Involutive left-continuous t-norms arising from completion of MV-chains

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Chang's MV-algebra

Linearly ordered MV-algebras (MV-chains) can be simple or non-simple: simple MV-chains are subalgebras of the standard MV-algebra $[0, 1]$.

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The basic example of a non simple MV-chain is Chang's MV-algebra.

It can be defined as

$$\mathcal{C} = \Gamma(\mathbb{Z} \operatorname{lex} \mathbb{Z}, (1, 0)),$$

where $\mathbb{Z} \operatorname{lex} \mathbb{Z}$ is the abelian ℓ-group obtained as the lexicographic product of two copies of the ℓ-group $\mathbb{Z}$ of the integer numbers, and Γ is Mundici's functor, which implements a categorical equivalence between abelian ℓ-groups with a distinguished strong unit and MV-algebras.

$\mathbb{DLMV}$

$\mathbb{DLMV}$ is the variety generated by the Chang's MV-algebra $\mathcal{C}$.

The variety $\mathbb{DLMV}$ is axiomatized from the variety of MV-algebras adding the axiom

$$(2x)^2 = 2x^2 \,.$$

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$\mathcal{C}$ is a subalgebra of $\Gamma(\mathbb{Z} \operatorname{lex} \mathbb{R}, (1,0))$.

$\Gamma(\mathbb{Z} \operatorname{lex} \mathbb{R}, (1,0))$ also generates $\mathbb{DLMV}$.

The MV-algebra $[0,1]_{(1/2)}$

We can represent $\Gamma(\mathbb{Z} \operatorname{lex} \mathbb{R}, (1,0))$ isomorphically as an MV-algebra

$$[0,1]_{(1/2)} = ([0,1/2) \cup (1/2,1], \widetilde{\odot}, \neg, 0)$$

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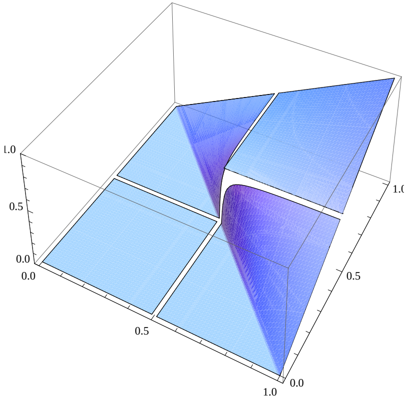
The monoidal operation $\widetilde{\odot}$ is given by

$$x \widetilde{\odot} y = \begin{cases} 1 - x - y + 2xy & \text{if } x,\ y \in (1/2, 1] \\ \frac{x+y-1}{2y-1} & \text{if } x \in [0,1/2),\ y \in (1/2,1] \text{ and } x+y>1 \\ 0 & \text{otherwise} \end{cases} .$$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Cancellative hoops

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The map $h : x \in (0,1] \rightarrow (x+1)/2 \in (1/2, 1]$ is a bijection and, so, h induces on $(1/2, 1]$ a structure of cancellative hoop.

Disconnected rotation

Definition

Let $(H, \cdot, \to, 1)$ be a hoop and H^- be a set disjoint from H, and let $^-$ be a bijection from H onto H^-. We denote by $\mathbf{DR}(H)$ the structure whose domain is $H \cup H^-$, whose constants are 1 and $0 = 1^-$ and whose operations $\circ$, $\Rightarrow$ and $\neg$ are defined, for all $x,\ y \in H$ by the following clauses:

$$x \circ y = \begin{cases} x \cdot y, & \text{if } x, y \in H \\ (x \to y^-)^- & \text{if } x \in H, y \in H^- \\ (y \to x^-)^- & \text{if } x \in H^-, y \in H \\ 0 & \text{otherwise.} \end{cases}$$

$$x \Rightarrow y = \begin{cases} x \to y, & \text{if } x, y \in H \\ (x \cdot y^-)^- & \text{if } x \in H, y \in H^- \\ 1 & \text{if } x \in H^-, y \in H \\ y^- \to x^- & \text{if } x, y \in H^- \end{cases}$$

This construction is called *disconnected rotation*.

Starting from cancellative hoops

The MV-algebra $[0,1]_{(1/2)}$ is, up to isomorphisms, the disconnected rotation of the standard cancellative hoop $((0,1],\cdot,\rightarrow,1)$.

