A betting metaphor for belief functions on MV-algebras and fuzzy epistemic states

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The extension problem: classical setting

Two players, Bookmaker (**B**) and Gambler (**G**), play the following game:

- **B** fixes a finite class of *events* $e_1, \ldots, e_k$ and a *Book* $\alpha : e_i \mapsto \alpha_i \in [0, 1]$;
- **G** chooses *stakes* $\sigma_1, \ldots, \sigma_k$ in $\mathbb{R}$ one for each event e_i and **G** pays to **B** the amount of $\sum_{i=1}^{k} \sigma_i \cdot \alpha_i$ euros.
- In a *future* possible word V, for each e_i, **B** pays to **G**:
  - 0 euros if e_i is *false* in V;
  - σ_i euros if e_i turns out to be *true* in V.
- Hence **G** and **B** are betting on unknown events and on the fact that they will turn out to be true.
- The total balance of the game for **B** is hence:

$$\sum_{i=1}^{k} \sigma_i \cdot \alpha_i - \sum_{i=1}^{k} \sigma_i \cdot V(e_i) = \sum_{i=1}^{k} \sigma_i \cdot (\alpha_i - V(e_i)).$$

The book α is said to be a *Dutch-Book* provided that Gambler **G** has a strategy of bets ensuring her a *sure win* in every possible world V.

Formalization of the problem

Let $X = \{V_1, V_2, \ldots, V_n\}$ be a finite set of possible worlds, and let $e_1, \ldots, e_k$ in 2^X. A *book* is a map

$$\alpha : e_i \mapsto \alpha_i \in [0, 1].$$

Then α is coherent iff for every $\sigma_1, \ldots, \sigma_k \in \mathbb{R}$, there exists a possible world (i.e. a Boolean homomorphism) $V_j : 2^X \to \{0, 1\}$ such that

$$\sum_{i=1}^{k} \sigma_i(\alpha(e_i) - V_j(e_i)) \geq 0.$$

By de Finetti's theorem the coherence of α is equivalent to the existence of a probability measure P_α on 2^X such that for each i,

$$P_\alpha(e_i) = \alpha(e_i) = \alpha_i.$$

For every possible world $V_j \in \{V_1, \ldots, V_n\}$ let

$$p_j = \langle V_j(e_1), \ldots, V_j(e_k) \rangle \in \{0,1\}^k$$

and let

$$\mathcal{H} = \mathrm{co}\{p_j : j \in \{1, 2, \ldots, n\}\} \subseteq [0,1]^k.$$

Then the book α is coherent (i.e. it extends to P_α) iff

$$\langle \alpha_1, \ldots, \alpha_k \rangle \in \mathcal{H}.$$

The case of many-valued events

MV-algebras are the equivalent algebraic semantics for Łukasiewicz logic. These algebras are systems $\mathbf{A} = (A, \oplus, \neg, 0, 1)$ of type $(2, 1, 0, 0)$. The class of MV-algebras forms a variety $\mathbb{MV}$.

(1) The typical example of MV-algebra is $[0,1]_{MV} = ([0,1], \oplus, \neg, 0, 1)$ where, for each $x, y \in [0,1]$, $x \oplus y = \min\{1, x+y\}$ and $\neg x = 1 - x$. The algebra $[0,1]_{MV}$ is generic for $\mathbb{MV}$.

(2) The class of all functions from $[0,1]^k$ to $[0,1]$ which are continuous, piecewise linear with integer coefficients, together with operations $\oplus$ and $\neg$ defined as in $[0,1]_{MV}$ pointwise, is the free MV-algebra with k generators.

De Finetti's coherence criterion can be stated in the frame of MV-algebras as follows (cf. Paris (7) and Mundici (6)):

Let A be an MV-algebra, and let $e_1, \ldots, e_k$ be *events* in A. Let further

$$\alpha : e_i \mapsto \alpha_i \in [0, 1]$$

be a book on the events e_i's published by the bookmaker.

Then α is coherent provided that for every choice of stakes $\sigma_1, \ldots, \sigma_k \in \mathbb{R}$, there exists a *many-valued possible world* $V : A \to [0, 1]_{MV}$ (i.e. an MV-homomorphism) such that

$$\sum_{i=1}^{k} \sigma_i \cdot \alpha(e_i) - \sum_{i=1}^{k} \sigma_i \cdot V(e_i) = \sum_{i=1}^{k} \sigma_i(\alpha(e_i) - V(e_i)) \geq 0.$$

A state on an MV-algebra A is a map $\mathbf{s} : A \to [0,1]$ such that:

- $\mathbf{s}(1) = 1$;
- Whenever $x \odot y = 0$, $\mathbf{s}(x \oplus y) = \mathbf{s}(x) + \mathbf{s}(y)$, (where $x \odot y = \neg(\neg x \oplus \neg y)$).

