On logics of formal inconsistency and fuzzy logics

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Motivation

Graham Priest, *Paraconsistent logic*, Handbook of Philosophical Logic, Volume 6, 2nd edition, 2002.

The major motivation behind paraconsistent logic has always been the thought that in certain circumstances we may be in a situation where our information or theory is inconsistent, and yet we are required to draw inferences in a sensible fashion.

Paraconsistency

Western Philosophy has been, in general, hostile to contradictions.

Aristotle's Law of Non-contradiction

It is impossible for the same thing to belong and not to belong at the same time to the same thing and in the same respect.

Therefore $\varphi, \neg\varphi \models \psi$ (Classical logic is explosive)
In the presence of contradictions, Classical Logic does not allow to *draw inferences in a sensible fashion*.

Definition

A logic is paraconsistent if it is not explosive.

History of paraconsistent logic – 1

Non-contradiction law is finally well established in the nineteenth century in classical logic with the systems of Boole and Frege.

Paraconsistent logics arrive in the twentieth century:

- Vasil'év (1910): Aristotelian syllogistic with "S is both P and not P".
- Orlov (1929): First axiomatization of relevant logic R.
- Łukasiewicz (1910): Critique of Aristotle's Law of Non-contradiction.
- Jaśkowski (1948): First non-adjunctive paraconsistent logic.

$$\Gamma \vdash_{J} \varphi \text{ iff } \Diamond\Gamma \vdash_{S5} \Diamond\varphi$$

- Asenjo (1954): First many-valued paraconsistent logic.

History of paraconsistent logic – 2

- Smiley (1959): Filter logic. Relevant paraconsistent logics. Pittsburgh school (Anderson, Belnap, Meyer, Dunn), Australian school (R. Routley, V. Routley, G. Priest).
- Da Costa (1963): Axiomatization of a family of paraconsistent logics (C systems) and first quantified paraconsistent logic. Campinas School.
- A. Avron and A. Zamansky, work also in Paraconsistency in the recent years.

Paraconsistency: basic references

- G. Priest, *Paraconsistent logic*, Handbook of Philosophical Logic, Volume 6, 2nd edition, 2002.

- W.A. Carnielli, M.E. Coniglio, and J. Marcos. *Logics of Formal Inconsistency (LFIs)*. In D. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic (2nd. edition)*, volume 14, pages 1–93. Springer, 2007.
  - Carnielli and Marcos (2002): Logics of Formal Inconsistency (LFIs) as *paraconsistent logics that internalize the notions of consistency and inconsistency at the object-language level.*

Paraconsistent fuzzy logics? – 1

We are concerned with logics for reasoning with imperfect information (imprecision (e.g. vagueness), uncertainty, inconsistency, ...).

Paraconsistent fuzzy logics would be a tool to deal with inconsistent and vague information.

To the best of our knowledge, paraconsistency has not been considered in the framework of Mathematical Fuzzy Logic (MFL).

Paraconsistent fuzzy logics? – 2

Usual (truth-preserving) fuzzy logics are explosive:

- $\varphi, \psi \vdash \varphi \,\&\, \psi$
- $\varphi \,\&\, \neg\varphi \vdash \overline{0}$
- $\overline{0} \vdash \psi$

Therefore:

- $\varphi, \neg\varphi \vdash \psi$

Degree-preserving fuzzy logics – 1

Given a ($\triangle$)-core fuzzy logic L, its degree-preserving companion $\mathrm{L}^{\leq}$ is defined as:

$\Gamma \vdash_{\mathrm{L}^{\leq}} \varphi$ iff for every L-chain $\boldsymbol{A}$, every $a \in A$, and every $\boldsymbol{A}$-evaluation v, if $a \leq v(\psi)$ for every $\psi \in \Gamma$, then $a \leq v(\varphi)$.