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Another case of the same construction permits to obtain product algebras from cancellative hoops:

Product standard algebra is given by the t-norm of product and its associated residuum

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It is easy to see that the product algebra $([0,1], \cdot, \rightarrow_., 0)$ can be obtained from the cancellative hoop $((0,1], \cdot, \rightarrow_., 1)$ by adding a bottom element and properly extending the operations.

Free algebras

In this section we give an explicit functional description of the free algebra in the variety $\mathbb{DLMV}$. It is know that

Theorem (CigTor)

$$\mathcal{F}^n_{\mathbb{DLMV}} \simeq \prod_{i=1}^{2^n} \mathbf{DR}(\mathcal{F}^n_{\mathbb{CH}})$$

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In order to give a $[0,1]$-functional representation of $\mathcal{F}^n_{\mathbb{DLMV}}$, we are going to use the fact that $\mathbb{DLMV}$ is generated by a disconnected rotation of the cancellative hoop $(0,1]$, together with resizing functions:

$$\beta_0 : x \in [0,1/2) \to 1-2x \in (0,1],$$
$$\beta_1 : x \in (1/2,1] \to 2x-1 \in (0,1].$$

Free cancellative hoops

Definition

A monomial n-variate function on $D \subseteq \mathbb{R}$ is a function $f : D^n \to D$ such that $f(x_1, \ldots, x_n) = 1 \wedge (x_1^{m_1} \cdot \ldots \cdot x_n^{m_n})$ where $m_i \in \mathbb{Z}$, for each $i = 1, \ldots, n$.
A piece-wise monomial function f on $D \subseteq \mathbb{R}$ is a continuous function f such that there exists a family $\{f_m\}_{m \in M}$ of monomial functions and $f = \bigvee_p \bigwedge_q f_{pq}$.

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Theorem

The free cancellative hoop $\mathcal{F}^n_{\mathbb{CH}}$ over n generators is the algebra of functions from $(0,1]^n \to (0,1]$ that are piecewise monomial.

Free $DLMV$-algebras

Definition

For every $\mathbf{b} = (b_1, \ldots, b_n) \in \{0,1\}^n$ consider

$$B_i^{\mathbf{b}} = \begin{cases} [0, 1/2) & \text{if } b_i = 0 \\ (1/2, 1] & \text{if } b_i = 1 \end{cases}$$

and let

$$D_{\mathbf{b}} = B_1^{\mathbf{b}} \times \ldots \times B_n^{\mathbf{b}} .$$

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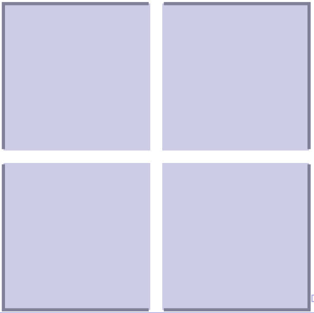

Let

$$\beta(D_{\mathbf{b}}) = \prod_{i=1}^{n} \beta_i(B_i^{\mathbf{b}}) \subseteq (0,1]^n .$$

Theorem

$\mathcal{F}^n_{\mathbb{DLMV}}$ is isomorphic to the MV-algebra of functions

$$f : {[0,1]_{(1/2)}}^n \to [0,1]_{(1/2)}$$

such that, for every $\mathbf{b} \in \{0,1\}^n$, there exists a piecewise monomial function

$$p_{\mathbf{b}} : (0,1]^n \to (0,1]$$

such that either

- $f \upharpoonright D_{\mathbf{b}} = \beta_0^{-1} \circ p_{\mathbf{b}} \circ \beta$ *or*
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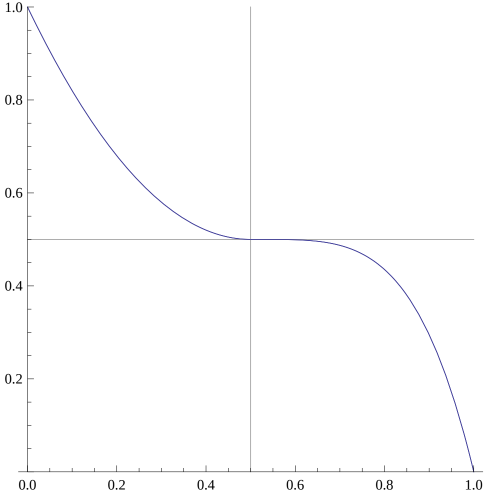
with operations defined pointwisely.

