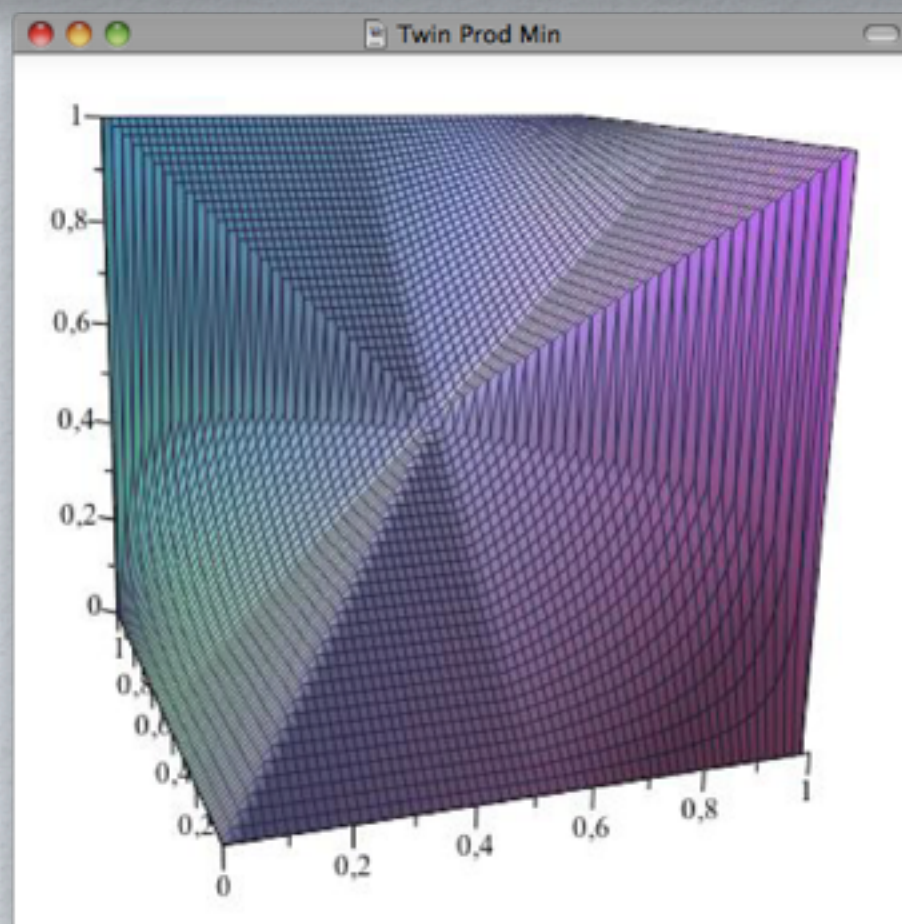
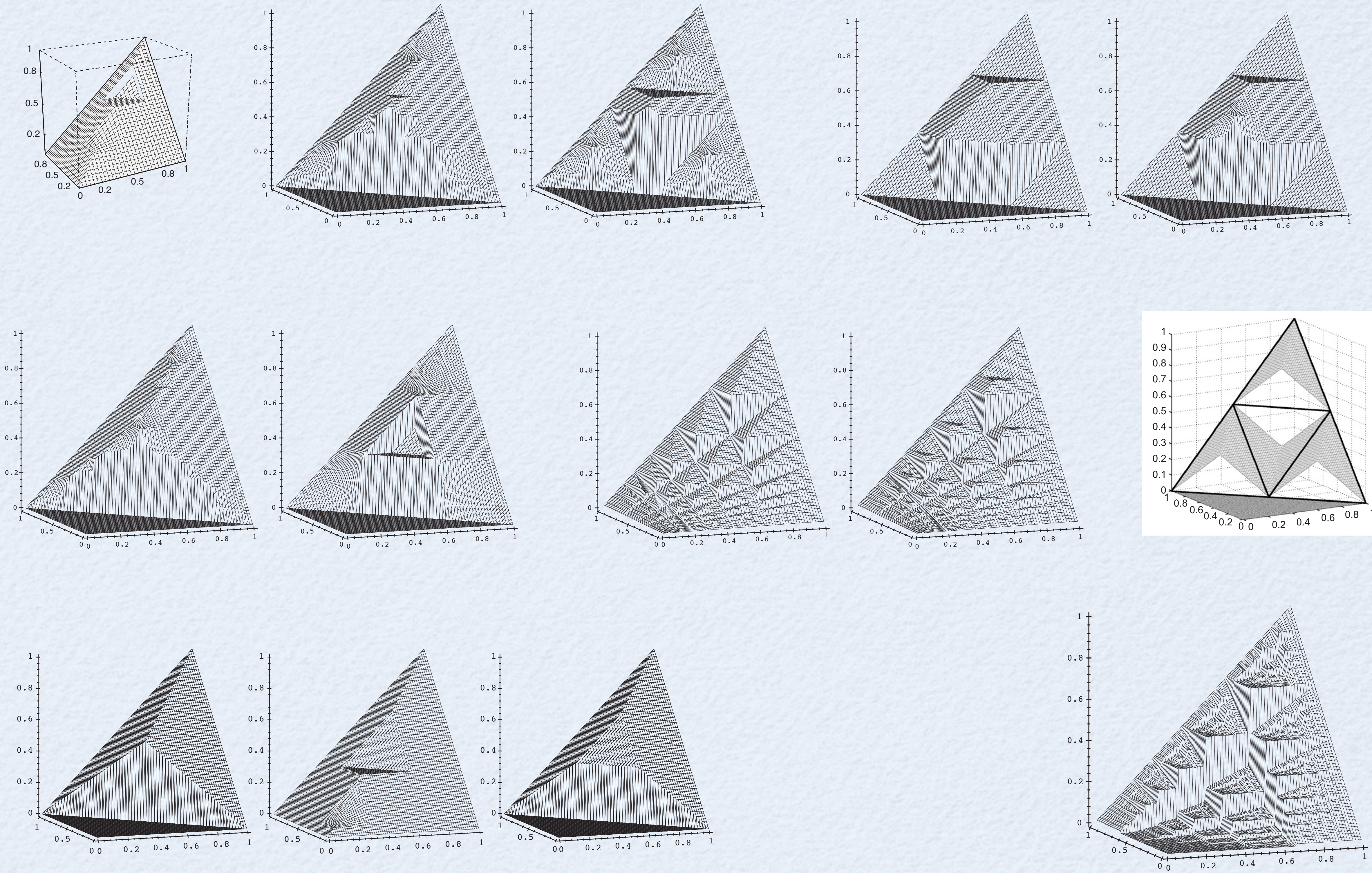
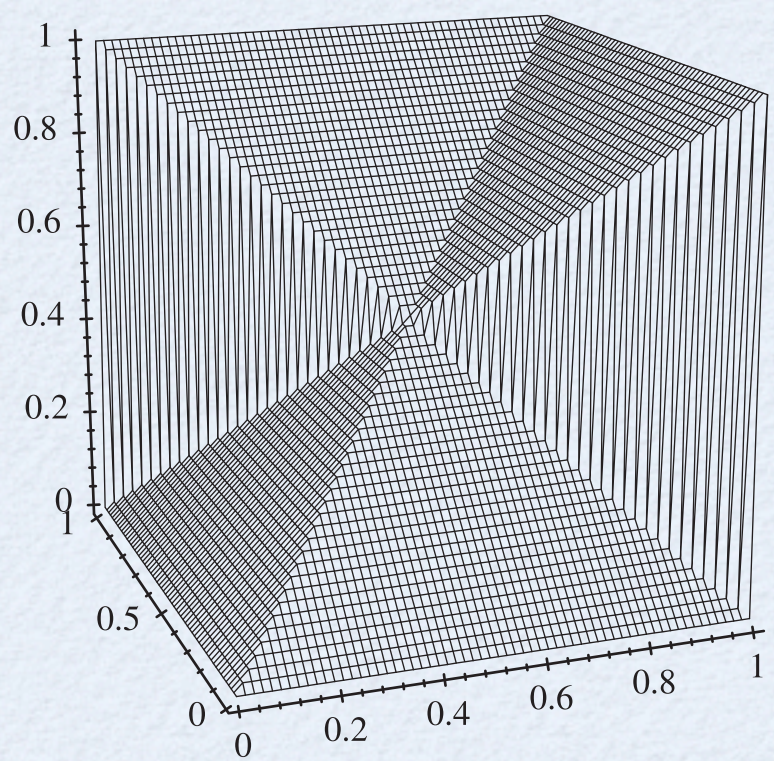


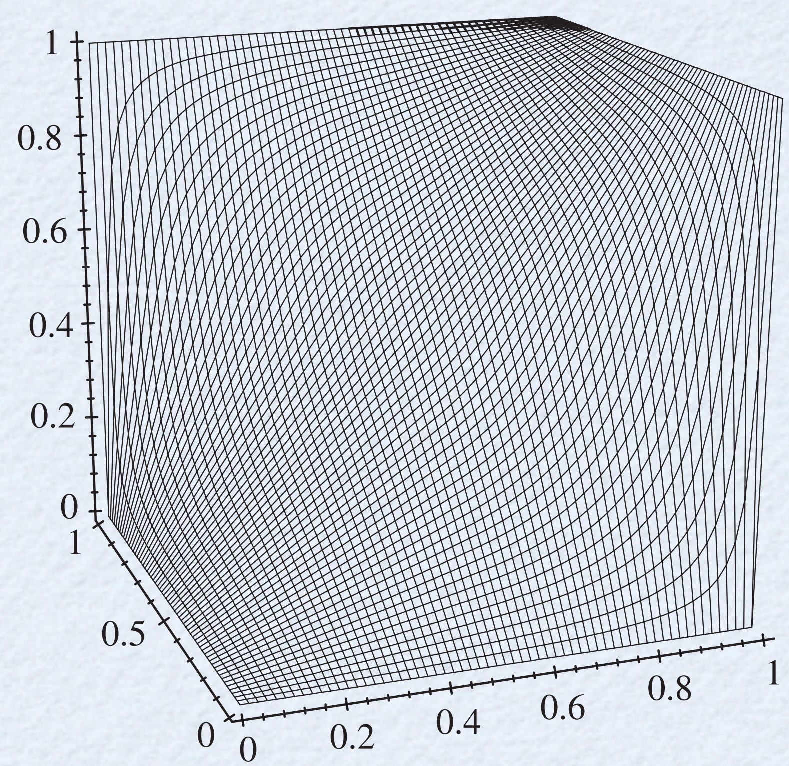
An Unchartable Wilderness

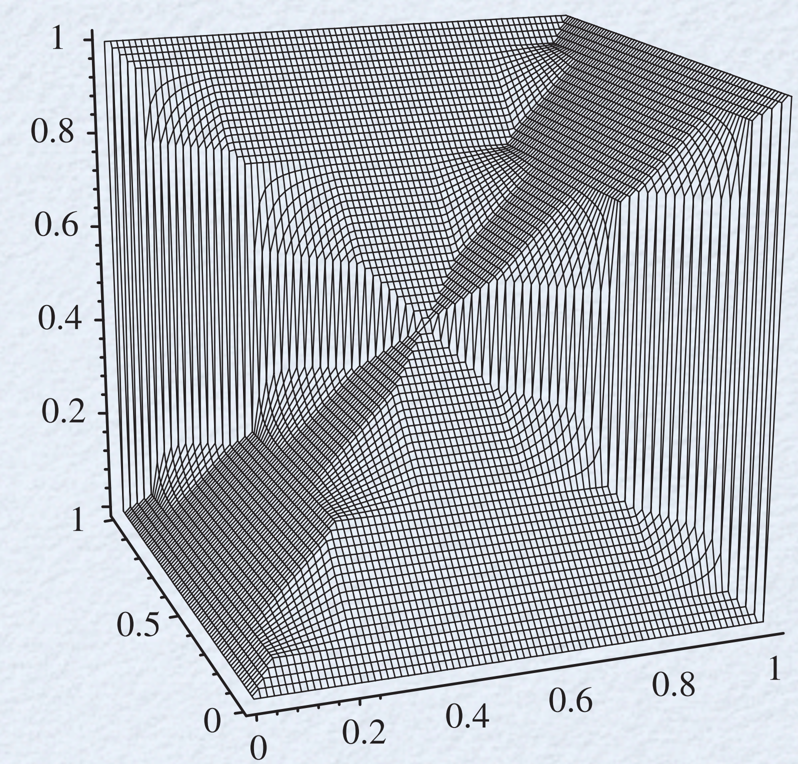


GROUP-LIKE CASE

Absorbent-continuous group-like FL_e-algebras on subreal chains







- [S. Jenei, F. Montagna, A classification of certain group-like FL_e-chains, submitted]

Absorbent-continuous group-like FL_e-algebras on subreal chains

- Call a chain $\langle X, \leq \rangle$ *weakly real* if X is order-dense and complete, there exists a dense $Y \subset X$ with $|Y| < |X|$, and for any $x,y \in Y$ there exist $u,v \in Y$ such that $u>x, v>y$, and there exists a strictly increasing function from [x,u] into [y,v].

- An order dense chain is said to be *subreal* if its Dedekind-MacNeille completion is weakly real.