- Font, Gil, Torrens, Verdú (AML, 2006): the case of Łukasiewicz logic
- Bou, Esteva, Font, Gil, Godo, Torrens, Verdú (JLC, 2009): the case of logics of bounded commutative integral residuated lattices

Degree-preserving fuzzy logics – 2

- The theorems of L and $\mathrm{L}^{\leq}$ coincide.
- $\psi_1, \ldots, \psi_n \vdash_{\mathrm{L}} \varphi$ iff $\psi_1 \& \ldots \& \psi_n \vdash_{\mathrm{L}} \varphi$.
- $\psi_1, \ldots, \psi_n \vdash_{\mathrm{L}^{\leq}} \varphi$ iff $\psi_1 \wedge \ldots \wedge \psi_n \vdash_{\mathrm{L}^{\leq}} \varphi$ iff $\vdash_{\mathrm{L}^{\leq}} \psi_1 \wedge \ldots \wedge \psi_n \to \varphi$ iff $\vdash_{\mathrm{L}} \psi_1 \wedge \ldots \wedge \psi_n \to \varphi$.
- $\mathrm{L}^{\leq}$ can be presented by the Hilbert system whose axioms are the theorems of L and the following deduction rules:

($\wedge$-adj) From φ and ψ, infer $\varphi \wedge \psi$.

$(\mathrm{MP})^{\leq}$ From φ, if $\varphi \to \psi$ is a theorem of L, infer ψ.

Paraconsistent fuzzy logics – 1

Theorem

$\mathrm{L}^{\leq}$ *is paraconsistent iff* L *is not pseudo-complemented.*

- $\varphi, \neg\varphi \vdash_{\mathrm{L}^{\leq}} \varphi \wedge \neg\varphi$
- $\vdash_{\mathrm{L}^{\leq}} \varphi \wedge \neg\varphi \to \overline{0}$ iff $\vdash_{\mathrm{L}} \varphi \wedge \neg\varphi \to \overline{0}$ iff L is pseudo-complemented

Therefore $\mathrm{L}^{\leq}$ is paraconsistent iff L is not an extension of SMTL.

Logics of Formal Inconsistency (LFI)

Definition

Let L be a logic containing a negation $\neg$, and let $\bigcirc(p)$ be a nonempty set of formulas depending exactly on the propositional variable p. Then L is an LFI if the following holds :

(i) $\varphi, \neg\varphi \nvdash \psi$ for some φ and ψ, i.e., L is not explosive w.r.t. $\neg$;

(ii) $\bigcirc(\varphi), \varphi \nvdash \psi$ for some φ and ψ;

(iii) $\bigcirc(\varphi), \neg\varphi \nvdash \psi$ for some φ and ψ; and

(iv) $\bigcirc(\varphi), \varphi, \neg\varphi \vdash \psi$ for every φ and ψ.

$\bigcirc(p)$ is what we need to *internalize the notions of consistency at the object-language level*.

Axiomatizing consistency operators over fuzzy logics I

Having in mind the properties that a consistency operator has to verify and that core fuzzy logics are logics complete with respect to the chains , it seems reasonable to define:

Consistency operators in non-SMTL chains

A consistency operator over a non-SMTL chain $\mathbf{A}$ is a unary operator $\circ : A \to A$ satisfying these minimal conditions:

(i) $x \wedge \circ(x) \neq 0$ for some $x \in A$;

(ii) $\neg x \wedge \circ(x) \neq 0$ for some $x \in A$;

(iii) $x \wedge \neg x \wedge \circ(x) = 0$ for every $x \in A$.

Such an operator $\circ$ can be thought as denoting the (fuzzy) degree of ‘classicality’ (or ‘reliability’, or ‘robustness’) of x with respect to the satisfaction of the law of explosion.

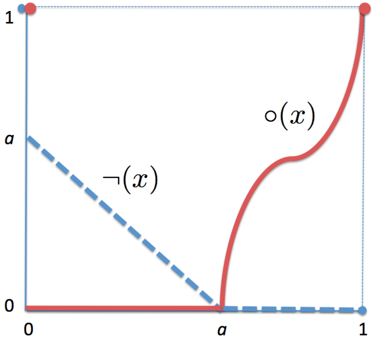
Axiomatizing consistency operators over fuzzy logics II

Proposed postulates:

(c1) If $x \wedge \neg x \neq 0$ then $\circ(x) = 0$;

(c2) If $x \in \{0, 1\}$ then $\circ(x) = 1$;

(c3) If $\neg x = 0$ and $x \leq y$ then $\circ(x) \leq \circ(y)$.