Example

This is an example for $n = 1$:



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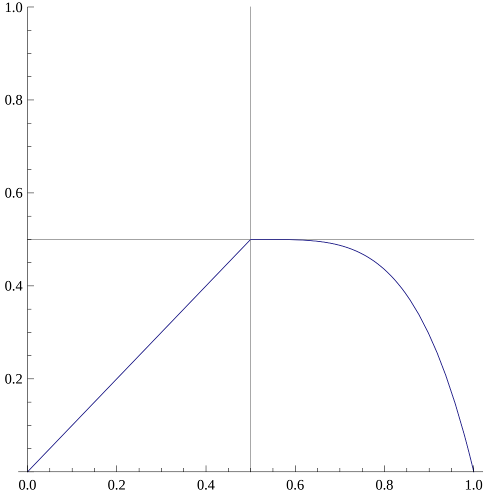


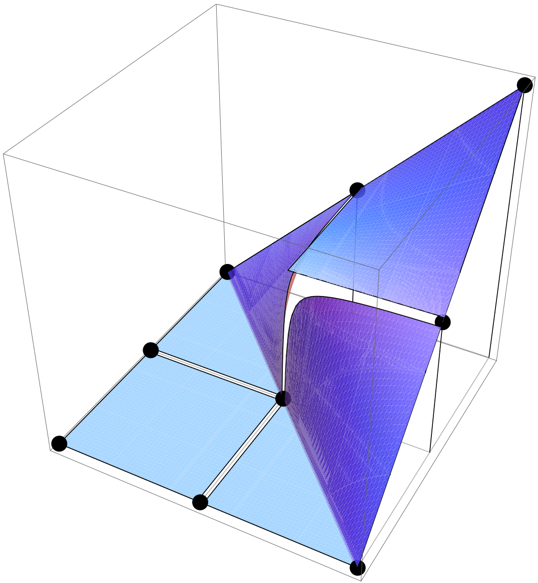
Adding $1/2$ to $[0,1] \setminus \{1/2\}$

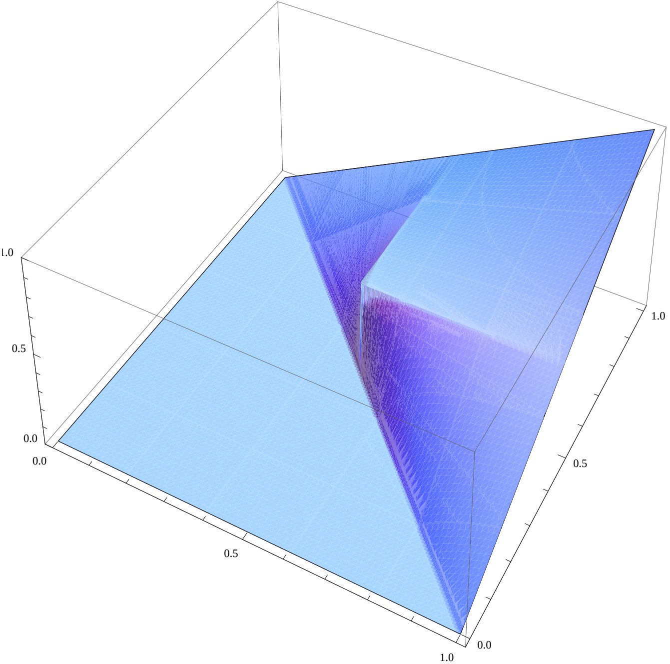
We want now to extend the operation $\widetilde{\odot}$ to the operation $\odot_{J\Pi}$ defined on the whole interval $[0,1]$:

$$x \odot_{J\Pi} y = \begin{cases} x \widetilde{\odot} y & \text{if } x, y \notin S_3 \\ x \odot_3 y & \text{if } x, y \in S_3 \\ x \odot_3 \lceil y \rceil_2 & \text{if } x \in S_3 \setminus \{1\}, y \notin S_3 \\ \lceil x \rceil_2 \odot_3 y & \text{if } x \notin S_3, y \in S_3 \setminus \{1\} \end{cases}$$