Mundici (6) (and Kühr-Mundici (5)) proved the following generalization of de Finetti's theorem:

Theorem. Let A be an MV-algebra, $\{e_1, \ldots, e_k\} \subseteq A$, and $\alpha : e_i \mapsto \alpha_i \in [0,1]$. Then the following are equivalent:

- α is coherent;
- There exists a state $\mathbf{s} : A \to [0,1]$ such that $\mathbf{s}(e_i) = \alpha_i$ for each $i = 1, \ldots, k$;
- There are MV-homomorphisms $V_1, \ldots, V_{k+1} : A \to [0,1]_{MV}$ such that

$$\langle \alpha_1, \ldots, \alpha_k \rangle \in \mathrm{co}\{p_j \mid j = 1, \ldots, k+1\}.$$

  where $p_j = \langle V_j(e_1), \ldots, V_j(e_k) \rangle \in [0,1]^k$.

Belief functions on Boolean algebras

Belief functions on Boolean algebras can be introduced as follows:
Let 2^X be a Boolean algebra of sets. For every $A \subseteq X$, consider the map

$$\beta_A : B \subseteq X \mapsto \begin{cases} 1 & \text{if } B \subseteq A \\ 0 & \text{otherwise.} \end{cases}$$

Then $bel : 2^X \to [0, 1]$ is a ***belief function on*** 2^X provided that there exists a probability measure $P : 2^{2^X} \to [0, 1]$ such that, for every $A \in 2^X$,

$$bel(A) = P(\beta_A).$$

A characterization of coherence in terms of extendability to a belief function was proved by Jaffray, 1989 (4). We will provide a similar result to the case of many-valued events.

Belief functions on MV-algebras of fuzzy sets

In order to generalize belief function to MV-algebras of the form $[0,1]^X$ (with X finite), consider, for every $a \in [0,1]^X$, the map ρ_a so defined:

$$\rho_a : \pi \in [0,1]^X \mapsto \inf\{\neg\pi(x) \oplus a(x) : x \in X\}.$$

Notice that the map ρ_a generalizes β_A: for every $A \in 2^X$, the restriction of ρ_A to 2^X coincides with β_A.

The MV-algebra $\mathcal{R}_X$ generated by all the functions ρ_a (for $a \in [0,1]^X$) is a separating MV-algebra of continuous functions.

The MV-algebra $\mathcal{R}_X$ is an MV-subalgebra of $[0,1]^{[0,1]^X}$.

Definition. A map $\mathbf{b} : [0,1]^X \to [0,1]$ is belief function if there exists a state $\mathbf{s} : \mathcal{R}_X \to [0,1]$ such that, for every $a \in [0,1]^X$,

$$\mathbf{b}(a) = \mathbf{s}(\rho_a).$$

A belief function $\mathbf{b}$ is said to be *normalized* provided that $\mathbf{b}(\mathbf{0}) = 0$.

The map $\rho_{(\cdot)}(\pi)$

For each $\pi \in [0,1]^X$, the map

$$N^\pi : a \in [0,1]^X \mapsto \rho_a(\pi) \in [0,1]$$

is a homogeneous necessity measure,
Moreover $N^\pi(\cdot)$ is normalized provided that there exists an $x \in X$ such that $\pi(x) = 1$.

Lemma. (1) The class of all necessity measures on $[0,1]^X$ coincides with the class $\{\rho_{(\cdot)}(\pi) : a \in [0,1]^X \mapsto \rho_a(\pi) \mid \pi \in [0,1]^X\}$.
(2) The class of all normalized necessity measures on $[0,1]^X$ coincides with the class $\{\rho_{(\cdot)}(\pi) : a \in [0,1]^X \mapsto \rho_a(\pi) \mid \pi \in [0,1]^X, \max_{x \in X} \pi(x) = 1\}$.

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Remark. In order to define (normalized) belief functions on $[0,1]^X$ we need two kind of mappings:

- A (normalized) necessity measure (equivalently a (normalized) possibility distribution);
- A state.

