Lattice BCK logics with modus ponens as the only rule

Joan Gispert and Antoni Torrens

Facultat de Matemàtiques. Universitat de Barcelona
jgispertb@ub.edu, atorrens@ub.edu

ManyVal 2013, Prague, September 2013

Outline

I. Preliminaries.

II. MainResults

III. Conclusions and Open Questions.

BCK Logic

BCK logic [Meredith 1962]

Axioms:

B $(\varphi \to \psi) \to ((\psi \to \xi) \to (\varphi \to \xi))$

C $(\varphi \to (\psi \to \xi)) \to (\psi \to (\varphi \to \xi))$

K $\varphi \to (\psi \to \varphi)$

Rules:

MP $\{\varphi, \varphi \to \psi\} \vdash \psi$

BCK Algebras

An algebra $\mathbf{B} = \langle B; \to, \top \rangle$ of type $(2, 0)$ is called **BCK-algebra** [Iseki 1966] provided that it satisfies:

$$(x \to y) \to ((y \to z) \to (x \to z) \approx \top.$$

$$\top \to x \approx x.$$

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If $x \to y \approx \top$ and $y \to x \approx \top$, then $x \approx y$

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The class of all BCK-algebras ($\mathbb{BCK}$) is a quasivariety.
In fact it is a strict quasivariety [Wronski 1983]

BCK logic is algebraizable with $\varphi \approx \top$ as defining equation and $\{\varphi \to \psi, \psi \to \varphi\}$ as set of equivalence formuli. Moreover $\mathbb{BCK}$ is its equivalent algebraic quasivariety.

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BCK logic is the $\top$-assertional logic of $\mathbb{BCK}$.
i.e. $\Gamma \vdash_{BCK} \varphi$ if and only if for every $\mathbf{A} \in \mathbb{BCK}$, $e[\Gamma] \subseteq \{\top\}$ implies $e(\varphi) = \top$ for every evaluation e on $\mathbf{A}$

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There is a one to one correspondence from the class of all finitary (axiomatic) extensions of BCK logic and the class of all subquasivarieties (relative subvarieties) of $\mathbb{BCK}$

Lattice BCK logic

Lattice BCK logic (*LBCK* to short).

Axioms:

B $(\varphi \to \psi) \to ((\psi \to \xi) \to (\varphi \to \xi))$

C $(\varphi \to (\psi \to \xi)) \to (\psi \to (\varphi \to \xi))$

K $\varphi \to (\psi \to \varphi)$

$\vee 1$ $\varphi \to \varphi \vee \psi$

$\vee 2$ $\psi \to \varphi \vee \psi$

$\wedge 1$ $\varphi \wedge \psi \to \varphi$

$\wedge 2$ $\varphi \wedge \psi \to \psi$

Rules:

M.P. $\{\varphi, \varphi \to \psi\} \vdash \psi$

$\vee$-rule $\{\varphi \to \xi, \psi \to \xi\} \vdash \varphi \vee \psi \to \xi$

$\wedge$-rule $\{\xi \to \varphi, \xi \to \psi\} \vdash \xi \to \varphi \wedge \psi$.

BCK Lattices

An algebra $\mathbf{A} = \langle A; \to, \wedge, \vee, \top \rangle$ of type $(2, 2, 2, 0)$ is a **BCK-lattice** [Idziak 1984] provided that

$\mathbf{A}^{-} = \langle A; \to \top \rangle$ is a BCK-algebra

$\mathbf{L(A)} = \langle A; \wedge, \vee \rangle$ is a lattice.