where $\widetilde{\odot}$ is the conjunction of $[0,1]_{(1/2)}$, S_3 is the MV-chain $\{0, 1/2, 1\}$, $\odot_3$ is the conjunction in S_3 and for each $x \in [0,1]$,

$$\lceil x \rceil_2 = \begin{cases} 0 & \text{if } x = 0 \\ 1/2 & \text{if } 0 < x \leq 1/2 \\ 1 & \text{if } 1/2 < x \leq 1 \end{cases}$$





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Rotation of product t-norm

Definition

Let T be a left continuous t-norm without zero divisors and T_1 the linear trasformation of T into $[1/2, 1]$. Define $T_J : [0,1]^2 \to [0,1]$ by

$$T_J(x,y) = \begin{cases} T_1(x,y) & \text{if } x, y > 1/2 \\ \neg I_{T_1}(x, \neg y) & \text{if } x > 1/2, y \leq 1/2 \\ \neg I_{T_1}(y, \neg x) & \text{if } x \leq 1/2, y > 1/2 \\ 0 & \text{if } x, y \leq 1/2 \end{cases},$$

where $I_{T_1}(x,y) = \sup\{s \in [1/2, 1] \mid T_1(x,s) \leq y\}$.

We call T_J the *connected rotation* of T.

Proposition

$\odot_{J\Pi}$ is a left-continuous t-norm. In particular it is the connected rotation of the product t-norm.

We can then consider the MTL-algebra

$$[0,1]_{J\Pi} = ([0,1], \odot_{J\Pi}, \rightarrow_{J\Pi}, \wedge, 0)$$

(that actually is an IMTL-algebra). Note that $[0,1]_{J\Pi}$ is not an MV-algebra.

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We further have

$[0,1]_{J\Pi}$ is the connected rotation of the cancellative hoop $((0,1], \cdot, \rightarrow_{\cdot}, 1)$.

Definition

Let $\mathbb{J}\Pi$ denote the variety of IMTL-algebras generated by

$$([0,1], \odot_{J\Pi}, \rightarrow_{J\Pi}, \wedge, 0).$$

Free algebras in $\mathbb{J}\Pi$

Theorem

$$\mathcal{F}^n_{\mathbb{J}\Pi} \cong DR(\mathcal{F}^n_{\mathbb{CH}})^{2^n} \times \prod_{j=1}^{n-1}(CR(\mathcal{F}^i_{\mathbb{CH}}))^{2^i\binom{n}{i}}$$

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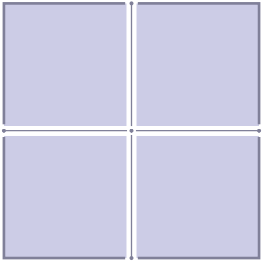
Functional description of $\mathcal{F}^n_{\mathbb{J}\Pi}$



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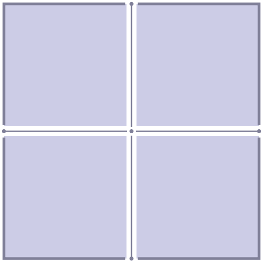

We define

$$\beta(D_{\mathbf{b}}) = \prod_{i \in I_{\mathbf{b}}} \beta_i(B_i^{\mathbf{b}}) \subseteq (0,1]^{n_{\mathbf{b}}}$$

where $I_{\mathbf{b}} = \{i \mid b_i \neq 1/2\}$ and $n_{\mathbf{b}} = |I_{\mathbf{b}}|$.