- Absorbent continuity = for $x \in X^-$, $a(x) \otimes x = x$, where $a(x) = inf\{ u \in X^- : u \otimes x = x \}$

Absorbent-continuous group-like FL_e-algebras on subreal chains

- BL-algebras = divisibility (continuity) **everywhere**
- Absorbent continuity = continuity only **at a few point** of the domain of $\otimes$ (viewed as a two-place function)

Absorbent-continuous group-like FL_e-algebras on subreal chains

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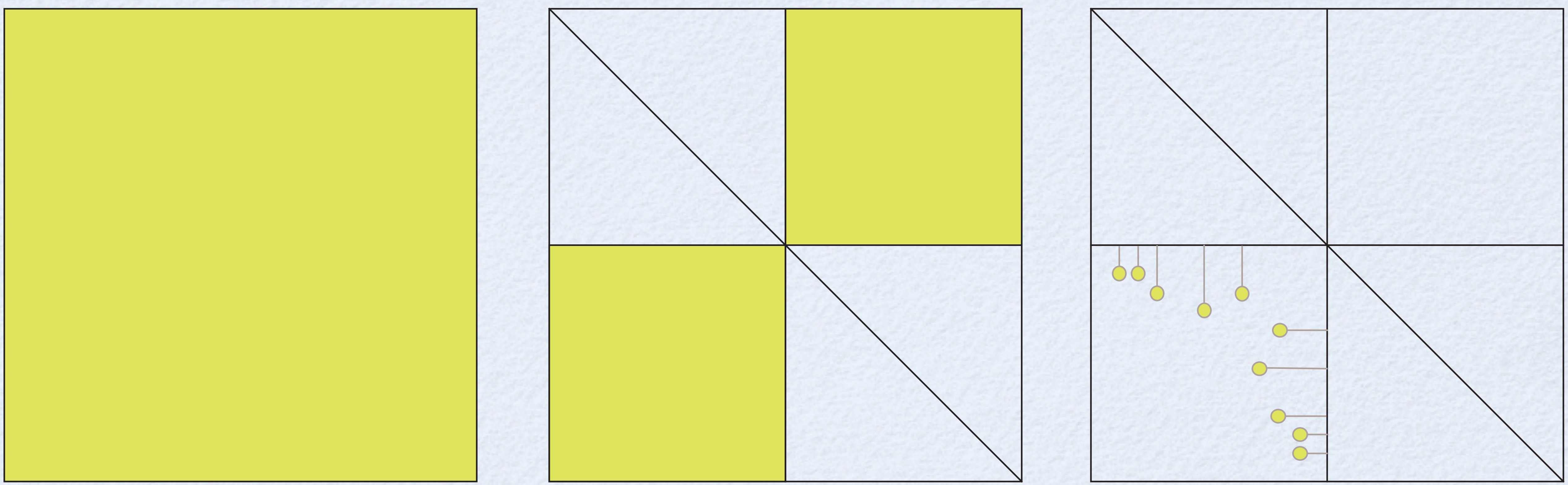

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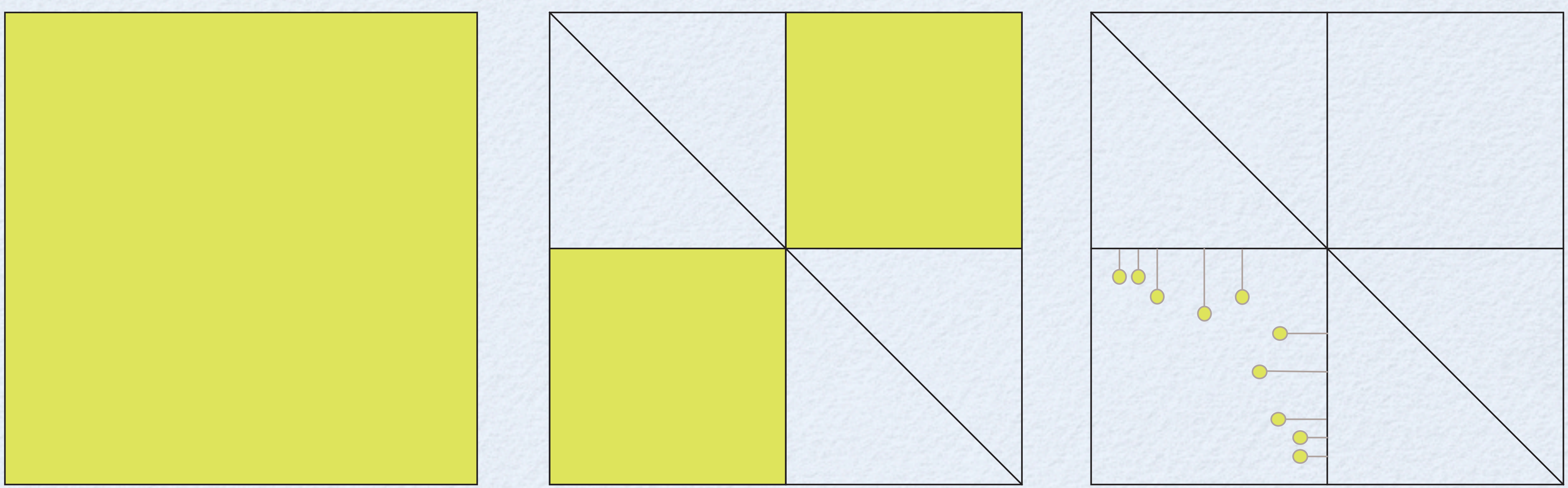
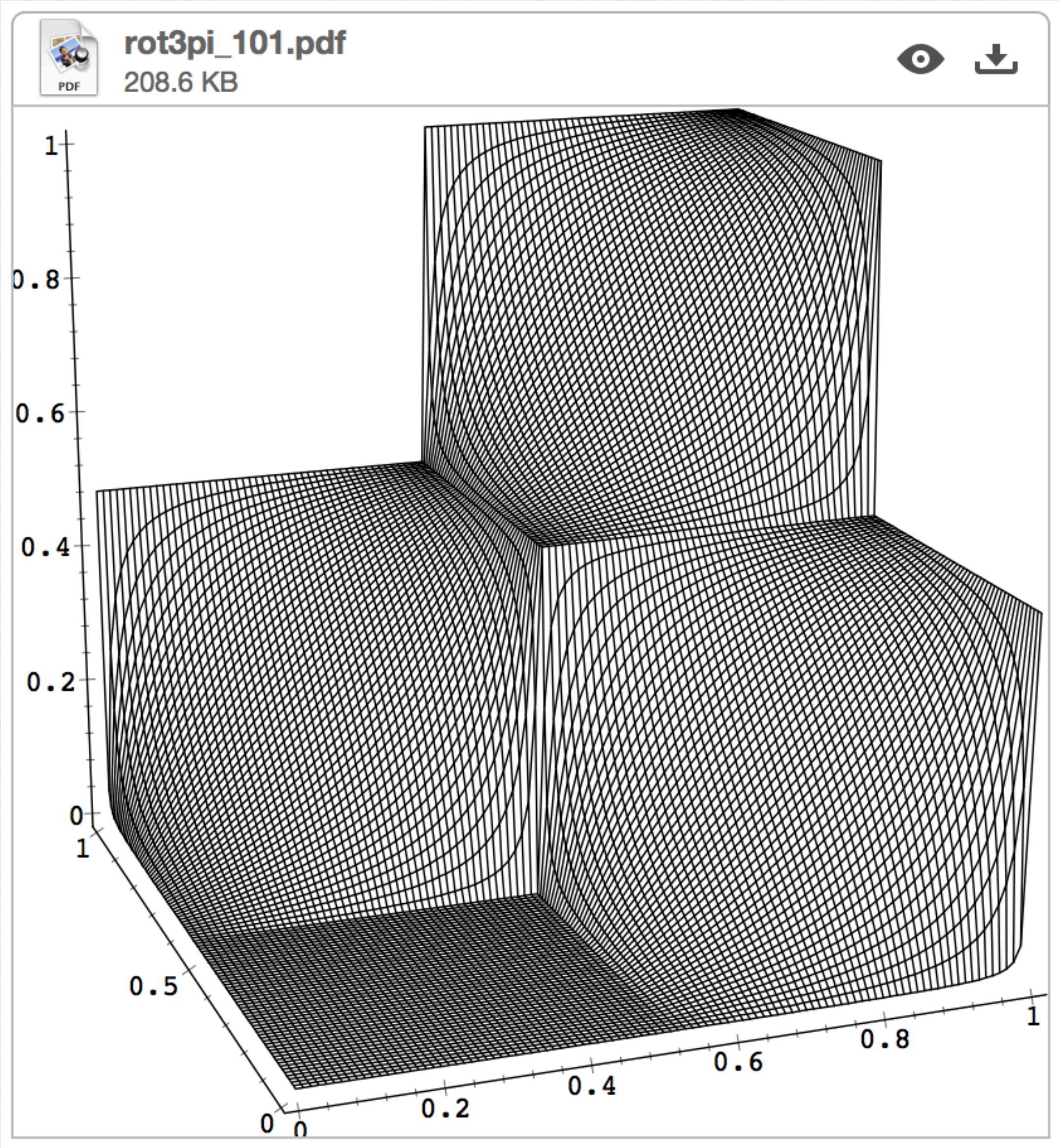


1: Involutive ordinal sums

- Theorem: The twin-rotation of the Clifford-style ordinal sum of any family of negative cones of group-like FL_e-chains and their skew-duals is a group-like FL_e-chain.