Axiomatizing consistency operators over fuzzy logics III

Definition

Let L be a non-*SMTL* logic. $\mathrm{L}_\circ$ is the expansion of L in a language which incorporates a new unary connective $\circ$ with the following axioms:

$$
\begin{array}{ll}
\text{(A1)} & \neg(\varphi \wedge \neg\varphi \wedge \circ\varphi) \\
\text{(A2)} & \circ\overline{1} \\
\text{(A3)} & \circ\overline{0}
\end{array}
$$

and the following inference rules:

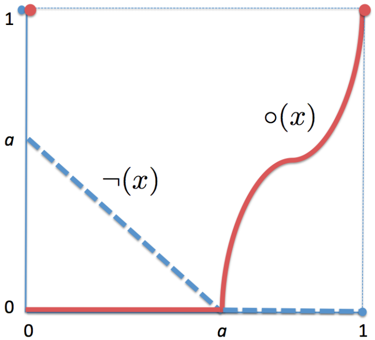
$$
\text{(sCng)} \quad \frac{(\varphi \leftrightarrow \psi) \vee \delta}{(\circ\varphi \leftrightarrow \circ\psi) \vee \delta} \qquad\qquad \text{(Coh)} \quad \frac{(\neg\neg\varphi \wedge (\varphi \to \psi)) \vee \delta}{(\circ\varphi \to \circ\psi) \vee \delta}
$$

Some properties of logics $\mathrm{L}_\circ$

- Chain-completeness: the logic $\mathrm{L}_\circ$ is strongly complete with respect to the class of $\mathrm{L}_\circ$-chains
- Conservativeness: $\mathrm{L}_\circ$ is a conservative expansion of L
- Real completeness preservation: a logic $\mathrm{L}_\circ$ is complete over $[0, 1]$-chains for deductions from a finite (resp. arbitrary) set of premises iff it is so the logic L.

Some interesting extensions / expansions

Recall the general form of $\circ$ operators in Ł chains:



$\circ(x)$ remains undetermined in the interval $\mathbb{I}_{\neg} = \{x < 1 \mid \neg(x) = 0\}$.

Next we consider some particular logics depending on $\circ$ in this interval

1) the case $\mathbb{I}_\neg = \emptyset$: the logic $L_\circ^{\neg\neg}$

The logic $L^{\neg\neg}$ is defined as the extension of L by adding the following rule:

$$(\neg\neg) \quad \frac{\neg\neg\varphi}{\varphi}$$

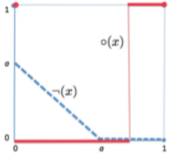
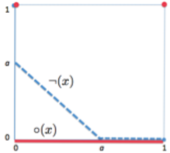
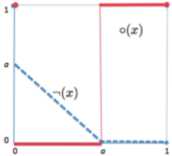
Then define the logic $L_\circ^{\neg\neg}$ as the expansion $L_\circ$ with the rule $(\neg\neg)$.

Observe that over chains, $\circ(x) = 1$ if $x \in \{0, 1\}$ and 0 otherwise.

Relation with Baaz-Monteiro's Δ operator:

- $\circ(\varphi) = \Delta(\varphi \vee \neg\varphi)$ and $\Delta(\varphi) = \circ(\varphi) \wedge \varphi$.
- $L_\circ^{\neg\neg}$ "equivalent" to $(L_\Delta)^{\neg\neg}$

2) the case of crisp ∘ operators

$L_\circ^c$	$L_\circ +$ (c) $\circ\varphi \vee \neg\circ\varphi$	
$L_\circ^{min}$	$L_\circ +$ (A4) $\varphi \vee \neg\varphi \vee \neg\circ\varphi$	
$L_\circ^{max}$	$L_\circ +$ $(\neg\neg_\circ)$ $\dfrac{\neg\neg\varphi \vee \delta}{\circ\varphi \vee \delta}$	

A family of Fuzzy LFIs

Our ultimate goal is the axiomatization of the expansions of paraconsistent logics $\mathrm{L}^{\leq}$ with a consistency operator $\circ$.