Let $FJ\Pi_n$ be the set of functions

$$f : [0,1]^n \to [0,1]$$

such that, for every $\mathbf{b} \in \{0, 1/2, 1\}^n$, there exists a piecewise monomial function $p_{\mathbf{b}} : (0,1]^{n_{\mathbf{b}}} \to (0,1]$ such that either

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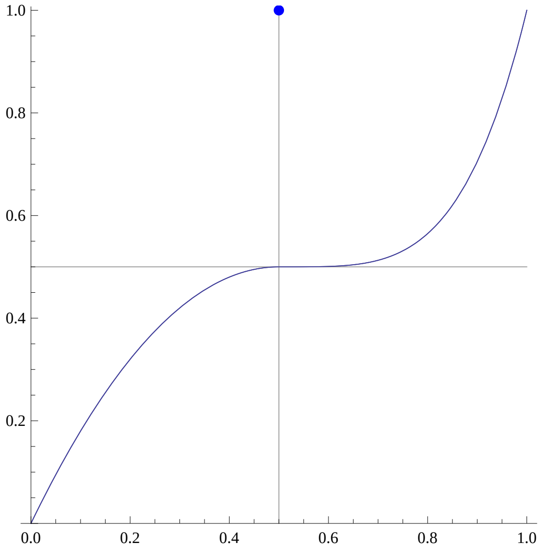

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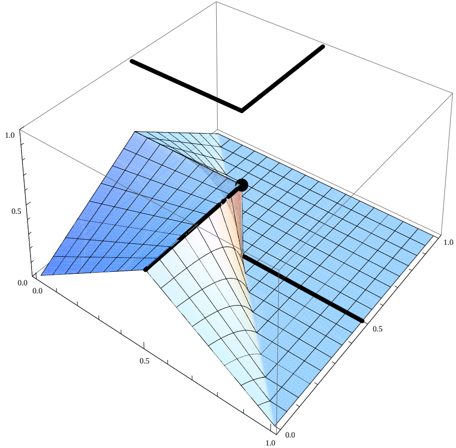

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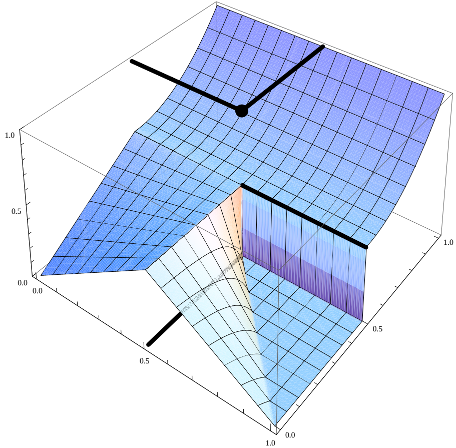

Axiomatising MV from IMTL

Let

$$B = [0,1]_{(1/2)} \qquad C = \{0, 1/2, 1\} \qquad D = [0,1]_{\mathbb{J}\Pi}$$

and note that B and C are MV-algebras, while D is not (we fix a common IMTL-language).

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Then, from a direct inspection of the functions involved, we have the following result

$$\mathcal{F}^1(\mathbb{V}(B, C)) \cong \mathcal{F}^1(\mathbb{V}(D)) \,.$$

As a consequence, the two varieties cannot be distinguished by equations with one variable.

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Hence, all one-variable equations holding for MV-algebras must also hold in D.

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The variety of BL-algebras admits no one-variable axiomatisation from the variety of MTL-algebras.

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The variety of BL-algebras admits no one-variable axiomatisation from the variety of MTL-algebras.

The variety $\mathbb{V}(B, C)$ can be axiomatized with one-variable equations from the axioms defining MV-algebras; hence:

Proposition

$$\mathbb{MV} \cap \mathbb{V}(D) = \mathbb{V}(B, C)\,.$$

Categorical equivalences

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Directly indecomposable algebras in Π are exactly cancellative hoops with an added bottom.

Directly indecomposable algebras in $\mathbb{DLMV}$ are exactly disconnected rotation of cancellative hoops.

Directly indecomposable algebras in $\mathbb{J}\Pi$ are either disconnected or connected rotation of cancellative hoops.

Categorical equivalences: directly indecomposable

We can hence establish a categorical equivalence among the following categories:

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We can then consider a category $\Pi^\flat$ whose objects are pairs

$$(P, b)$$

of a Π-algebra P and an element b in the Boolean skeleton of P, and whose arrows $(P_1, b_1) \to (P_2, b_2)$ are product algebras homomorphisms $f : P_1 \to P_2$ such that $f(b_1) \leq b_2$.