MOTIVATION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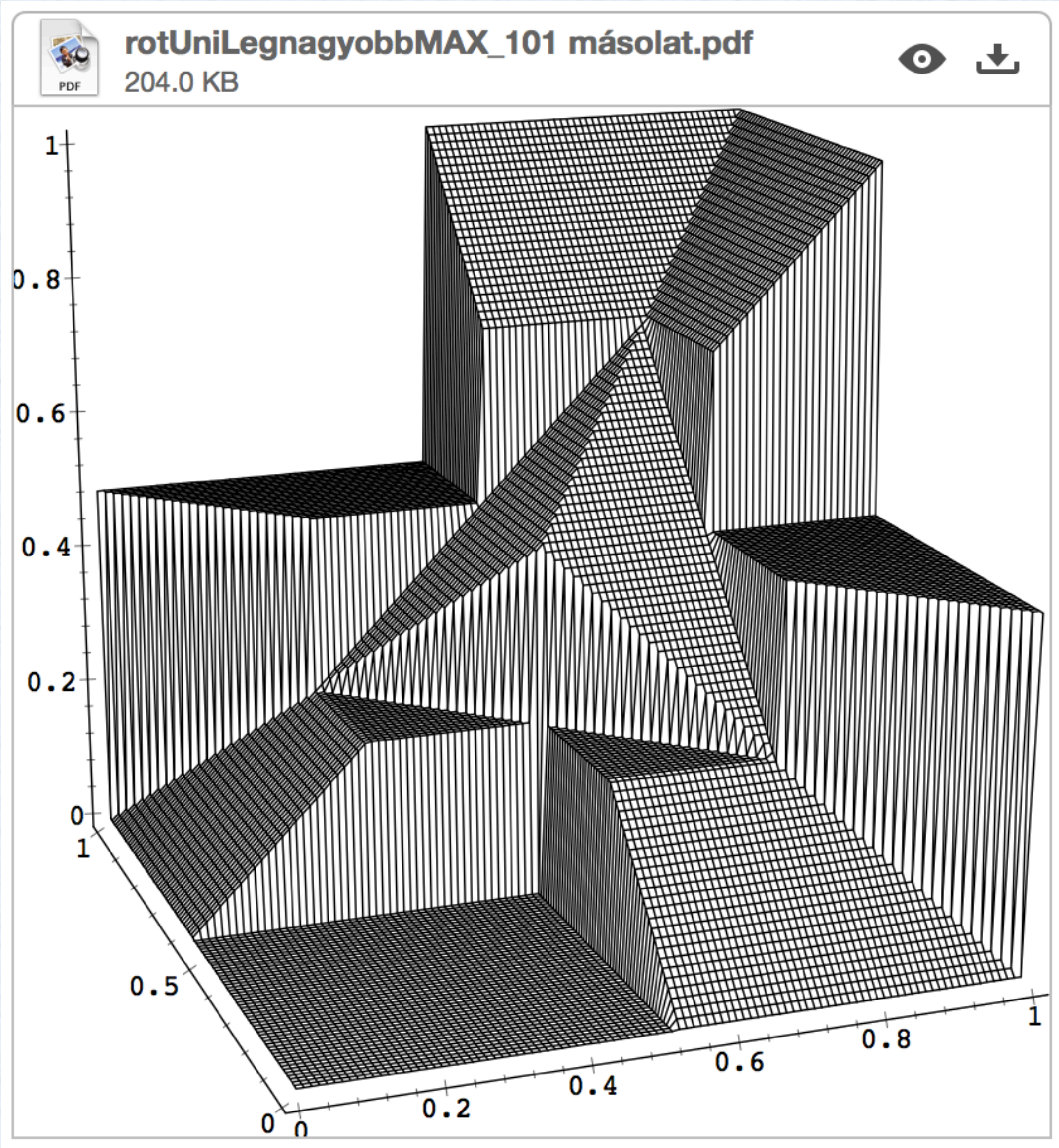
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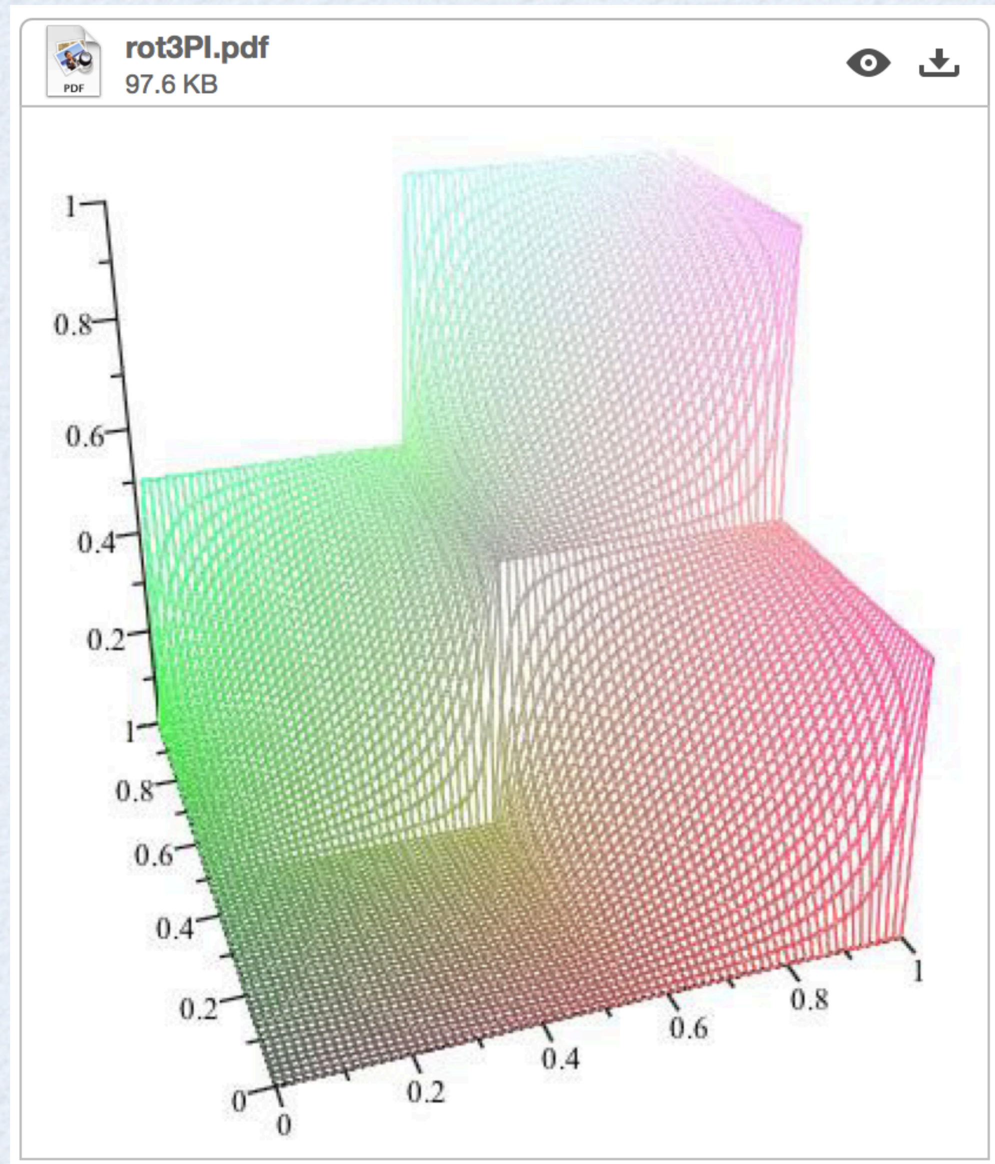
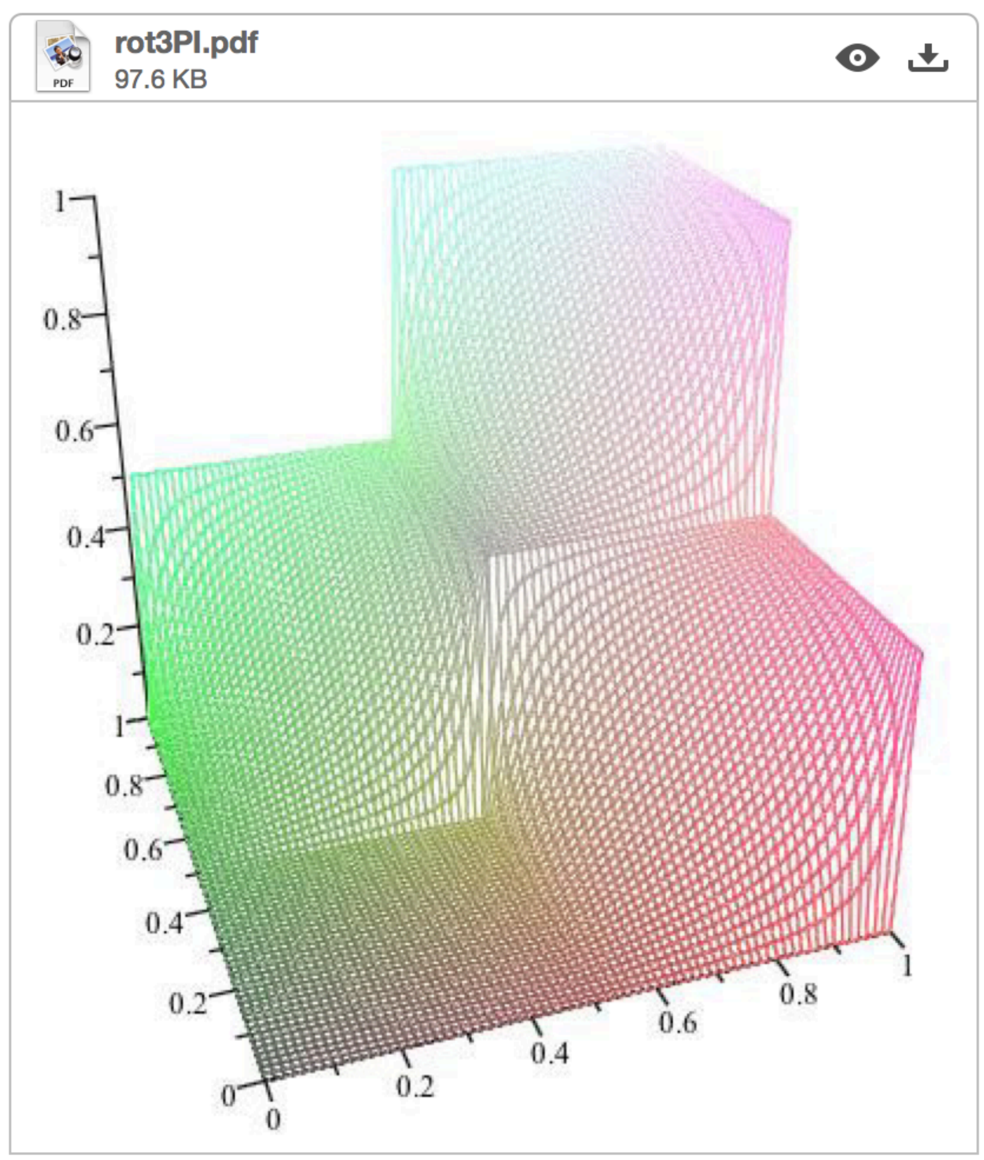
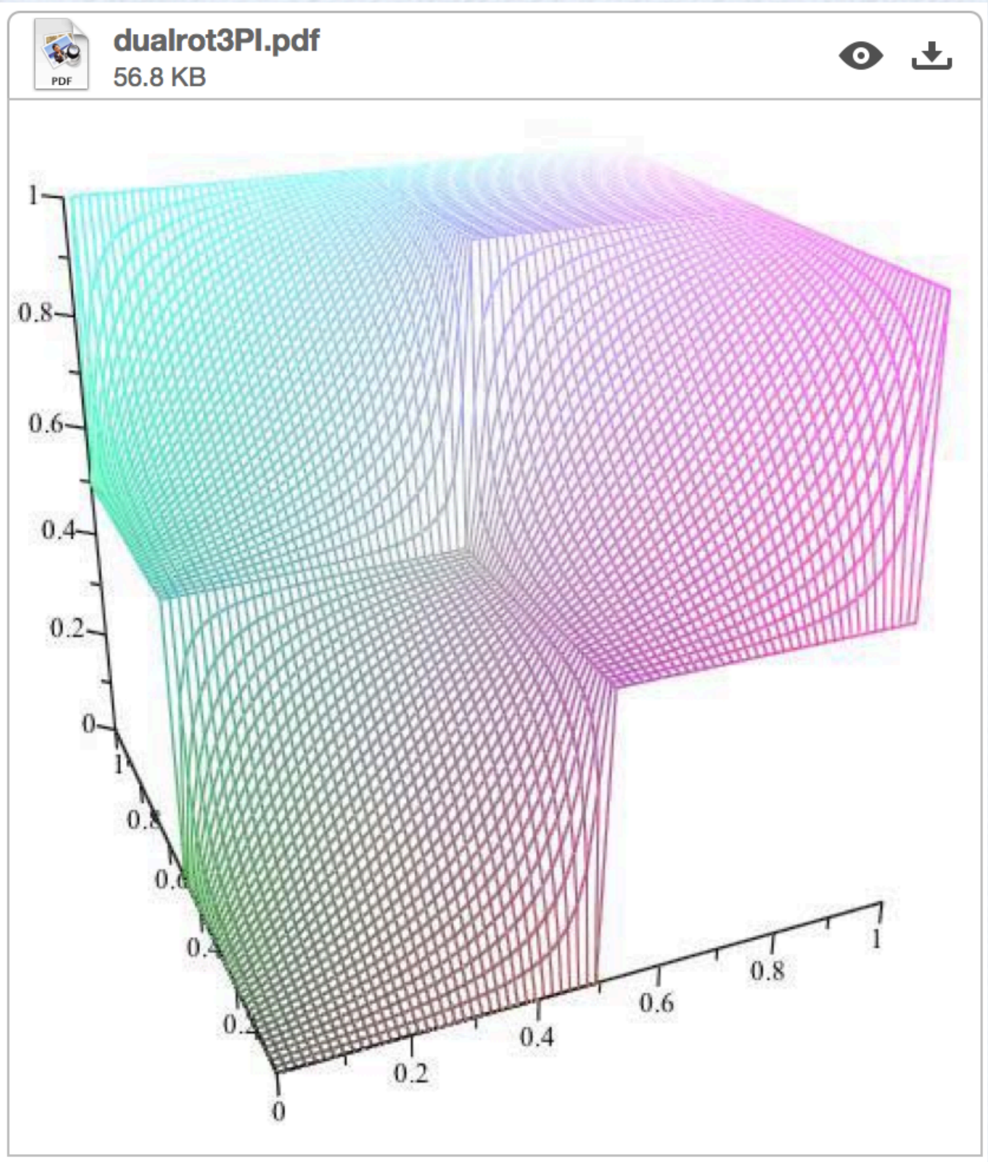
Motivation