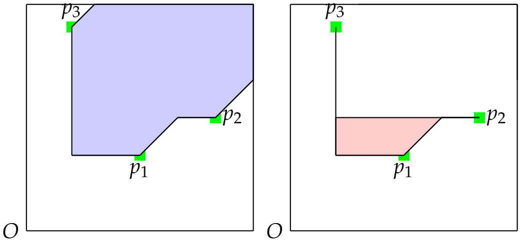
Idempotent (tropical) convex combinations

Fix $p_1, \ldots, p_n \in [0,1]^k$. A point $x \in [0,1]^k$ is a

bounded (normalized) min-plus convex combination of $p_1, \ldots, p_n$ if there exist $\lambda_1, \ldots \lambda_n \in [-1, 0]$ (with $\bigvee_{i \leq n} \lambda_i = 0$) such that

$$x(j) = \bigwedge_{i \leq n} (\lambda_i + p_i(j)), \text{ for every } j = 1, \ldots, k.$$

The *bounded min-plus convex hull* of $\{p_1, \ldots, p_n\}$ is denoted $\text{bmp-co}(p_1, \ldots, p_n)$, The *bounded normalized min-plus convex hull* of $\{p_1, \ldots, p_n\}$ is denoted $\text{nmp-co}(p_1, \ldots, p_n)$,



Theorem. [F -Godo, (3)] Let $e_1, \ldots, e_k \in [0,1]^X$, and let $\alpha : e_i \mapsto \alpha_i$ be an assignment. Then the following hold:

1. α extends to a belief function $\mathbf{b}$ on $[0,1]^X$ iff there are MV-homomorphisms $V_x : [0,1]^X \to [0,1]_{MV}$ (for $x \in X$) such that

$$\langle \alpha_1, \ldots, \alpha_k \rangle \in \overline{\text{co}}(\text{bmp-co}(\{p_x : x \in X\})).$$

2. α extends to a normalized belief function $\mathbf{b}$ on $[0,1]^X$ iff there are MV-homomorphisms $V_x : [0,1]^X \to [0,1]_{MV}$ (for $x \in X$) such that

$$\langle \alpha_1, \ldots, \alpha_s \rangle \in \overline{\text{co}}(\text{nmp-co}(\{p_x : x \in X\})).$$

(For every $x \in X$, $p_x = \langle V_x(e_1), \ldots, V_x(e_k) \rangle$)

Let $X = \{V_1, V_2, V_3\}$, and let $e_1, e_2 \in [0,1]^X$ be:

$$e_1 = \langle 1/2, 5/6, 1/5 \rangle \text{ and } e_2 = \langle 1/3, 1/2, 9/10 \rangle,$$

and the following assignments

$$\alpha_1(e_1) = 1/3, \alpha_1(e_2) = 2/5 \tag{1}$$

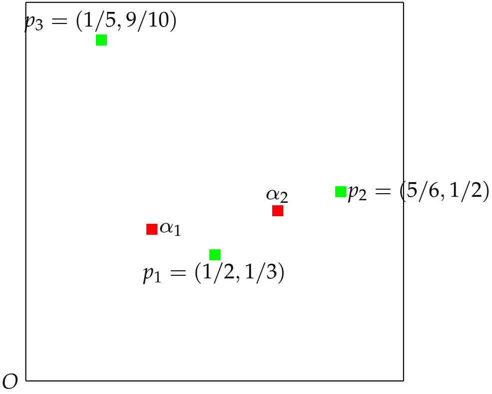
and

$$\alpha_2(e_1) = 2/3, \alpha_2(e_2) = 18/40 \tag{2}$$

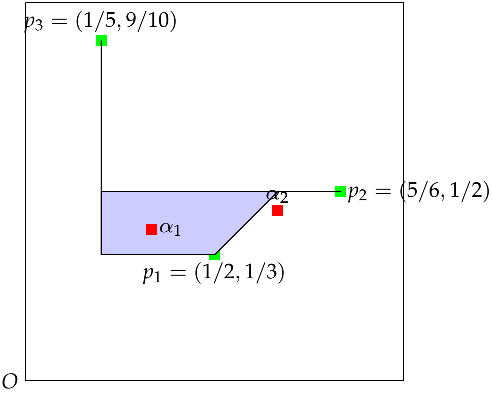
The events e_1 and e_2 corresponds, in $[0,1]^2$, to the points:

$$\begin{aligned} p_1 &= \langle V_1(e_1), V_1(e_2) \rangle = \langle 1/2, 1/3 \rangle \\ p_2 &= \langle V_2(e_1), V_2(e_2) \rangle = \langle 5/6, 1/2 \rangle \\ p_3 &= \langle V_3(e_1), V_3(e_2) \rangle = \langle 1/5, 9/10 \rangle \end{aligned}$$