Further, let $\Pi^\flat_1$ be the subcategory of $\Pi^\flat$ in which the product algebras are directly indecomposable. Then:

Categorical equivalences: directly indecomposable

We can hence establish a categorical equivalence among the following categories:

The categories of directly indecomposable Π algebras, directly indecomposable $\mathbb{DLMV}$ algebras and cancellative hoops are equivalent.

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Further, let $\Pi_1^\flat$ be the subcategory of $\Pi^\flat$ in which the product algebras are directly indecomposable. Then:

The category of directly indecomposable $\mathbb{J}\Pi$ algebras is equivalent to $\Pi_1^\flat$.

Categorical equivalences: finitely presented

Finitely presented algebras are direct product of finitely many directly indecomposable algebras, hence we have:

The category of finitely presented $\mathbb{DLMV}$ algebras is equivalent to the category of finitely presented Π algebras.

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The category of finitely presented $\mathbb{DLMV}$ algebras is equivalent to the category of finitely presented Π algebras.

Further, let $\Pi_2^\flat$ be the subcategory of $\Pi^\flat$ in which the product algebras are finitely presented.

The category of finitely presented $\mathbb{J}\Pi$ algebras is equivalent to $\Pi_2^\flat$.

Generalising

Let us fix some notation: we set

$$\begin{aligned} S_n^{\omega} &= \Gamma(\mathbb{Z} \operatorname{lex} \mathbb{Z}, (n-1, 0)), \\ S_n^{\mathfrak{c}} &= \Gamma(\mathbb{Z} \operatorname{lex} \mathbb{R}, (n-1, 0)), \\ S_n &= \Gamma(\mathbb{Z}, n-1). \end{aligned}$$

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Note that

$$S_n \cong L_n = \left\{0, \frac{1}{n-1}, \frac{2}{n-1} \ldots, \frac{n-2}{n-1}, 1\right\}.$$

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$$S_n \cong L_n = \left\{0, \frac{1}{n-1}, \frac{2}{n-1} \ldots, \frac{n-2}{n-1}, 1\right\}.$$

For each integer $n > 1$, we can find an MV-chain $L_n^{\mathfrak{c}}$ with universe

$$[0,1] \setminus \left\{\frac{1}{n}, \frac{2}{n} \ldots, \frac{n-2}{n}\right\}$$

such that

$$S_n^{\omega} \subseteq S_n^{\mathfrak{c}} \cong L_n^{\mathfrak{c}}.$$

Left-continuous t-norms $\odot_n^*$

Clearly, $[0,1] = L_n^{\mathfrak{c}} \cup L_{n+1}$.

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We can define for each integer $n > 1$ the operation $\odot_n^*$ setting, for every $x, y \in [0,1]$:

$$x \odot_n^* y = \begin{cases} x \odot_n^{\mathfrak{c}} y & \text{if } x, y \notin L_{n+1} \\ x \odot_{n+1} y & \text{if } x, y \in L_{n+1} \\ x \odot_{n+1} \lceil y \rceil_{n+1} & \text{if } x \in L_{n+1}, y \notin L_{n+1} \\ \lceil x \rceil_{n+1} \odot_{n+1} y & \text{if } x \notin L_{n+1}, y \in L_{n+1} \end{cases}$$

where $\odot_n^{\mathfrak{c}}$ is the monoidal conjunction of $L_n^{\mathfrak{c}}$, $\odot_{n+1}$ is the monoidal conjunction of L_{n+1} and for each $x \in [0,1]$, $\lceil x \rceil_{n+1}$ is the smallest element of L_{n+1} greater or equal to x.