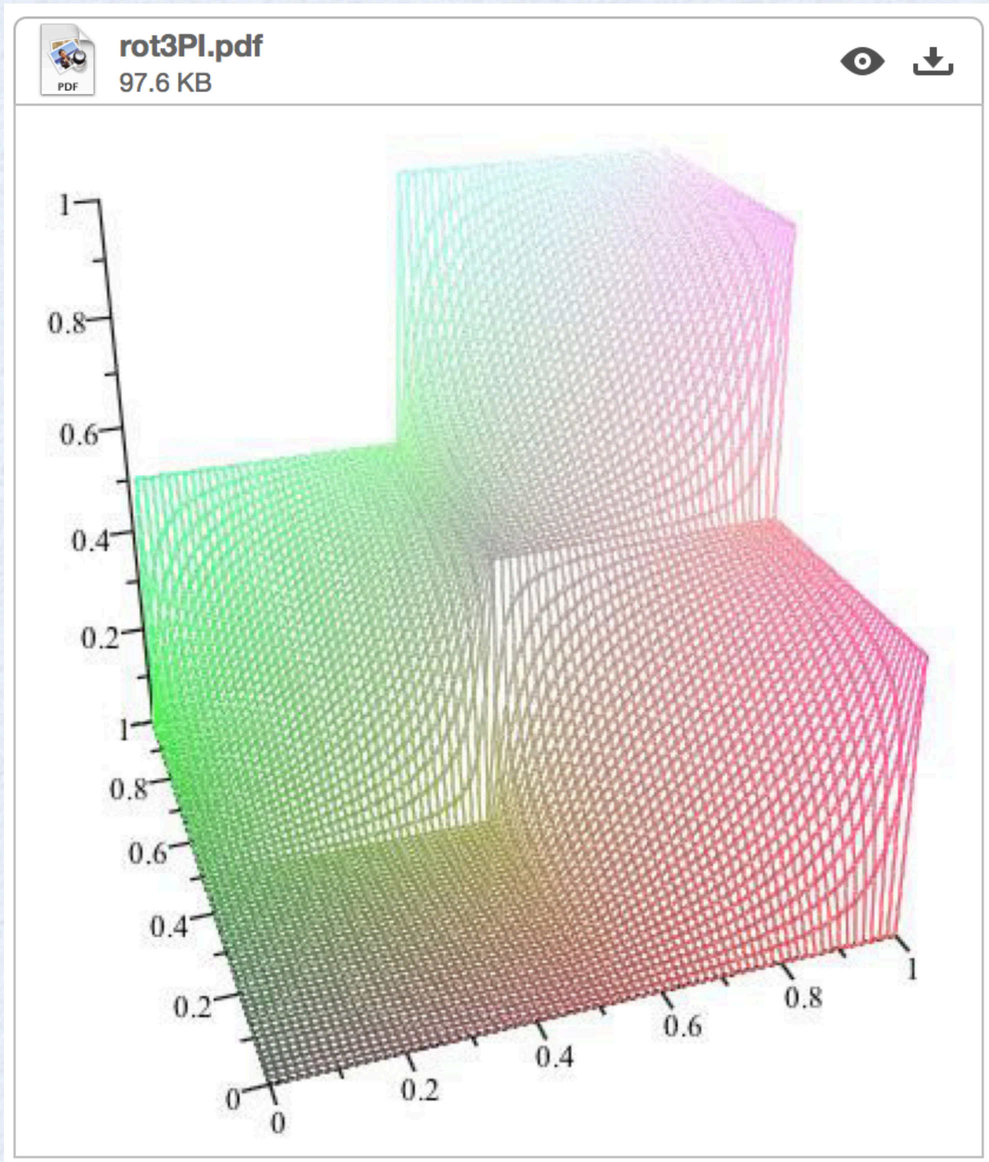


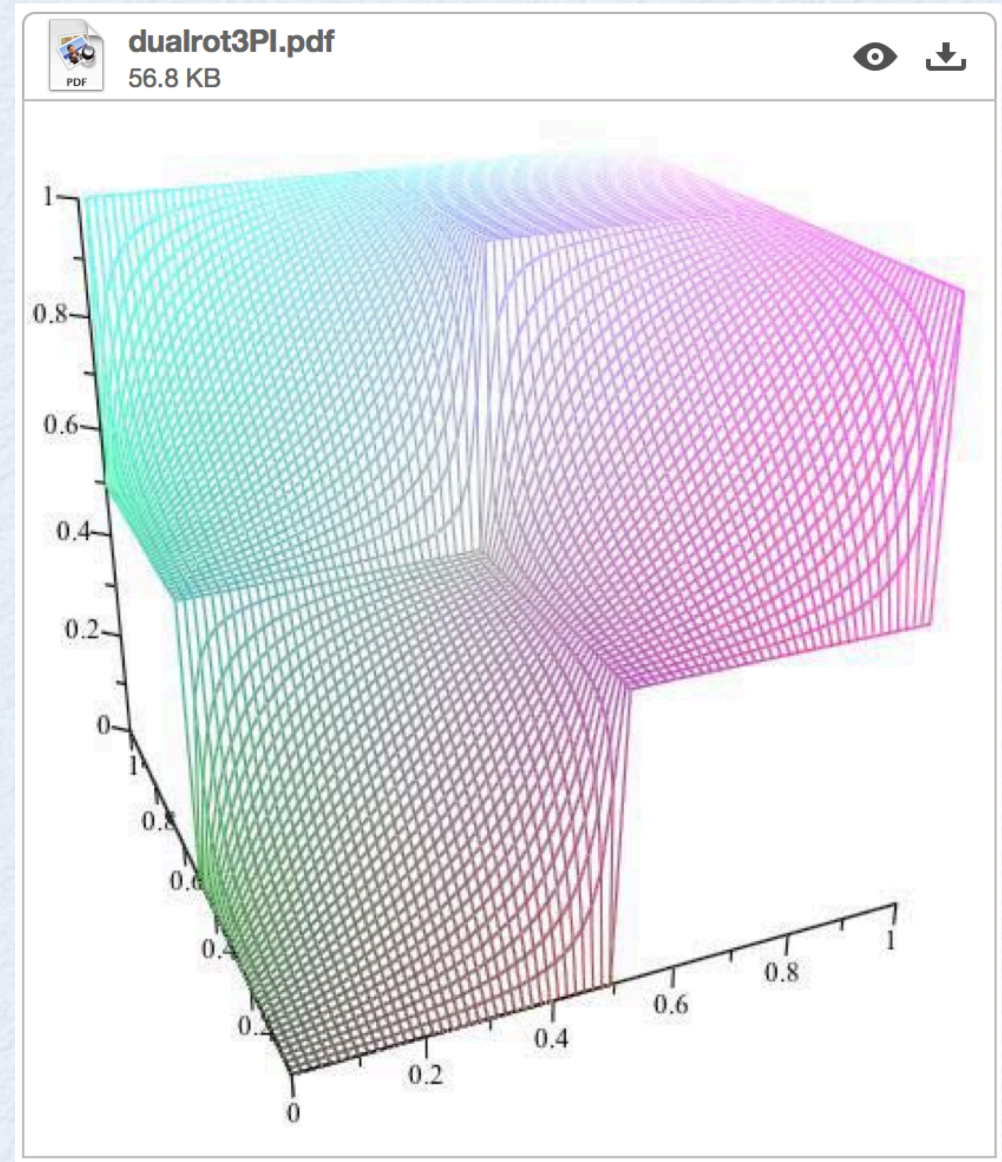
Motivation





MOTIVATION

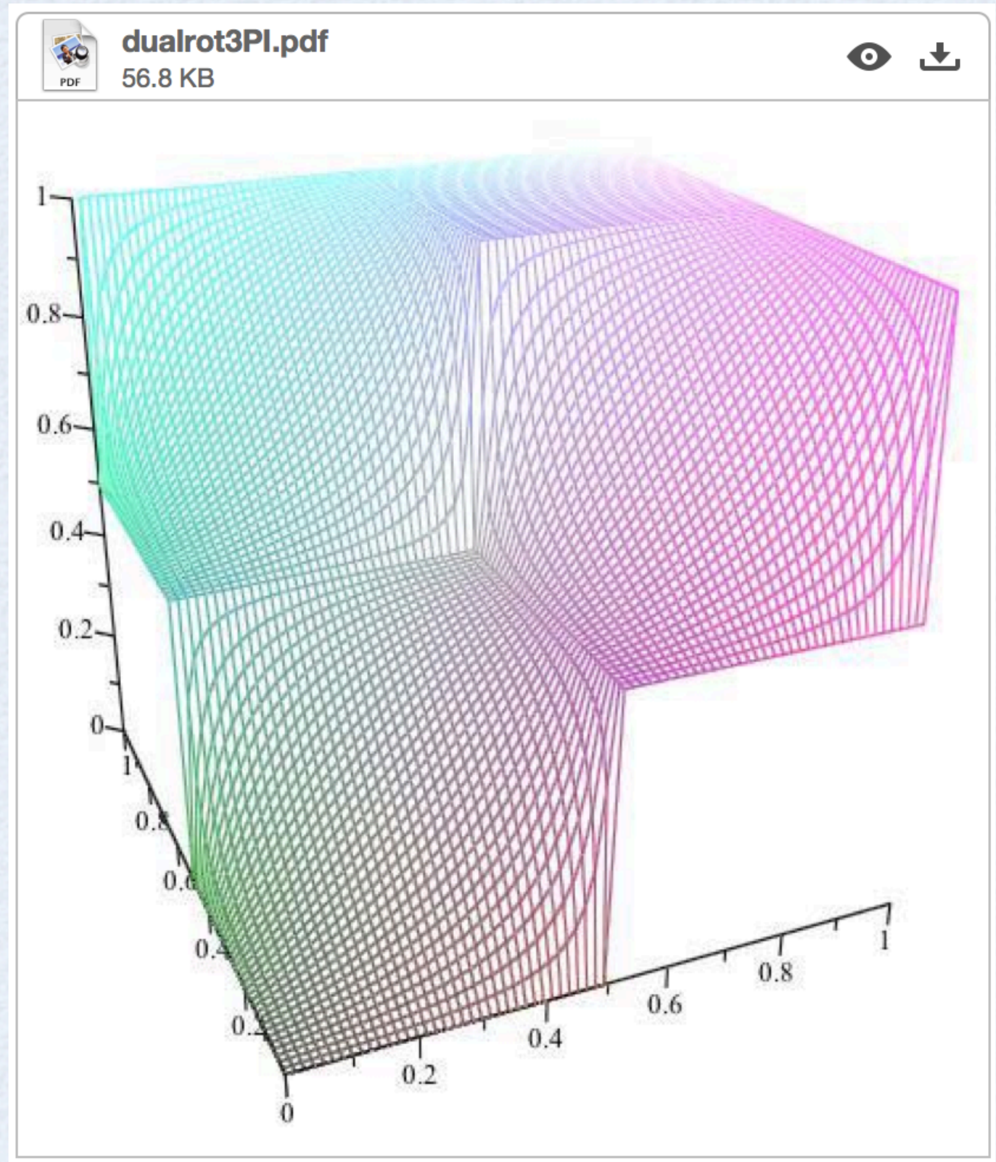




$$x \odot y = \sup\{u \circledbullet v \mid u < x, v < y\}$$

skewed modification of $\odot$.

Motivation



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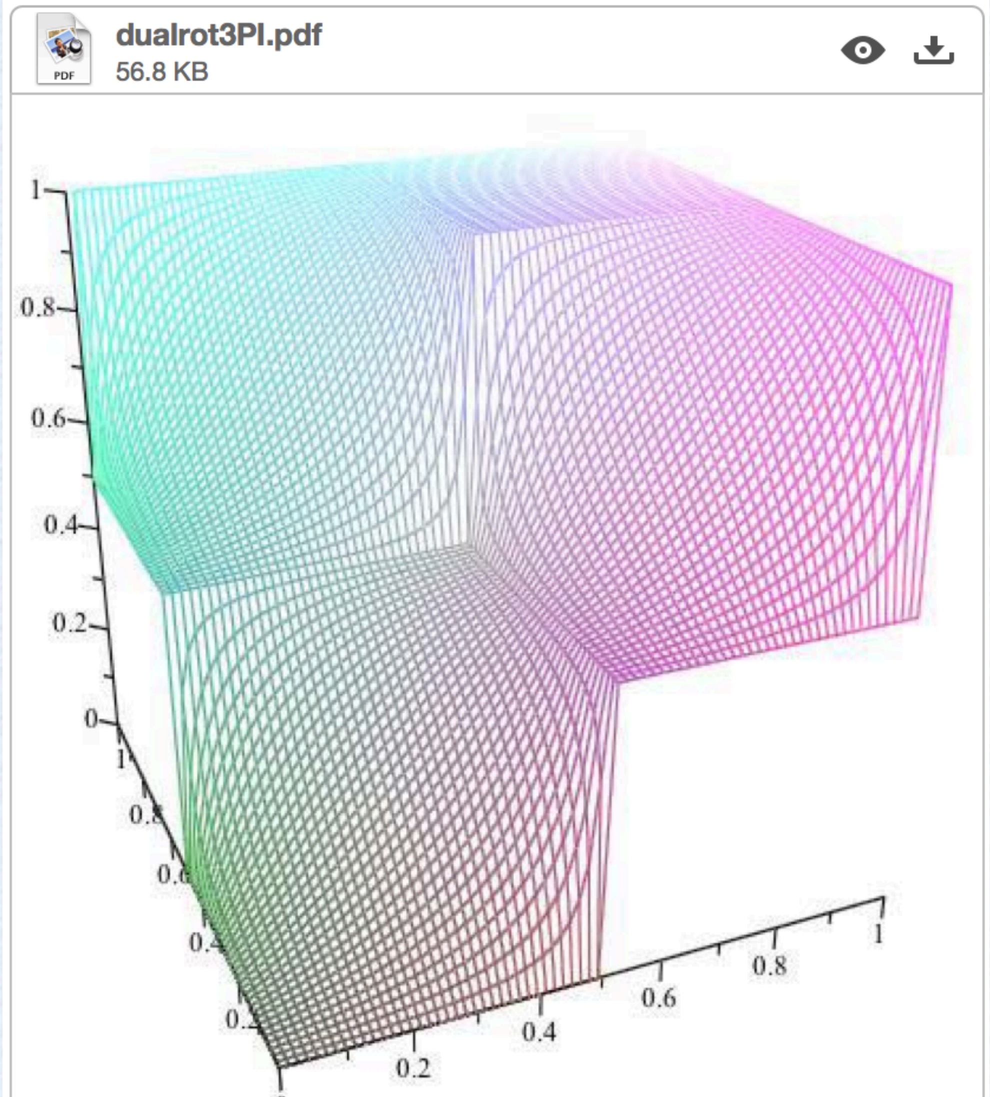
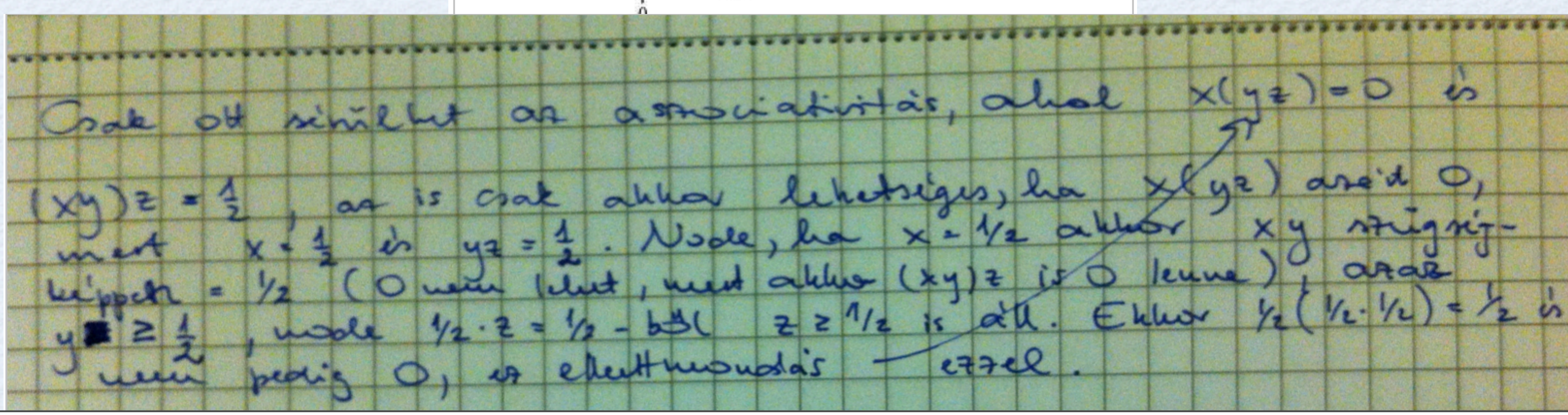
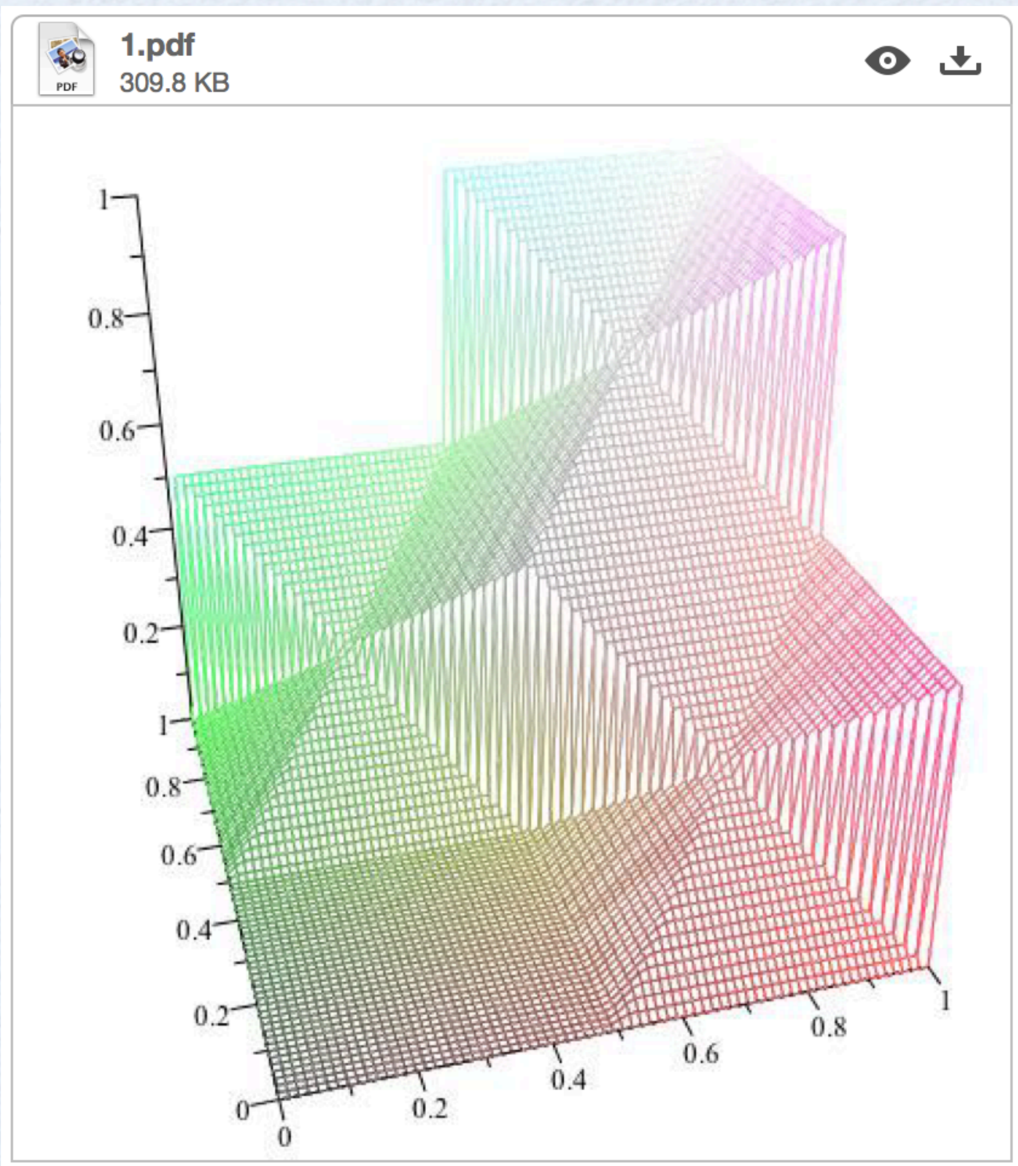



Csak ott sérülhet az asszociativitás, ahol $x(yz) = 0$ és $(xy)z = \frac{1}{2}$, az is csak akkor lehetséges, ha $x(yz)$ azért 0, mert $x \leq \frac{1}{2}$ és $yz = \frac{1}{2}$. Node, ha $x = 1/2$ akkor xy szükségképpen $= 1/2$ (0 nem lehet, mert akkor $(xy)z$ is 0 lenne), azaz y [redacted] $\geq \frac{1}{2}$, node $1/2 \cdot z = 1/2$-ből $z \geq 1/2$ is áll. Ekkor $1/2(1/2 \cdot 1/2) = 1/2$ és nem pedig 0, az ellentmondás ezzel.