Natural order given by $\mathbf{A}^{-}$ coincides with lattice order.
i.e. For every $a, b \in A$, $a \to b = \top$ iff $a \wedge b = a$

The class of all BCK lattices ($\mathbb{LBCK}$) is a variety. [Idziak 1984]

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$$x \to \top \approx \top$$
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$$(x \to y) \to ((y \to z) \to (x \to z)) \approx \top$$

$x \wedge y \to y \approx \top$	$x \to x \vee y \approx \top$
$x \wedge ((x \to y) \to y) \approx x$	$x \vee ((x \to y) \to y) \approx (x \to y) \to y$
$x \wedge x \approx x$	$x \vee x \approx x$
$x \wedge y \approx y \wedge x$	$x \vee y \approx y \vee x$
$x \wedge (y \wedge z) \approx (x \wedge y) \wedge z$	$x \vee (y \vee z) \approx (x \vee y) \vee z$

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New presentation of LBCK

Axioms:

$$\begin{array}{rl}
\text{B} & (\varphi \to \psi) \to ((\psi \to \xi) \to (\varphi \to \xi)) \\
\text{C} & (\varphi \to (\psi \to \xi)) \to (\psi \to (\varphi \to \xi)) \\
\text{K} & \varphi \to (\psi \to \varphi) \\
\vee 1 & \varphi \to \varphi \vee \psi \\
\vee 2 & \psi \to \varphi \vee \psi \\
\wedge 1 & \varphi \wedge \psi \to \varphi \\
\wedge 2 & \varphi \wedge \psi \to \psi
\end{array}$$

Rules:

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\vee\text{-rule} & \{\varphi \to \xi, \psi \to \xi\} \vdash \varphi \vee \psi \to \xi \\
\wedge\text{-rule} & \{\xi \to \varphi, \xi \to \psi\} \vdash \xi \to \varphi \wedge \psi.
\end{array}$$

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$\vee 3$ $(\varphi \to \xi) \wedge (\psi \to \xi) \to (\varphi \vee \psi \to \xi)$

$\wedge 1$ $\varphi \wedge \psi \to \varphi$

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Rules:

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$\wedge$-rule $\{\xi \to \varphi, \xi \to \psi\} \vdash \xi \to \varphi \wedge \psi.$

How about the $\wedge$-rule?

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$$\nvdash (\varphi \to \psi) \wedge (\varphi \to \chi) \to (\varphi \to \psi \wedge \chi)$$

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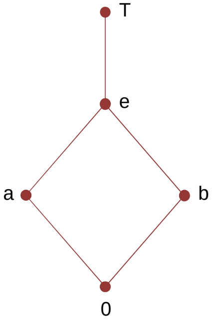
Is it possible to obtain an axiomatic presentation of LBCK with Modus Ponens as the only rule?

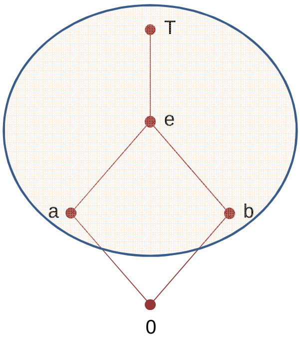
$\mathbf{C} = <\{0,a,b,e,T\}; \rightarrow, \wedge, \vee, T>$



L(**C**)

$\rightarrow$	0	a	b	e	T
0	T	T	T	T	T
a	0	T	e	T	T
b	0	e	T	T	T
e	0	e	e	T	T
T	0	a	b	e	T

F={ a, b, e, T }



→	0	a	b	e	T
0	T	T	T	T	T
a	0	T	e	T	T
b	0	e	T	T	T
e	0	e	e	T	T
T	0	a	b	e	T

C is a BCK-lattice.

F is closed under Modus Ponens.

F is not closed under $\wedge$-rule, because $e \to a = e \to b = e \in F$, but $e \to a \wedge b = e \to 0 = 0 \notin F$.

Hence $\langle \mathbf{C}, F \rangle$ is a model of all theorems of $LBCK$, modus ponens and it is not a model of $\wedge$-rule.

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Hence $\langle \mathbf{C}, F \rangle$ is a model of all theorems of $LBCK$, modus ponens and it is not a model of $\wedge$-rule.

There is no axiomatic presentation of $LBCK$ with Modus Ponens as the only rule.