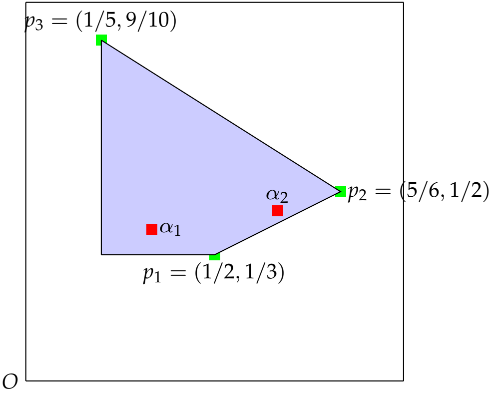
Extending to a Normalized Belief Function



Extending to a Normalized Necessity Measure



Extending to a Normalized Belief Function



Towards a betting interpretation

Turning back to the previous result, given a finite class of events in $[0,1]^X$, and a book

$$\alpha : e_i \mapsto \alpha_i,$$

the following are equivalent:

- There exists a (normalized) belief function $\mathbf{b} : [0,1]^X \to [0,1]$ such that

  $$\mathbf{b}(e_i) = \alpha_i$$

  for each i;
- There exists a state $\mathbf{s} : \mathcal{R}_X \to [0,1]$ such that, for each $i = 1, \ldots, k$

  $$\mathbf{s}(\rho_{e_i}) = \alpha_i.$$

- The book

  $$\alpha_R : \rho_{e_i} \mapsto \alpha_i$$

  is coherent (in terms of *states*), i.e. for every stakes $\sigma_1, \ldots, \sigma_k \in \mathbb{R}$, there exists a MV-homomorphism $V : \mathcal{R}_X \to [0,1]_{MV}$ (i.e. a MV-possible world) such that

  $$\sum_{i=1}^{k} \sigma_i(\alpha(\rho_{e_i}) - V(\rho_{e_i})) \geq 0.$$

Lemma. For every homomorphisms $V : \mathcal{R}_X \to [0,1]_{MV}$ there is a point $\pi \in [0,1]^X$ such that $V(\rho_a) = \rho_a(\pi) = N^\pi(a)$.

Hence we can state the coherence criterion for belief functions as follows:

Definition. A book $\alpha : e_i \in [0,1]^X \to [0,1]$ is **b**-*coherent* iff for all stakes $\sigma_1, \ldots, \sigma_k$, there exists a possibility distribution $\pi : X \to [0,1]$ such that

$$\sum_{i=1}^{k} \sigma_i(\alpha(e_i) - N^\pi(e_i)) \geq 0$$

Then

Theorem. A book $\alpha : e_i \mapsto \alpha_i \in [0,1]$ is **b**-coherent iff there exists a belief function $\mathbf{b} : [0,1]^X \to [0,1]$ such that, for each $i = 1, \ldots, k$,

$$\mathbf{b}(e_i) = \alpha(e_i).$$

Back to betting games

Two players, Bookmaker (**B**) and Gambler (**G**), play the following game:

- **B** fixes a finite class of *events* $e_1, \ldots, e_k \in [0,1]^X$ and a *book* $\alpha : e_i \mapsto \alpha_i$;
- **G** chooses *stakes* $\sigma_1, \ldots, \sigma_k$ in $\mathbb{R}$ one for each event e_i and **G** pays to **B** $\sum_{i=1}^k \sigma_i \cdot \alpha(e_i)$.
- Now **G** and **B** are betting on unknown events and on the fact that they will turn out to be necessarily true in a *fuzzy epistemic state* π:
  - **B** and **G** receive, for every event e_i, a truth value $V_x(e_i)$ *from every* $x \in X$ (not only one truth-value as in the case of states!).
  - Given π, they aggregate the truth values of each e_i by the necessity measure N^π as

$$N^\pi(e_i) = \bigwedge_{x \in X} \neg\pi(e_i) \oplus V_x(e_i).$$

- The total balance of the game for **B** is hence:

$$\sum_{i=1}^k \sigma_i \cdot \alpha(e_i) - \sum_{i=1}^k \sigma_i \cdot N^\pi(e_i) = \sum_{i=1}^k \sigma_i \cdot (\alpha(e_i) - N^\pi(e_i)).$$

The book α is said to be a **b**-*Dutch-Book* provided that Gambler **G** has a winning strategy ensuring a *sure win* in every possibility distribution of worlds (i.e. fuzzy epistemic state) $\pi : X \to [0,1]$.