Motivation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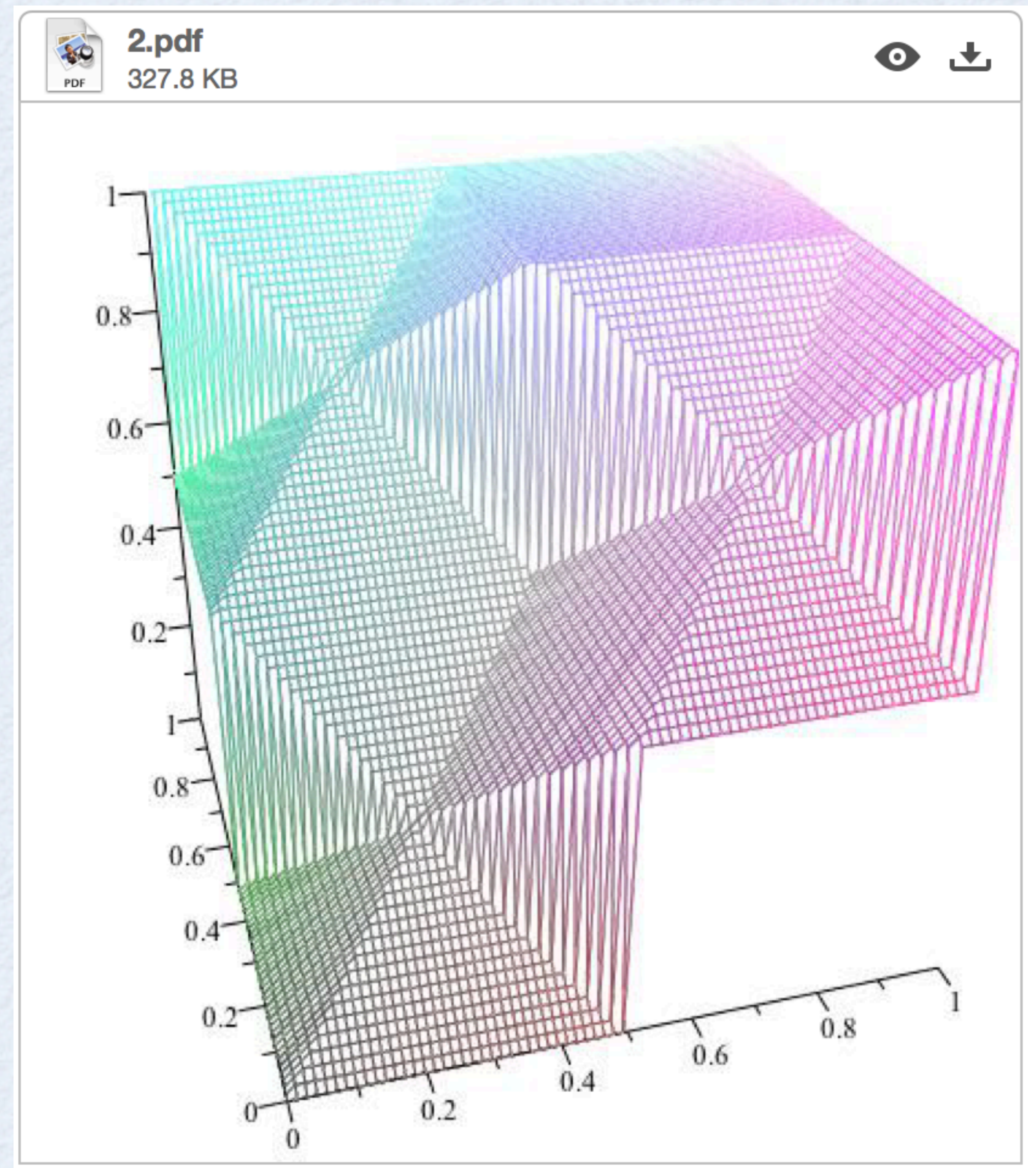
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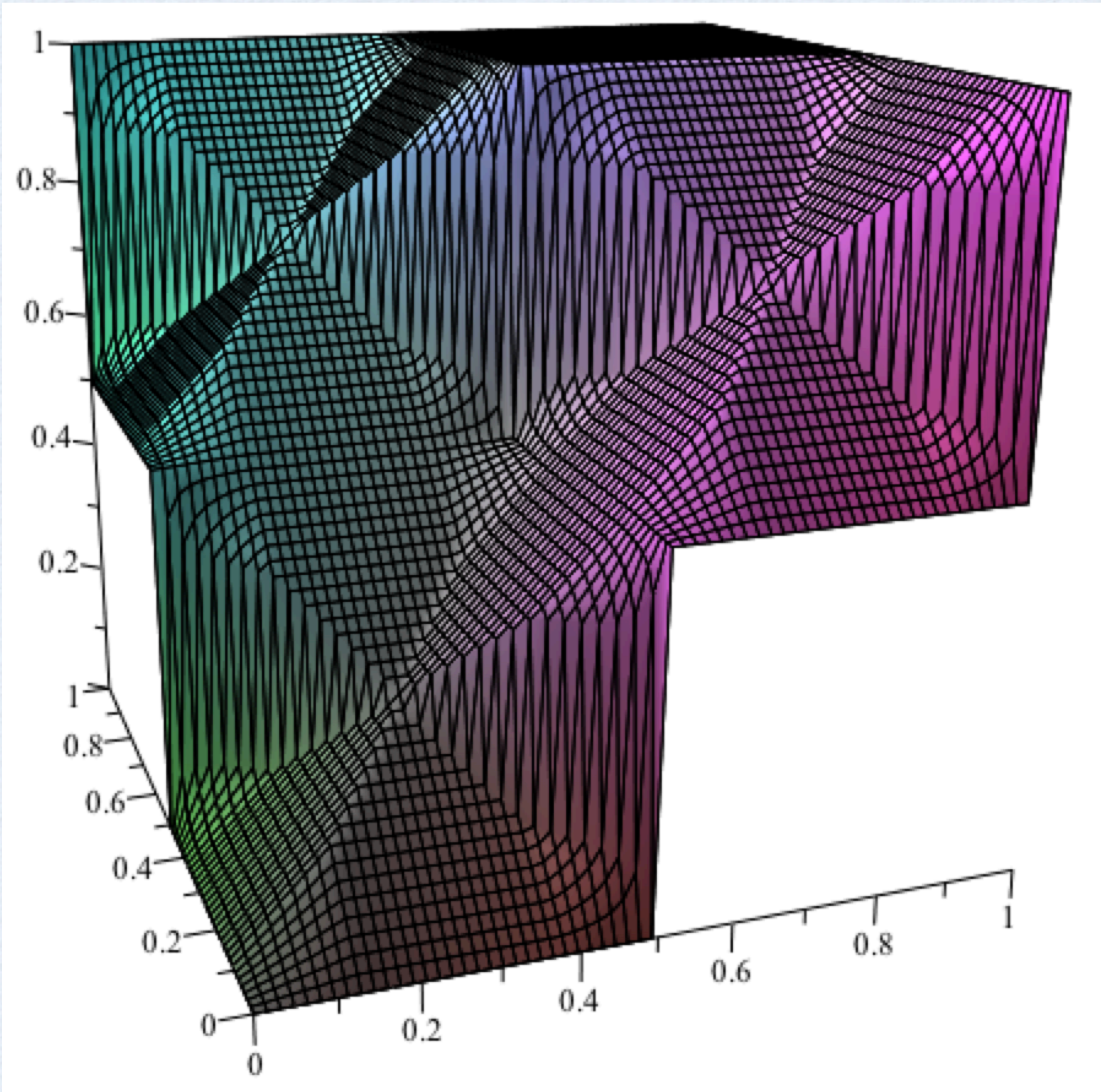
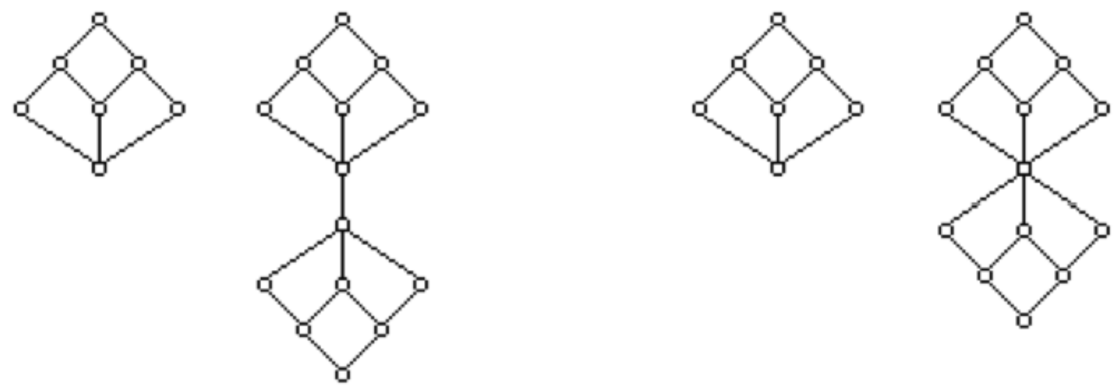
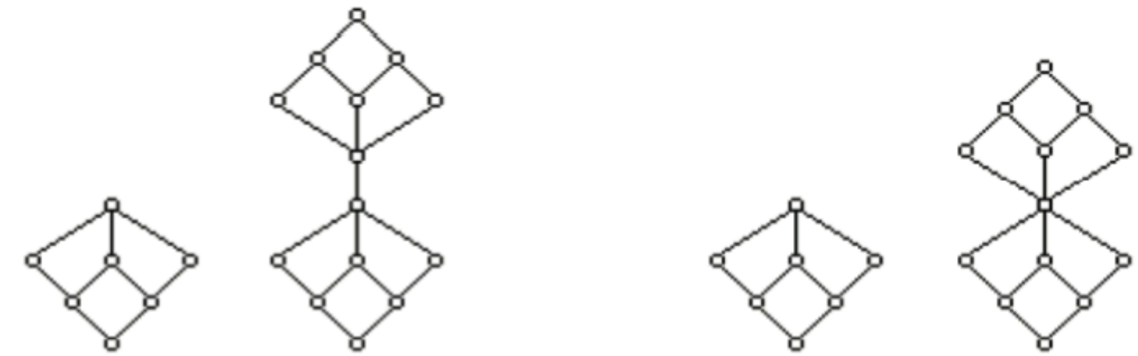


2: Co-Rotations



let $x \circledast_\vartheta y =$

$$\begin{cases} x \circledast y & \text{if } x, y \in M^+ \\ (x \rightarrow_\circledast y')' & \text{if } x \in M^+ \text{ and } y \in M^- \\ (y \rightarrow_\circledast x')' & \text{if } x \in M^- \text{ and } y \in M^+ \\ \bot & \text{if } x, y \in M^- \end{cases}$$



let $x \circledast_\eta y =$

$$\begin{cases} \bot & \text{if } x \text{ or } y \text{ is } \bot \\ x \circledast y & \text{if } x, y \in M^- \setminus \{\bot\} \\ (x \rightarrow_\circledast y')' & \text{if } x \in M^- \setminus \{\bot\} \text{ and } y \in M^+ \\ (y \rightarrow_\circledast x')' & \text{if } x \in M^+ \text{ and } y \in M^- \setminus \{\bot\} \\ \top & \text{if } x, y \in M^+ \end{cases}$$

- disconnected

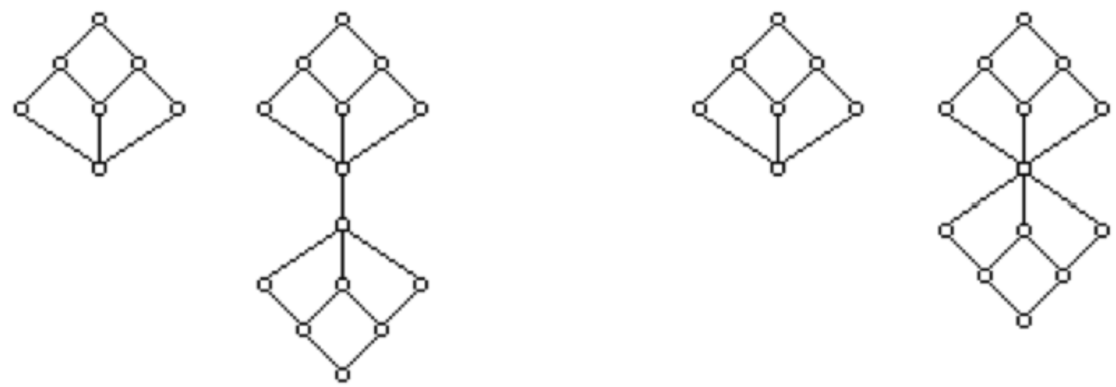
  commutative, residuated po-semigroup

- connected

  commutative, residuated po-semigroup either

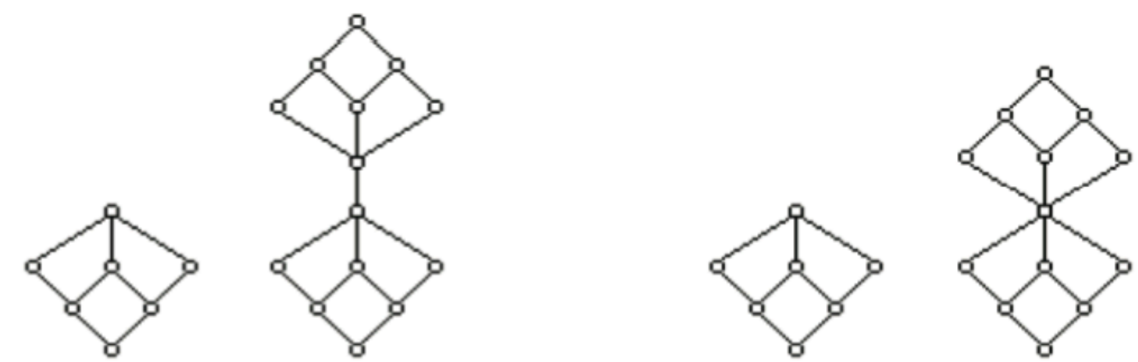
  1. *without zero divisors or*
  2. *with zero divisors. In this case suppose that there exist $c \in M$ such that for any zero divisor x, $x \rightarrow_\circledast \iota = c$ holds.*

2: Co-Rotations



let $x \circledast_{\vartheta} y =$

$$\begin{cases} x \circledast y & \text{if } x, y \in M^{+} \\ (x \rightarrow_{\circledast} y')' & \text{if } x \in M^{+} \text{ and } y \in M^{-} \\ (y \rightarrow_{\circledast} x')' & \text{if } x \in M^{-} \text{ and } y \in M^{+} \\ \bot & \text{if } x, y \in M^{-} \end{cases}$$



let $x \circledast_{\eta} y =$

$$\begin{cases} \bot & \text{if } x \text{ or } y \text{ is } \bot \\ x \circledast y & \text{if } x, y \in M^{-} \setminus \{\bot\} \\ (x \rightarrow_{\circledast} y')' & \text{if } x \in M^{-} \setminus \{\bot\} \text{ and } y \in M^{+} \\ (y \rightarrow_{\circledast} x')' & \text{if } x \in M^{+} \text{ and } y \in M^{-} \setminus \{\bot\} \\ \top & \text{if } x, y \in M^{+} \end{cases}$$