Residuated Lattice BCK logic (*RLBCK* to short) is the axiomatic extension of LBCK adding the axiom

$$(\wedge 3) \qquad (\varphi \to \psi) \wedge (\varphi \to \chi) \to (\varphi \to \psi \wedge \chi)$$

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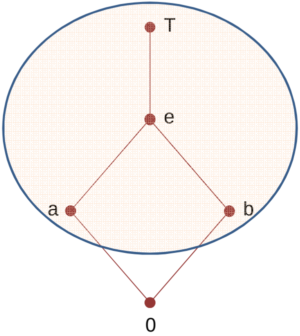
RLBCK is the $\{\wedge, \vee, \to\}$-fragment of the FLew logic.

RLBCK can be axiomatized by
Axioms: B, C, K, $\vee$ 1, $\vee$ 2, $\vee$ 3, $\wedge$ 1, $\wedge$ 2, $\wedge$ 3 + $\varphi \to (\psi \to \varphi \wedge \psi)$
Rule: M.P.

Objective

Our purpose is to study axiomatic extensions of *LBCK* with Modus Ponens as the only rule

F={ a, b, e, T }



→	0	a	b	e	T
0	T	T	T	T	T
a	0	T	e	T	T
b	0	e	T	T	T
e	0	e	e	T	T
T	0	a	b	e	T

i-filters and $\wedge$-i-filters

We recall that an **implicative filter** (**i-filter** to short) of a BCK-lattice **B** is a subset F of B such that

(f1) $\top \in F$.

(f2) For every $a, b \in \mathbf{B}$, $a, a \to b \in F$ implies $b \in F$.

We say that an i-filter F of **B** is a $\wedge$-**implicative filter** ($\wedge$-**i-filter**) if and only if

(f3) For every $a, b, c \in \mathbf{B}$,
$a \to b, a \to c \in F$ implies $a \to b \wedge c \in F$

(Kühr 2007)

In every BCK-lattice the posets of congruence relations and $\wedge$-i-filters are isomorphic, both ordered by inclusion.

Let L be an axiomatic extension of LBCK and let V_L be its associated variety. If L admits an axiomatic presentation with Modus Ponens as the only rule then for every $\mathbf{A} \in V_L$, every i-filter on $\mathbf{A}$ is an $\wedge$-i-filter.

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The converse is also true

Characterization Theorem

Theorem

An axiomatic extension L of LBCK admits Modus Ponens as the only rule if and only if there are n, m non negative integers such that

$$\vdash_L (\varphi \to \psi)^n \to ((\varphi \to \chi)^m \to (\varphi \to \psi \wedge \chi)).$$

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where $x^n \to y$ is defined recursively: $x^0 \to y := y$ and $x^{n+1} \to y := x \to (x^n \to y)$ for any $n \geqslant 0$.

For every $n, m \in \omega$ we denote by $\mathbf{LBCK_{n,m}}$ the axiomatic extension of $LBCK$ obtained by adding the axiom

$$(\varphi \to \psi)^n \to ((\varphi \to \chi)^m \to (\varphi \to \psi \wedge \chi))$$

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Given two logics L, K we denote by $L \leq K$ the usual relation of K beeing stronger than L (L beeing weaker than K) that is: For every set of formuli Γ and every formula φ, if $\Gamma \vdash_L \varphi$ then $\Gamma \vdash_K \varphi$. Then,

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For any $n, m, k \in \omega$, we have

(a) $LBCK_{n,m} = LBCK_{m,n}$.

(b) *If* $k \leq n$ *then* $LBCK_{n,m} \leq LBCK_{k,m}$

(c) *If* $n, m > 0$ *then* $LBCK_{n,m} \leq RBCK$.

(d) $LBCK_{n,0}$ *is the inconsistent logic.*

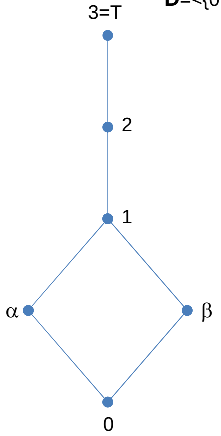
Our aim is to see that for every $n > 0$,

$$RLBCK > LBCK_{n,n} > LBCK_{n+1,n+1}.$$

The first strict inclusion follows from the next result.