The above criterion is stated with respect to the whole class $\mathscr{P}(X) = [0,1]^X$ of possibility distribution. Let

$$\mathscr{N}(X) = \{\pi \in \mathscr{P}(X) \mid \exists x \in X, \pi(x) = 1\}.$$

and

$$\mathscr{D}(X) = \{\pi \in \mathscr{N}(X) \mid \exists! x \in X, \pi(x) = 1 \text{ and } \pi(x') = 0 \text{ if } x' \neq x\}.$$

Then

$$\mathscr{D}(X) \subseteq \mathscr{N}(X) \subseteq \mathscr{P}(X).$$

For a subset $\mathscr{S}(X)$ of $\mathscr{P}(X)$ let us call $\mathscr{S}(X)$-*coherent* any book α on $e_1, \ldots, e_k$, for which the betting game fixes the possibility distributions to be in $\mathscr{S}(X)$.

Theorem. Let $e_1, \ldots, e_k$ be events in $[0,1]^X$ and let $\alpha : e_i \mapsto \alpha_i$ be a book. Then:

- α is $\mathscr{P}(X)$-coherent iff there exists a **belief function** $\mathbf{b} : [0,1]^X \to [0,1]$ which extends α.
- α is $\mathscr{N}(X)$-coherent iff there exists a **normalized** belief function $\mathbf{b} : [0,1]^X \to [0,1]$ which extends α.
- α is $\mathscr{D}(X)$-coherent iff there exists a **state** $\mathbf{s} : [0,1]^X \to [0,1]$ which extends α.

Indeterminacy degree and conditional bets (work in progress)

Given a possibility distribution π on worlds, it is natural to define, for every event e, its *indeterminacy degree* as:

$$I^{\pi}(e) = \Pi^{\pi}(e) - N^{\pi}(e).$$

Then we can consider a game in which **B**, given a possibility distribution π, is *obliged* to call off a bet on each event e_i involved in a book, for which $I^{\pi}(e_i) = 1$.

In this frame, the total balance for **B**, in a distribution π, is given by

$$\sum_{i=1}^{k}(1 - I^{\pi}(e_i)) \cdot (\sigma_i \cdot (\alpha(e_i) - N^{\pi}(e_i))).$$

A book $\alpha : e_i \to \alpha_i$, is said to be *coherent under indeterminacy* iff there is no way for **G** to make **B** incur in a sure loss, i.e. iff there is no system of bets $\sigma_1, \ldots, \sigma_k \in \mathbb{R}$ such that, for every possibility distribution π,

$$\sum_{i=1}^{k}(1 - I^{\pi}(e_i)) \cdot (\sigma_i \cdot (\alpha(e_i) - N^{\pi}(e_i))) < 0.$$

Then

Expected result. A book $\alpha : e_i \to \alpha_i$, is coherent under indeterminacy iff there is a *conditional state* **s** such that, for every $i = 1, \ldots, n$

$$\mathbf{s}(\Box e_i \mid \Diamond e_i \to \Box e_i) = \alpha_i.$$

Future work

We have presented a betting metaphor for belief functions on MV-algebras of fuzzy sets which can be uniformly applied to recover similar results w.r.t. normalized belief functions and states.

- Provide an *intuitive interpretation* for possibility distributions (fuzzy-epistemic states for **B** and **G**).
- Defining a *decision procedure* for the players which could suggest them to chose a particular subset of $\mathscr{P}(X)$ for their game (*reliability degree* on agents/possible words).
- Study much more in details the protocol of *coherence under indeterminacy*.

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Thank you.

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(Normalized) homogeneous necessity measures

A map $N^b : [0,1]^X \to [0,1]$ is a homogeneous necessity measure, i.e.

- $N^b(\top) = 1$, $[\rho_\top(b) = 1]$;
- $N^b(x \wedge y) = \min\{N^b(x), N^b(y)\}$, $[\rho_{x\wedge y}(b) = \min\{\rho_x(b), \rho_y(b)\}]$;
- $N^b(\bar{r} \oplus x) = r \oplus N^b(x)$, $[\rho_{\bar{r}\oplus x}(b) = r \oplus \rho_x(b)]$.