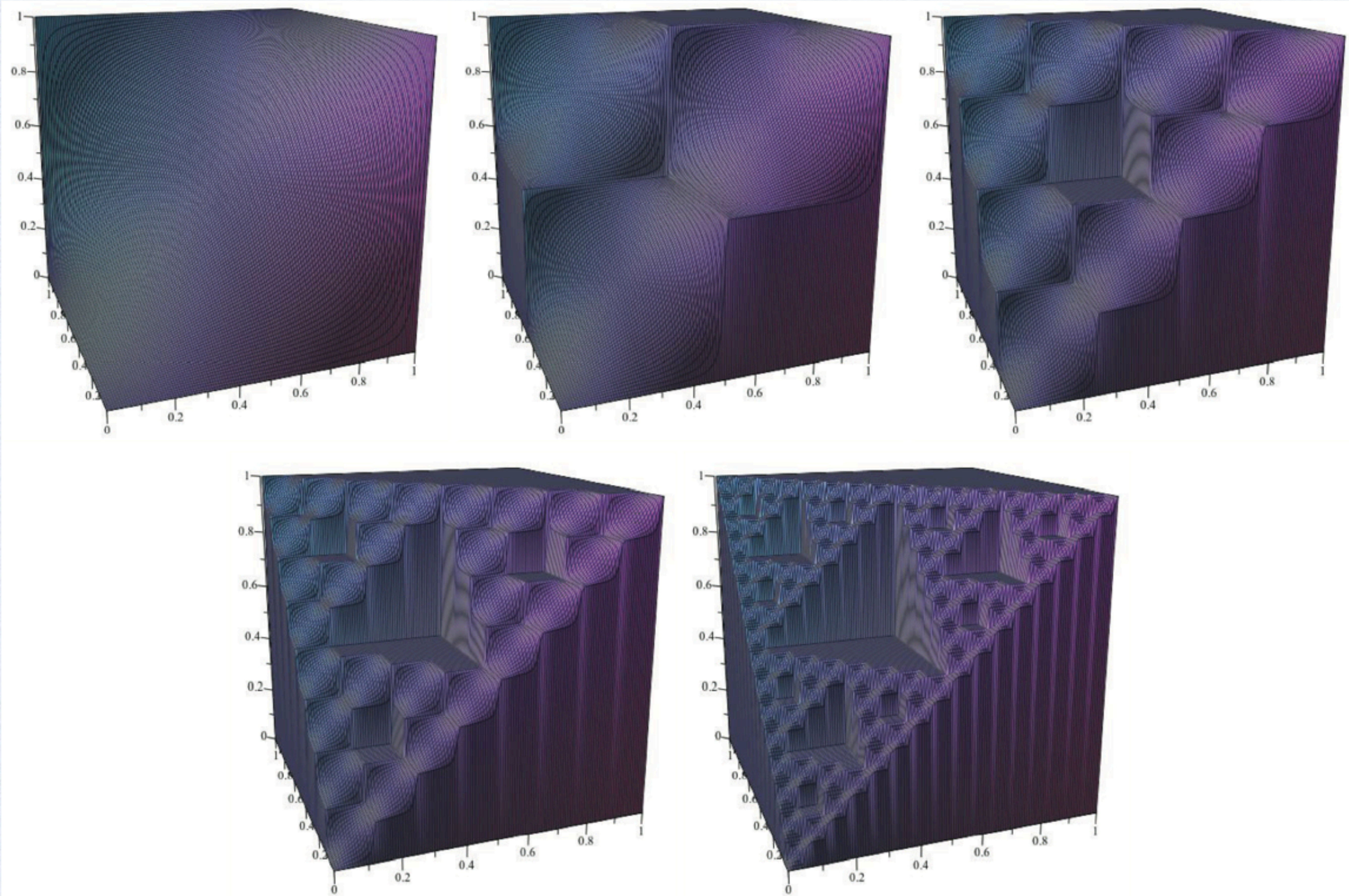
- disconnected

commutative, residuated po-semigroup *without zero divisors.*

- connected

commutative, residuated po-semigroup without zero divisors and satisfying

$$\iota \circledast x = \iota \quad \text{for } x > \bot. \tag{8}$$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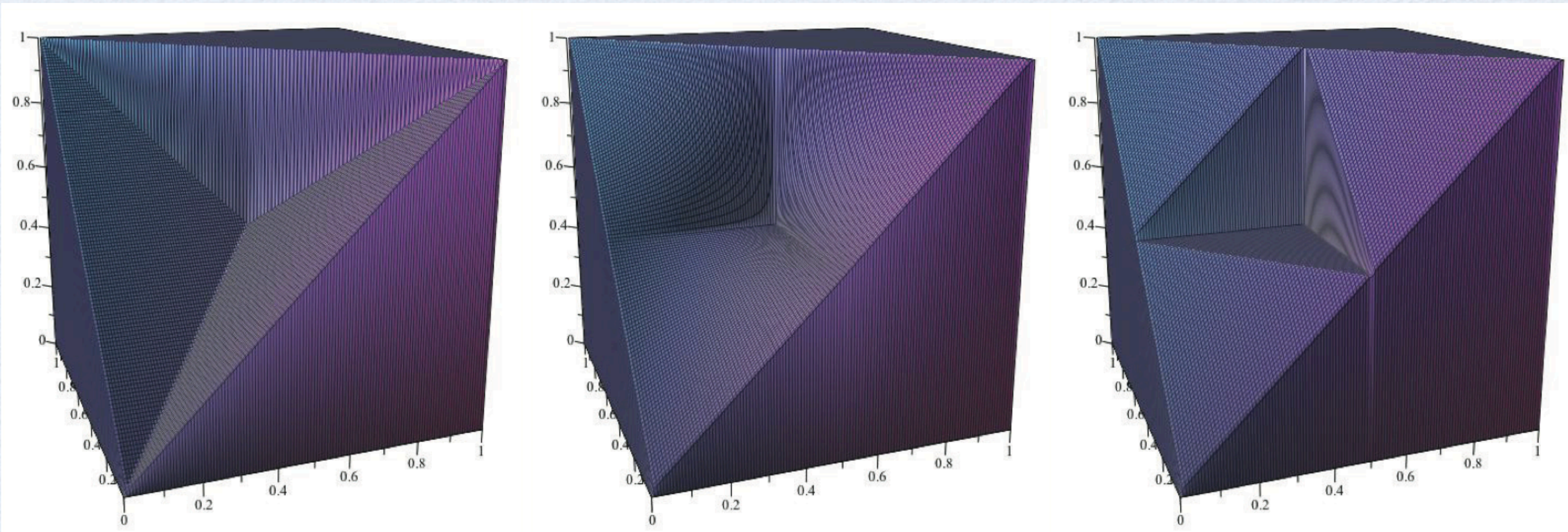
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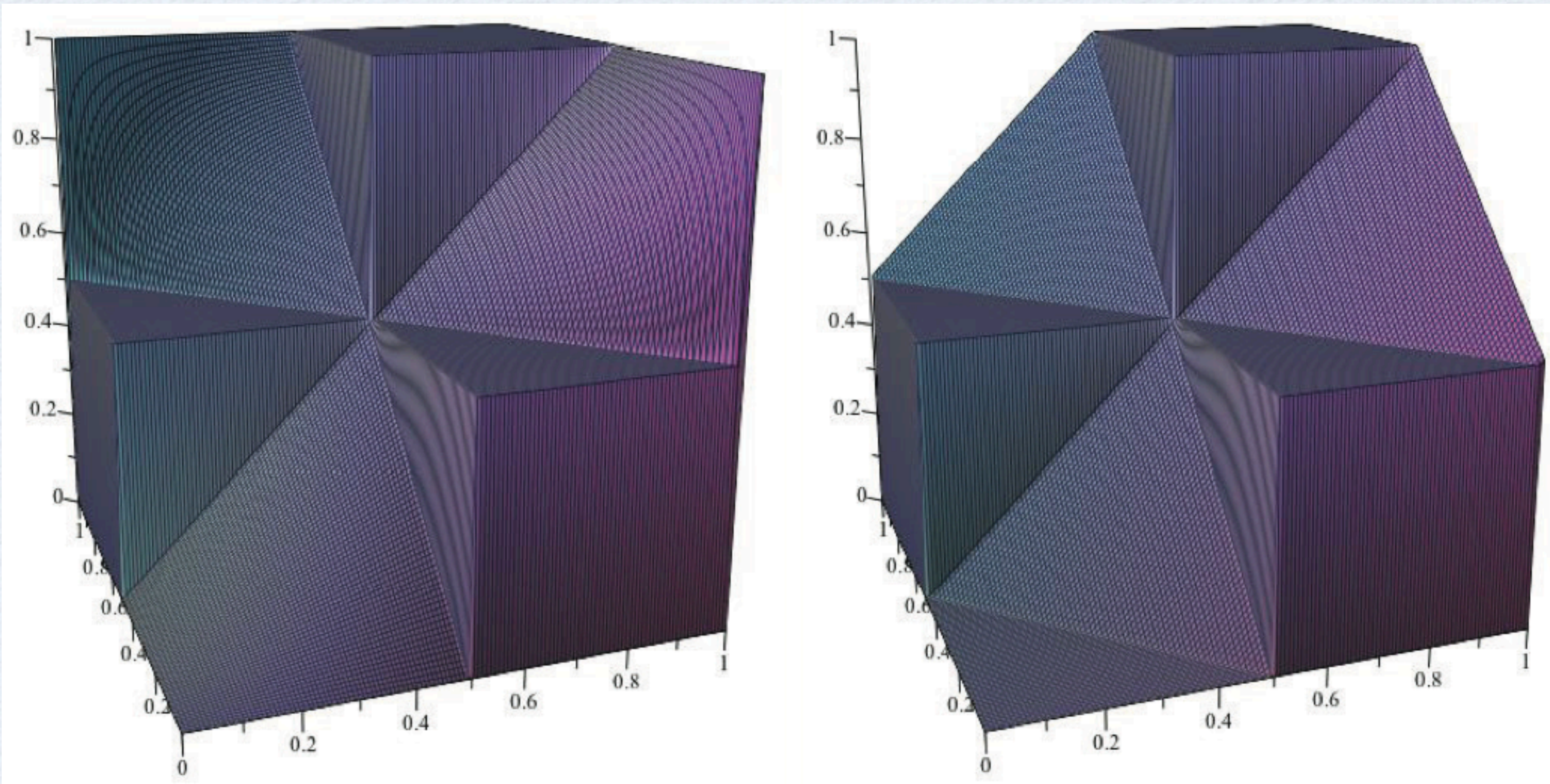
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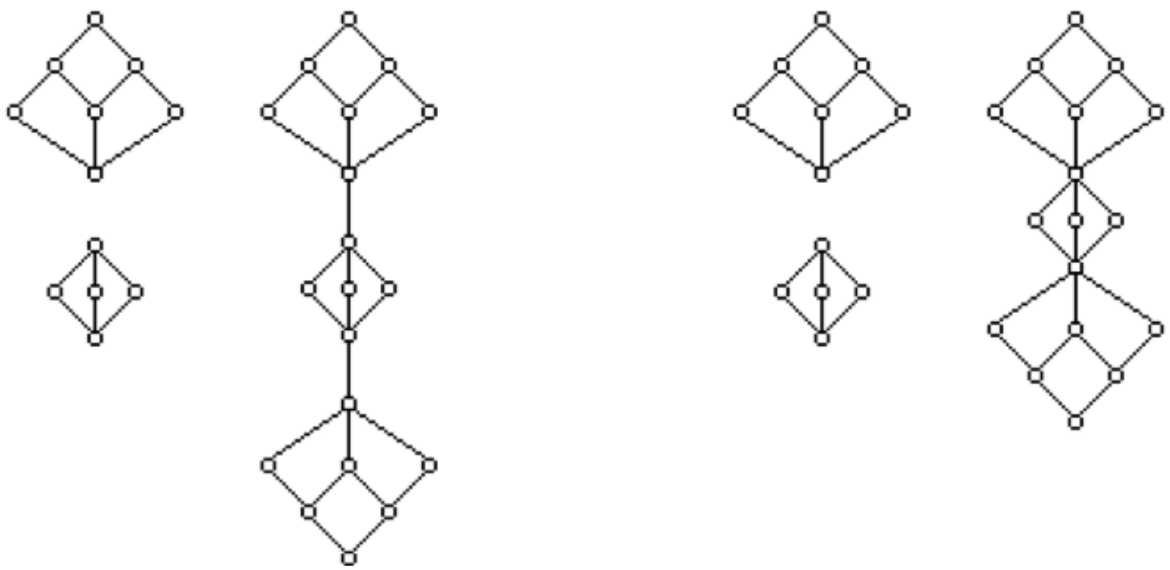
Applications of the rotation construction

- in the structural description of
  - Perfect and bipartite IMTL-algebras
    [C. Noguera, F. Esteva, J. Gispert, Perfect and bipartite IMTL-algebras and disconnected rotations of basic semihoops, Archive for Mathematical Logic, 44 (2005), 869–886.]
  - Free nilpotent minimum algebras
    [M. Busaniche, Free nilpotent minimum algebras, Mathematical Logic Quartely 52 (3) (2006) 219–236.]
  - Free Glivenko MTL-algebras
    [R. Cignoli, A. Torrens, Free algebras in varieties of Glivenko MTL-algebras satisfying the equation 2(x2) = (2x)2, Studia Logica 83 (1-3) (2006) 157-181]

Applications of the Rotation Construction

- Nelson algebras
  [M. Busaniche, R. Cignoli, Constructive Logic with Strong Negation as a Substructural Logic, Journal of Logic and Computation 20 (4) (2010) 761–793.]
- in establishing a spectral duality for finitely generated nilpotent minimum algebras
  [S. Aguzzoli, M. Busaniche, Spectral duality for finitely generated nilpotent minimum algebras, with applications, Journal of Logic and Computation 17 (4) (2007) 749–765.]
- in the previous talk (on one-variable axiomatizations)