$RLBCK \neq LBCK_{1,1}$.

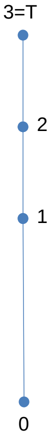
D=<{0,1, 2, 3, α, β}; →, ∧, ∨, T>



L(**D**)

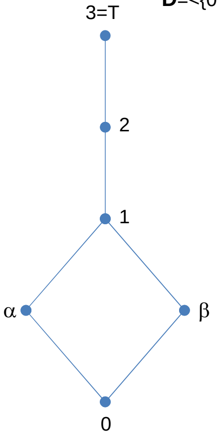
→	0	1	2	T	α	β
0	T	T	T	T	T	T
1	2	T	T	T	2	2
2	1	2	T	T	2	2
T	0	1	2	T	α	β
α	2	T	T	T	T	2
β	2	T	T	T	2	T

Ł$_4$



$\rightarrow$	0	1	2	T
0	T	T	T	T
1	2	T	T	T
2	1	2	T	T
T	0	1	2	T

D=<{0,1, 2, 3, α, β}; →, ∧, ∨, T>



L(**D**)

→	0	1	2	T	α	β
0	T	T	T	T	T	T
1	2	T	T	T	2	2
2	1	2	T	T	2	2
T	0	1	2	T	α	β
α	2	T	T	T	T	2
β	2	T	T	T	2	T

-**D** is a BCK- lattice.

-$(\varphi \to \psi) \to ((\varphi \to \chi) \to (\varphi \to \psi \wedge \chi))$ is a **D** tautology.

Moreover, since

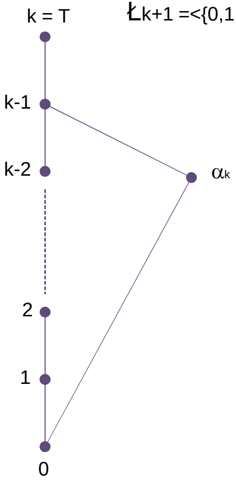
$$(2 \to \alpha)\wedge(2 \to \beta) \to (2 \to (\alpha\wedge\beta)) = 2\wedge 2 \to (2 \to 0) = 2 \neq \top,$$

-$(\varphi \to \psi) \wedge (\varphi \to \chi) \to (\varphi \to \psi \wedge \chi)$ is not a **D** tautology

Hence, $LBCK_{1,1} \neq RBCK$

To prove that $LBCK_{n,n} < LBCK_{n+1,n+1}$, it suffices to prove $LBCK_{n,n} \neq LBCK_{n+1,n+1}$

$Ł^{+}_{k+1} = <\{0,1,2,\ldots,k-1,k,\alpha\};\ ;\ \rightarrow, \wedge, \vee, T>$



$\mathbf{L}(Ł^{+}_{k+1})$

⟶	0	1	2	…	k-2	k-1	k	α_k
0	k	k	k	…	k	k	k	k-1
1	k-1	k	k	…	k	k	k	k-1
2	k-2	k-1	k	…	k	k	k	k-1
…	…	…	…	…	…	…	…	…
k-2	2	3	4	…	k	k	k	k-1
k-1	1	2	3	…	k-1	k	k	k-1
k	0	1	2	…	k-2	k-1	k	α_k
α_k	1	2	3	…	k-1	k	k	k

$Ł_{k+1}$

→	0	1	2	...	k-2	k-1	k
0	k	k	k	...	k	k	k
1	k-1	k	k	...	k	k	k
2	k-2	k-1	k	...	k	k	k
...	...	...	...	...	...	...	...
k-2	2	3	4	...	k	k	k
k-1	1	2	3	...	k-1	k	k
k	0	1	2	...	k-2	k-1	k