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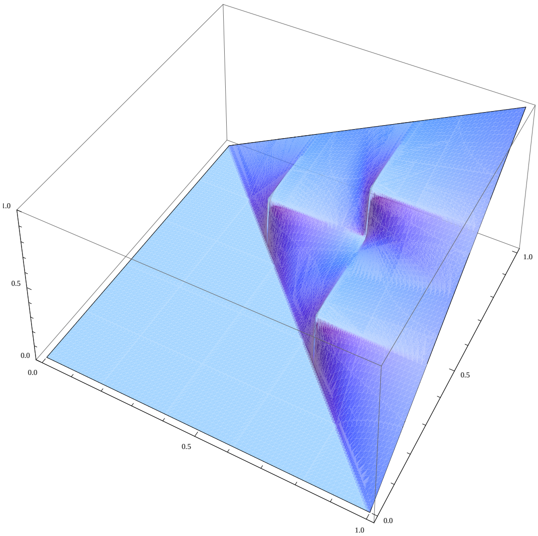
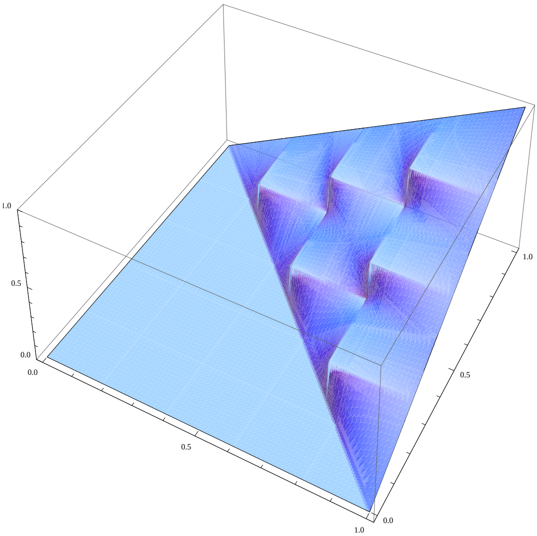
$\odot_n^*$ is a left-continuous t-norm.

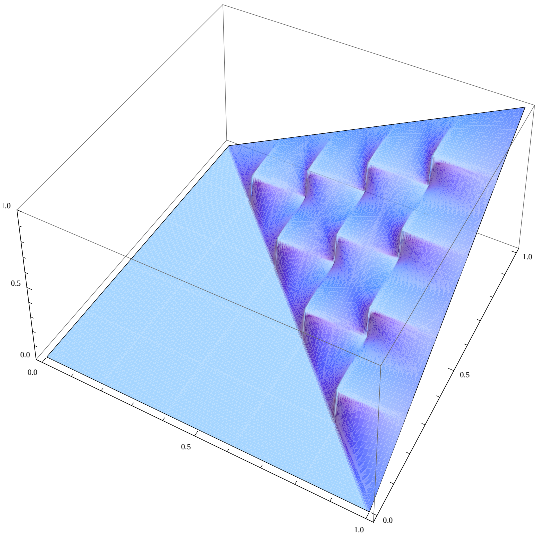
This is an example for $n = 3$:



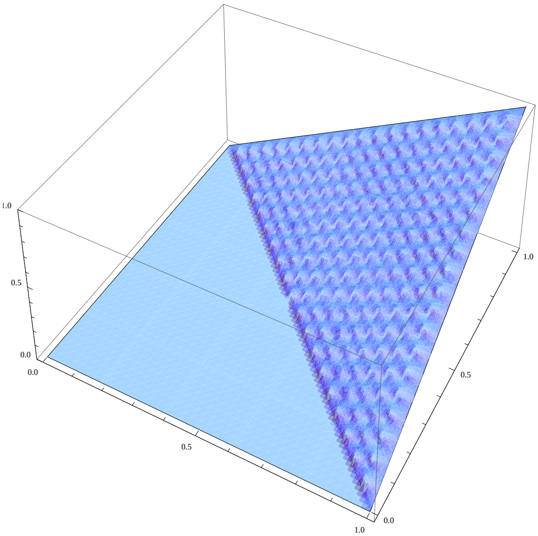
This is an example for $n = 4$:



This is an example for $n = 5$:



This is an example for $n = 20$:



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Note that, for each $n > 1$,

$$\mathcal{F}^1(\mathbb{V}([0,1], \odot_n^{\mathfrak{c}}, \rightarrow_n^{\mathfrak{c}}, \wedge, 0)) \cong \mathcal{F}^1(\mathbb{V}(S_n^{\omega}, S_{n+1})),$$

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Theorem

Given a subvariety of MV-algebras $\mathbb{V}$, if there exists a standard IMTL-algebra L such that:

$$\mathcal{F}_1(\mathbb{V}) \cong \mathcal{F}_1(\mathbb{V}(L))$$

then either $\mathbb{V} = \mathbb{MV}$ (and the L is the standard MV-algebra) or there is n such that $\mathbb{V} = \mathbb{V}(S_n^{\omega}, S_{n+1})$.

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