3: Co-Rotation-Annihilations



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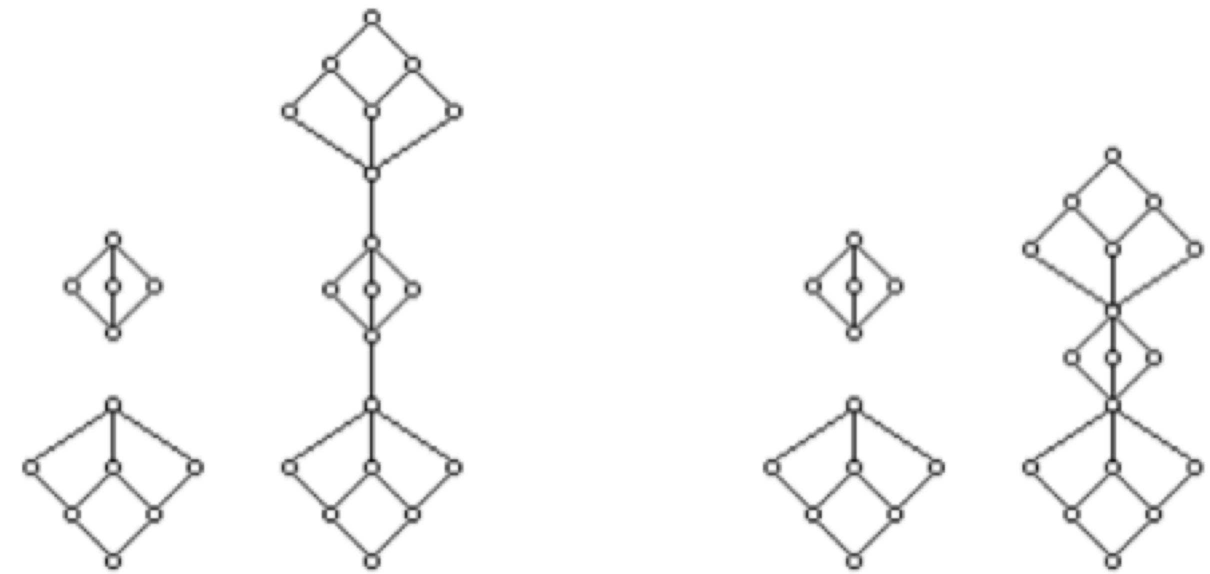
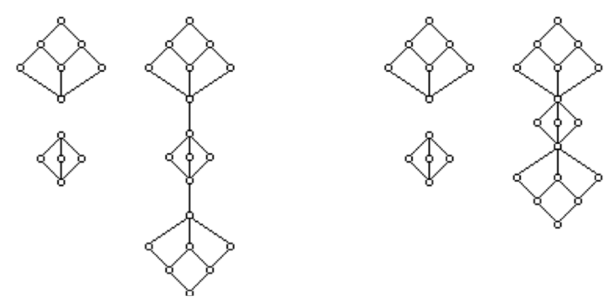
\begin{cases}
x \circledast y & \text{if } x, y \in M^{+} \\
(x \rightarrow_{\circledast} y')' & \text{if } x \in M^{+}, y \in M^{-} \\
(y \rightarrow_{\circledast} x')' & \text{if } x \in M^{-}, y \in M^{+} \\
0 & \text{if } x, y \in M^{-} \\
x \odot y & \text{if } x, y \in M^{0} \text{ and } x > y' \\
\top & \text{if } x, y \in M^{0} \text{ and } x \leq y' \\
y & \text{if } x \in M^{+}, y \in M^{0} \\
x & \text{if } x \in M^{0}, y \in M^{+} \\
0 & \text{if } x \in M^{-}, y \in M^{0} \\
0 & \text{if } x \in M^{0}, y \in M^{-}
\end{cases}
$$

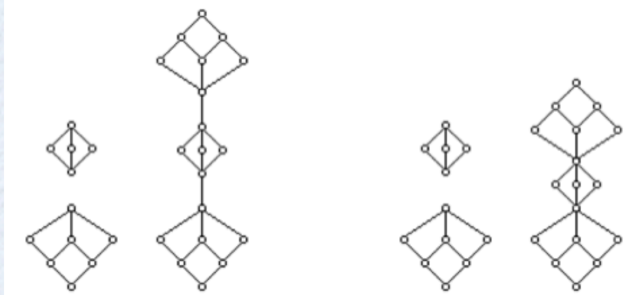


$$
\begin{cases}
\bot & \text{if } x \text{ or } y \text{ is } \bot \\
x \circledast y & \text{if } x, y \in M^{-} \setminus \{\bot\} \\
(x \rightarrow_{\circledast} y')' & \text{if } x \in M^{-} \setminus \{\bot\} \text{ and } y \in M^{+} \\
(y \rightarrow_{\circledast} x')' & \text{if } x \in M^{+} \text{ and } y \in M^{-} \setminus \{\bot\} \\
\top & \text{if } x, y \in M^{+} \\
x \odot y & \text{if } x, y \in M^{0} \text{ and } x \leq y' \\
\top & \text{if } x, y \in M^{0} \text{ and } x > y' \\
\top & \text{if } x \in M^{+}, y \in M^{0} \\
\top & \text{if } x \in M^{0}, y \in M^{+} \\
y & \text{if } x \in M^{-} \setminus \{\bot\}, y \in M^{0} \\
x & \text{if } x \in M^{0}, y \in M^{-} \setminus \{\bot\}
\end{cases}
$$

3: Co-Rotation-Annihilations



$$\begin{cases} x \circledast y & \textit{if } x,y \in M^+ \\ (x \to_{\circledast} y')' & \textit{if } x \in M^+, y \in M^- \\ (y \to_{\circledast} x')' & \textit{if } x \in M^-, y \in M^+ \\ 0 & \textit{if } x,y \in M^- \\ x \odot y & \textit{if } x,y \in M^0 \textit{ and } x > y' \\ \top & \textit{if } x,y \in M^0 \textit{ and } x \le y' \\ y & \textit{if } x \in M^+, y \in M^0 \\ x & \textit{if } x \in M^0, y \in M^+ \\ 0 & \textit{if } x \in M^-, y \in M^0 \\ 0 & \textit{if } x \in M^0, y \in M^- \end{cases}$$



$$\begin{cases} \bot & \textit{if } x \textit{ or } y \textit{ is } \bot \\ x \circledast y & \textit{if } x,y \in M^- \setminus \{\bot\} \\ (x \to_{\circledast} y')' & \textit{if } x \in M^- \setminus \{\bot\} \textit{ and } y \in M^+ \\ (y \to_{\circledast} x')' & \textit{if } x \in M^+ \textit{ and } y \in M^- \setminus \{\bot\} \\ \top & \textit{if } x,y \in M^+ \\ x \odot y & \textit{if } x,y \in M^0 \textit{ and } x \le y' \\ \top & \textit{if } x,y \in M^0 \textit{ and } x > y' \\ \top & \textit{if } x \in M^+, y \in M^0 \\ \top & \textit{if } x \in M^0, y \in M^+ \\ y & \textit{if } x \in M^- \setminus \{\bot\}, y \in M^0 \\ x & \textit{if } x \in M^0, y \in M^- \setminus \{\bot\} \end{cases}$$

- disconnected

commutative, residuated, po-semigroup,

commutative, conjunctive, rotation-invariant po-semigroup,

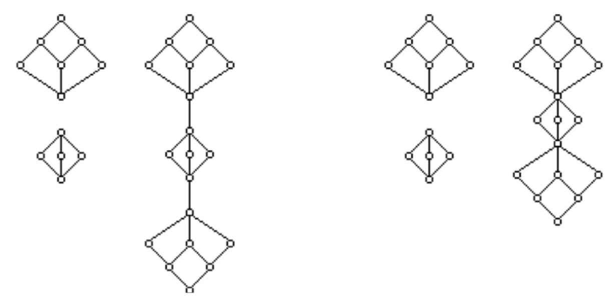
- connected

commutative, residuated, po-semigroup, *commutative, rotation-invariant, integral po-monoid,*

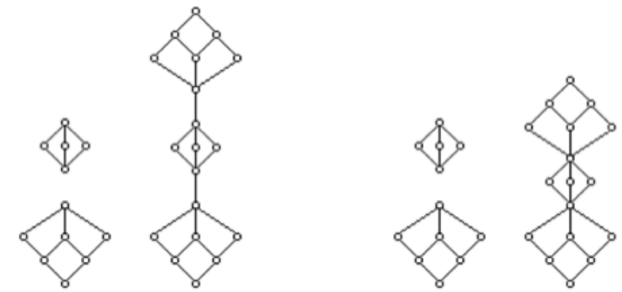
commutative, residuated, po-semigroup without zero divisors,

commutative, conjunctive, rotation-invariant po-semigroup,

3: co-rotation-annihilations



$$\begin{cases} x \circledast y & \text{if } x, y \in M^{+} \\ (x \to_{\circledast} y')' & \text{if } x \in M^{+}, y \in M^{-} \\ (y \to_{\circledast} x')' & \text{if } x \in M^{-}, y \in M^{+} \\ 0 & \text{if } x, y \in M^{-} \\ x \odot y & \text{if } x, y \in M^{0} \text{ and } x > y' \\ \top & \text{if } x, y \in M^{0} \text{ and } x \leq y' \\ y & \text{if } x \in M^{+}, y \in M^{0} \\ x & \text{if } x \in M^{0}, y \in M^{+} \\ 0 & \text{if } x \in M^{-}, y \in M^{0} \\ 0 & \text{if } x \in M^{0}, y \in M^{-} \end{cases}$$



$$\begin{cases} \bot & \text{if } x \text{ or } y \text{ is } \bot \\ x \circledast y & \text{if } x, y \in M^{-} \setminus \{\bot\} \\ (x \to_{\circledast} y')' & \text{if } x \in M^{-} \setminus \{\bot\} \text{ and } y \in M^{+} \\ (y \to_{\circledast} x')' & \text{if } x \in M^{+} \text{ and } y \in M^{-} \setminus \{\bot\} \\ \top & \text{if } x, y \in M^{+} \\ x \odot y & \text{if } x, y \in M^{0} \text{ and } x \leq y' \\ \top & \text{if } x, y \in M^{0} \text{ and } x > y' \\ \top & \text{if } x \in M^{+}, y \in M^{0} \\ \top & \text{if } x \in M^{0}, y \in M^{+} \\ y & \text{if } x \in M^{-} \setminus \{\bot\}, y \in M^{0} \\ x & \text{if } x \in M^{0}, y \in M^{-} \setminus \{\bot\} \end{cases}$$

- disconnected

commutative, residuated po-semigroup without zero divisors.

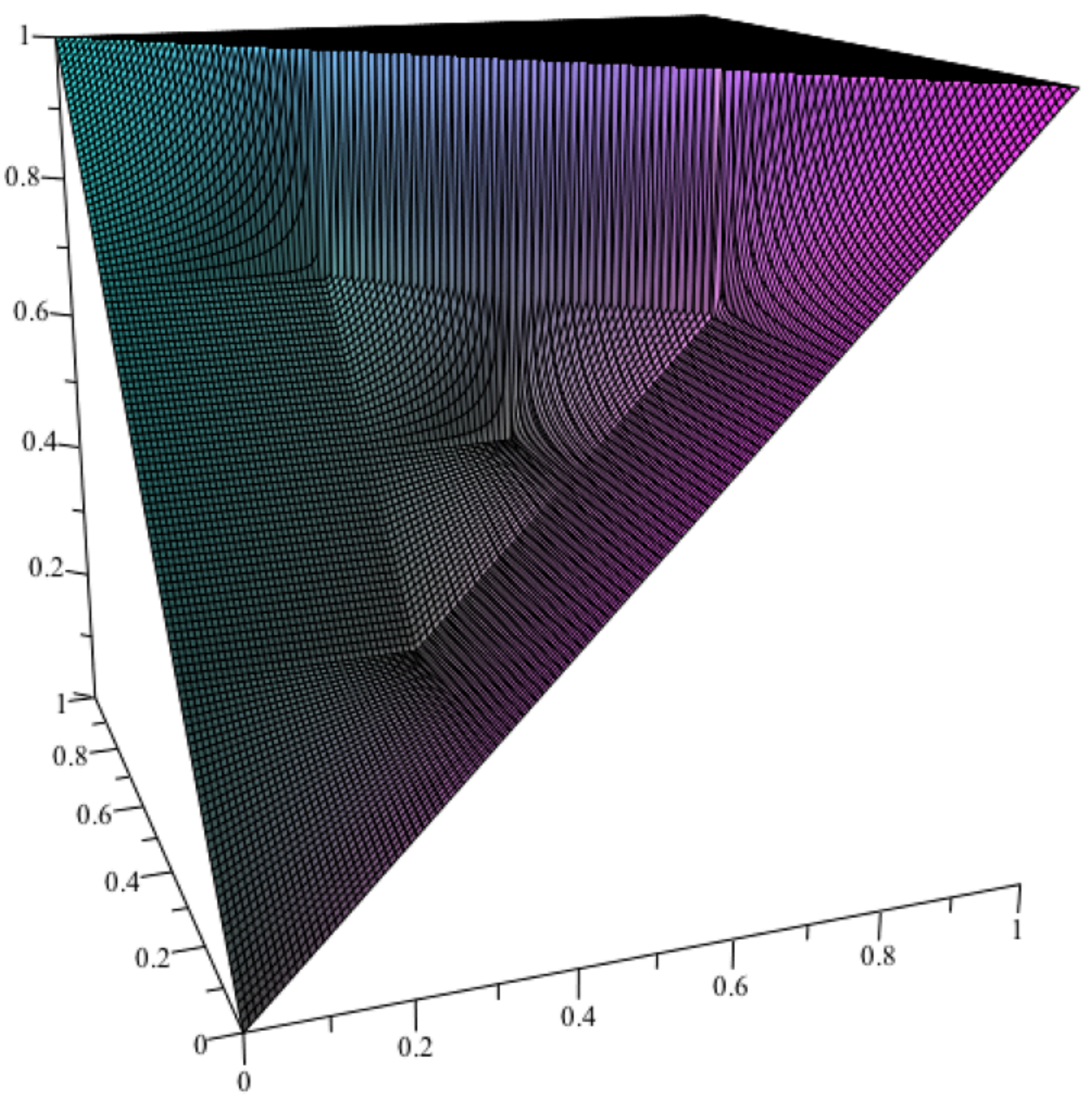
commutative, weakly disjunctive, rotation-invariant po-semigroup,