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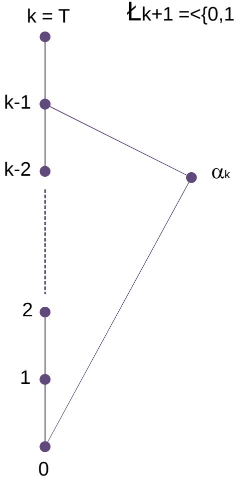

$\mathbf{L}(Ł^{+}_{k+1})$

$\longrightarrow$	0	1	2	…	k-2	k-1	k	α_k
0	k	k	k	…	k	k	k	k-1
1	k-1	k	k	…	k	k	k	k-1
2	k-2	k-1	k	…	k	k	k	k-1
…	…	…	…	…	…	…	…	…
k-2	2	3	4	…	k	k	k	k-1
k-1	1	2	3	…	k-1	k	k	k-1
k	0	1	2	…	k-2	k-1	k	α_k
α_k	1	2	3	…	k-1	k	k	k

-$\mathbf{L}^{+}_{\mathbf{k+1}}$ is a BCK lattice

-$(\varphi \to \psi)^{k-1} \to ((\varphi \to \chi)^{k-1} \to (\varphi \to \psi \wedge \chi))$ is a $\mathbf{L}^{+}_{\mathbf{k+1}}$ tautology.

-$(\varphi \to \psi)^{k-2} \to ((\varphi \to \chi)^{k-2} \to (\varphi \to \psi \wedge \chi))$ is not a $\mathbf{L}^{+}_{\mathbf{k+1}}$ tautology.

because, for every $m \in \omega$,
$(\alpha_k \to \alpha_k)^m \to ((\alpha_k \to k-2)^{k-2} \to (\alpha_k \to \alpha_k \wedge k-2)) = k-1 \neq \top$.

Theorem

For every $n > 0$, $LBCK_{n+1,n+1} < LBCK_{n,n}$

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Corollary

There is no weakest consistent axiomatic extension of LBCK with modus ponens as the only rule.

Local Deduction Theorem

Let L be an axiomatic extension of $LBCK_{n,m}$. Then for every set of formuli Σ and every formuli φ and ψ,

$$\Sigma \cup \{\varphi\} \vdash_L \psi \text{ if and only if } \Sigma \vdash_L \varphi^k \to \psi \text{ for some } k \in \omega$$

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This is a special case of local deduction theorem, which we call the **natural local deduction theorem**

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Theorem

Let L be an axiomatic extension of LBCK, then L admits Modus Ponens as the only rule if and only if L satisfies the natural local deduction theorem.

LBCK does not enjoy local deduction theorem.

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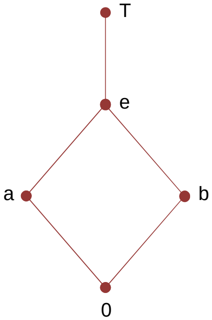
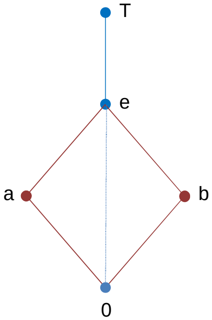
Algebraic proof: $\mathbb{LBCK}$ does not satisfy the congruence extension property CEP (equivalently filter extension property FEP).

C



$G_3 < C$



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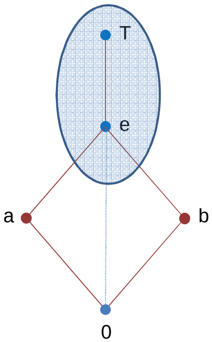

Conclusions

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- Natural local deduction theorem also characterizes axiomatic extensions of LBCK with modus ponens as the only rule. While *LBCK* does not satisfy any local deduction theorem.

Open questions

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- Is the local deduction theorem in the frame of $LBCK$ axiomatic extensions, equivalent to the property of admitting modus ponens as unique rule?

THANK YOU FOR YOUR ATTENTION