- connected

commutative, residuated, po-semigroup satisfying

$$\iota \circledast x = \iota \quad \textit{for } x > \bot.$$

commutative, rotation-invariant, weakly disjunctive monoid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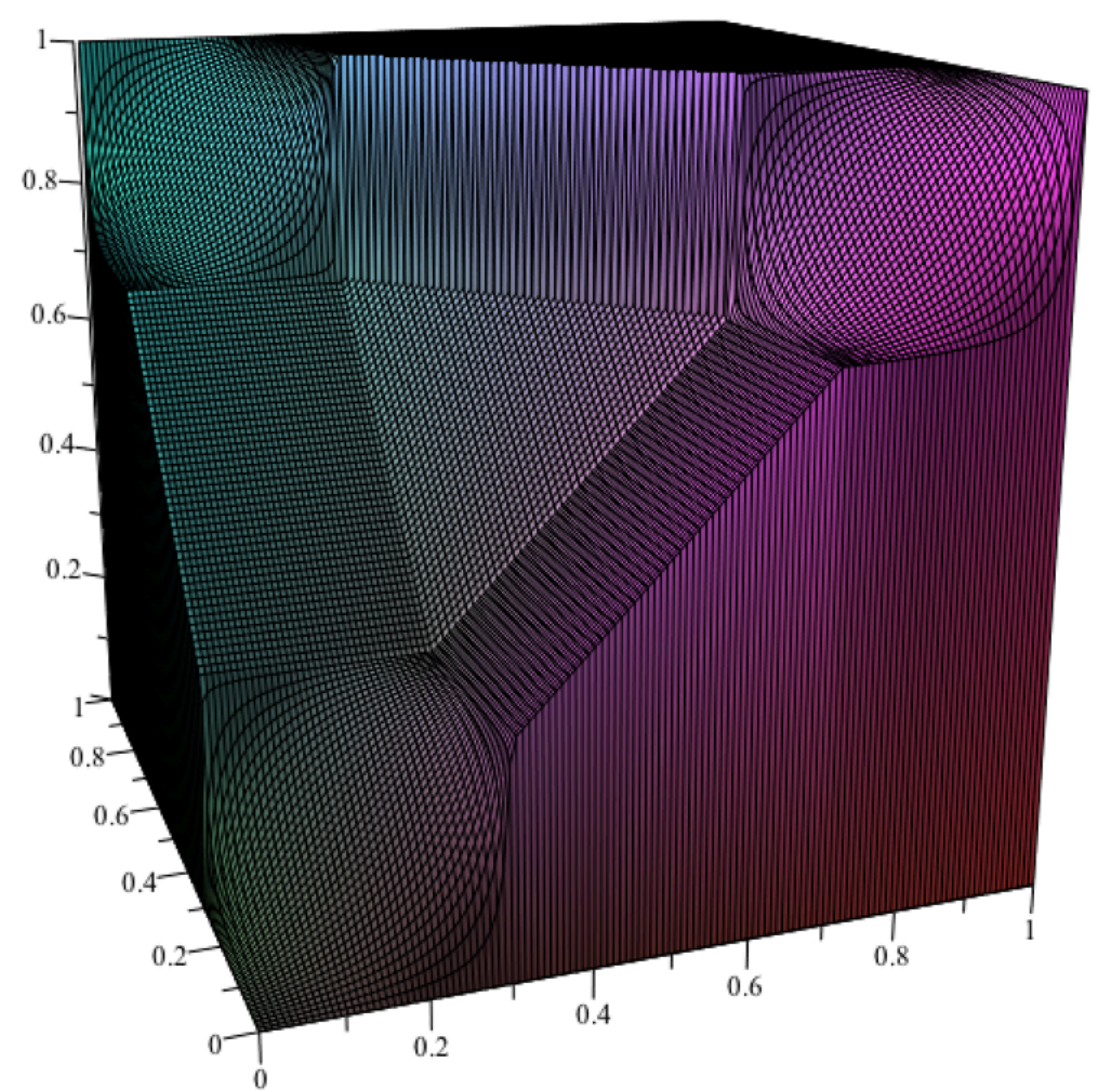
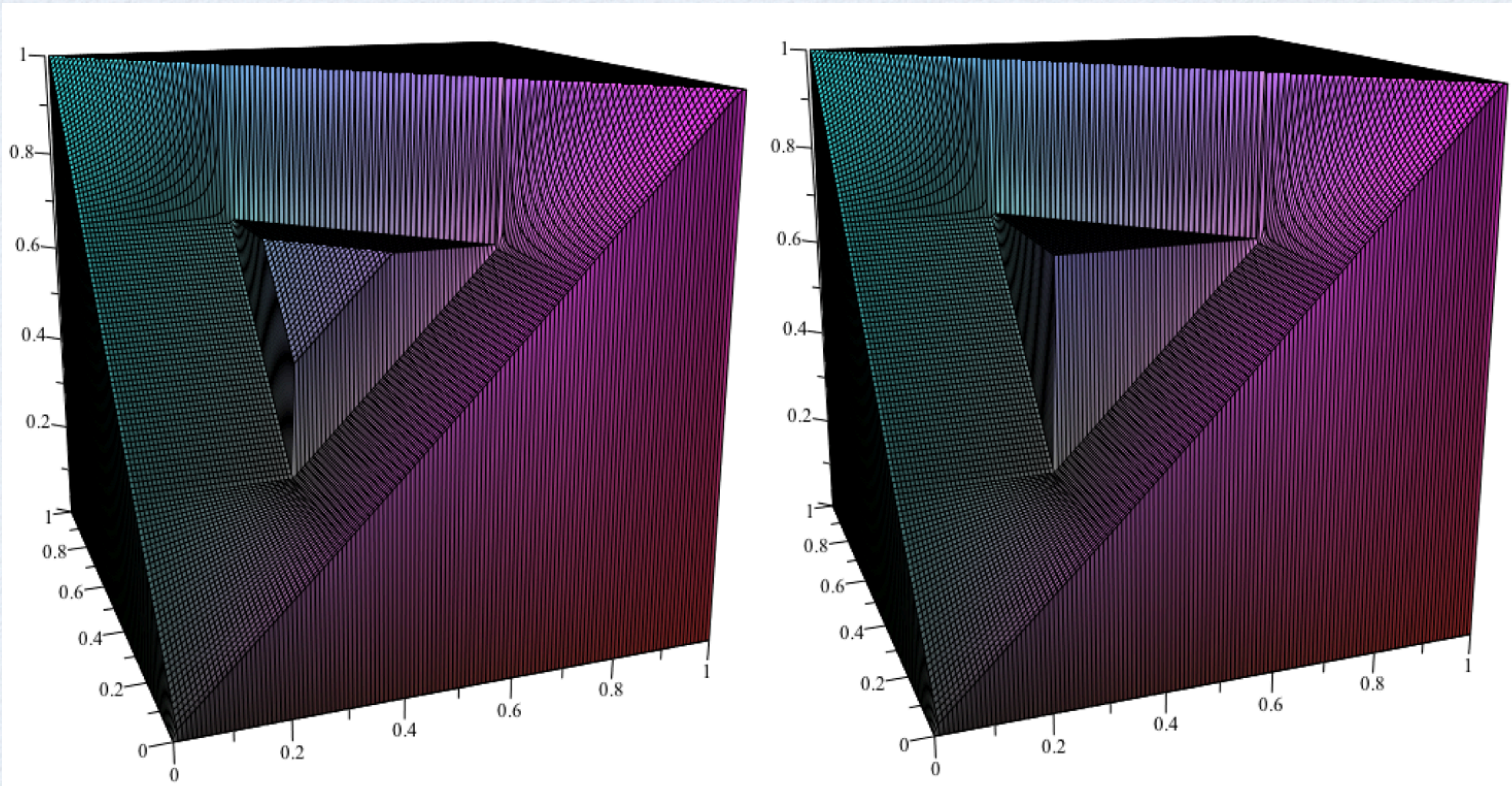
0.2
0.4
0.6
0.8
1


0
0.2
0.4
0.6
0.8
1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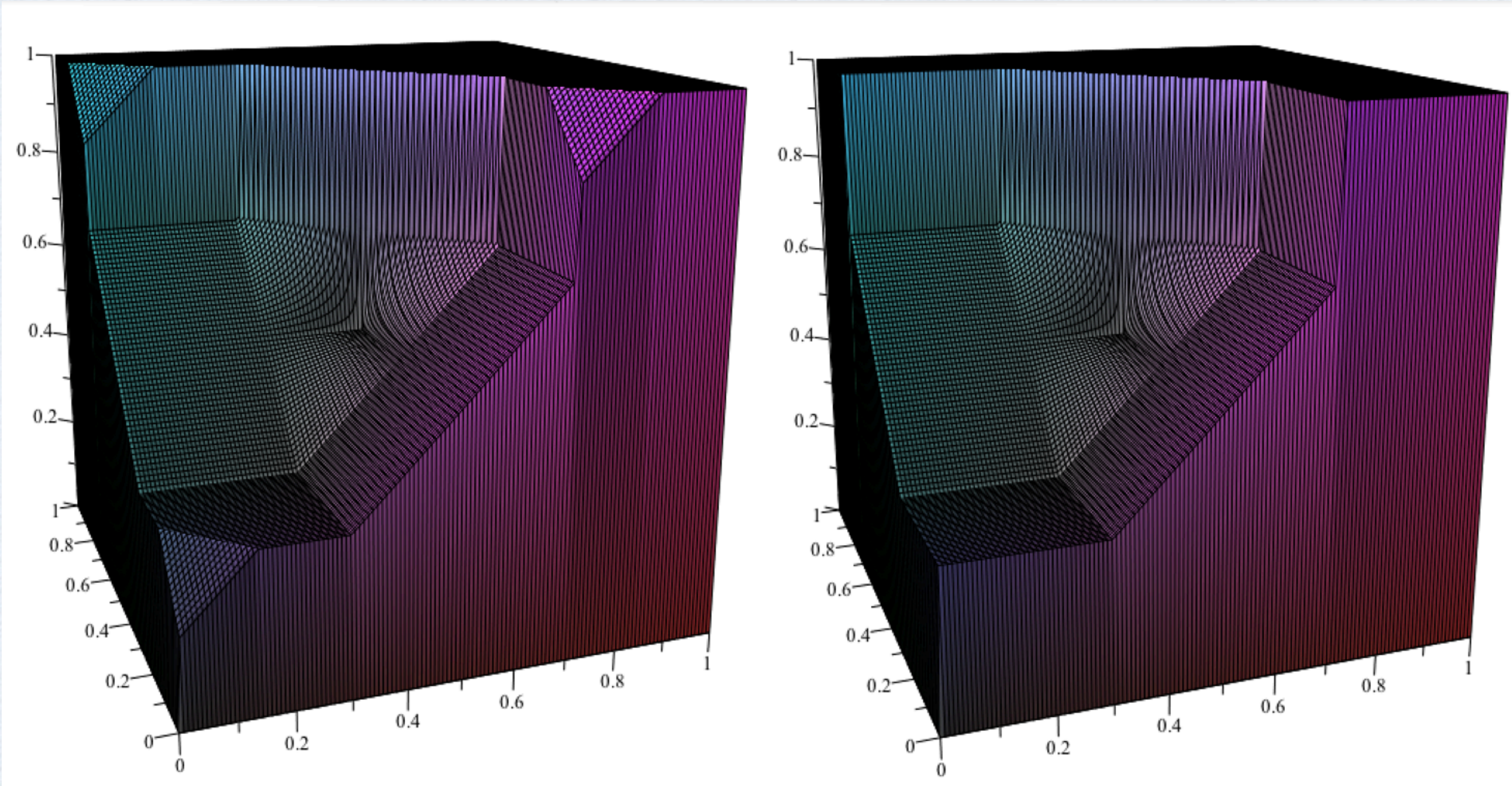
0.8
1


1
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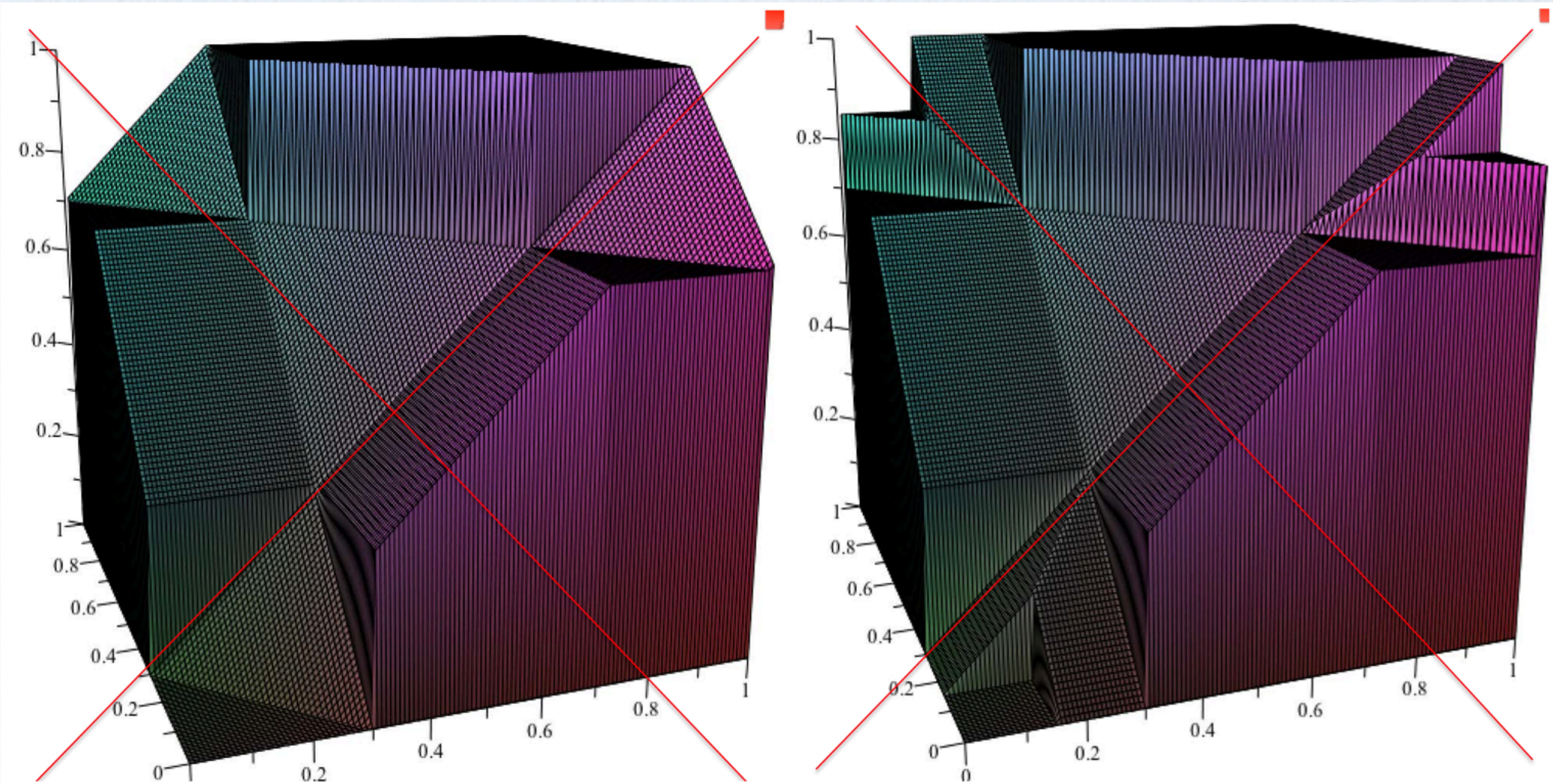
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0.8
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0.4
0.2
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1

THANK YOU FOR YOUR ATTENTION!