Axiomatization of $\mathrm{L}^{\leq}_{\circ}$

It is obtained by taking the same axioms of $\mathrm{L}_{\circ}$ and adding the following inference rules:

(Adj-$\wedge$) from φ and ψ deduce $\varphi \wedge \psi$

(MP-r) if $\vdash_{\mathrm{L}_{\circ}} \varphi \to \psi$, then from φ derive ψ

(Cong-r) if $\vdash_{\mathrm{L}_{\circ}} (\varphi \leftrightarrow \psi) \vee \delta$ then derive $(\circ\varphi \leftrightarrow \circ\psi) \vee \delta$

(Coh-r) if $\vdash_{\mathrm{L}_{\circ}} (\neg\neg\varphi \wedge (\varphi \to \psi)) \vee \delta$ then derive $(\circ\varphi \to \circ\psi) \vee \delta$

Similarly, when we replace $\mathrm{L}_{\circ}$ by any of the above consideres expansions / extensions.

A family of Fuzzy LFIs

Logic	Inference rules
$L_\circ^{\leq}$	rules of $L^{\leq}$ + (Cong-r) $\dfrac{\vdash_{L_\circ} (\varphi \leftrightarrow \psi) \vee \delta}{(\circ\varphi \leftrightarrow \circ\psi) \vee \delta}$ (Coh-r) $\dfrac{\vdash_{L_\circ} (\neg\neg\varphi \wedge (\varphi \to \psi)) \vee \delta}{(\circ\varphi \to \circ\psi) \vee \delta}$
$(L_\circ^{\neg\neg})^{\leq}$	rules of $L_\circ^{\leq}$ + $(\neg\neg\text{-}r)$ $\dfrac{\vdash_{L_\circ^{\neg\neg}} \neg\neg\varphi}{\varphi}$
$(L_\circ^{c})^{\leq}$	rules of $L_\circ^{\leq}$
$(L_\circ^{min})^{\leq}$	rules of $L_\circ^{\leq}$
$(L_\circ^{max})^{\leq}$	rules of $L_\circ^{\leq}$ + $(\neg\neg_\circ\text{-}r)$ $\dfrac{\vdash_{L_\circ^{max}} \neg\neg\varphi \vee \delta}{\circ\varphi \vee \delta}$

Recovering classical logic in LFIs

In the context of LFIs, it is a desirable property to recover classical reasoning by means of the consistency connective $\circ$:

$$\text{(DAT)} \quad \Gamma \vdash_{\mathbf{CPL}} \varphi \text{ iff } \circ(\Theta), \Gamma \vdash_{\mathrm{L}} \varphi.$$

where Θ, Γ and φ are in the language of **CPL**. This is known as *Derivability Adjustment Theorem* (DAT).

When the operator $\circ$ *suitably propagates* through connectives of a LFI logic L the DAT reduces to this simplified form:

PDAT

$$\text{(PDAT)} \quad \Gamma \vdash_{\mathbf{CPL}} \varphi \text{ iff } \{\circ p_1, \ldots, \circ p_n\} \cup \Gamma \vdash_{\mathrm{L}} \varphi$$

where $\{p_1, \ldots, p_n\}$ is the set of propositional variables occurring in $\Gamma \cup \{\varphi\}$.

Is there a DAT for the LFI logics $\mathrm{L}_\circ^\leq$?

Consider this (simplified form) of the translation:

$$(\text{PDAT}^*) \quad \vdash_{\mathbf{CPL}} \varphi \quad \text{iff} \quad \{\circ p_1, \ldots, \circ p_n\} \vdash_{\mathrm{L}_\circ^\leq} \varphi$$

$$\left(\text{iff} \quad \vdash_{\mathrm{L}_\circ} \left(\bigwedge_{i=1}^{n} \circ p_i\right) \to \varphi\right)$$

Unfortunately, this does not hold in general:
$\vdash_{\mathbf{CPL}} p \vee \neg p$ but, in general, $\nvdash_{\mathrm{L}_\circ^\leq} \circ p \to (p \vee \neg p)$

Define $\mathrm{L}_\circ^{dat}$ as the extension of $\mathrm{L}_\circ$ with the axiom $\circ\varphi \to (\varphi \vee \neg\varphi)$

A DAT property for $\mathrm{L}_\circ^\leq$

$\Gamma \vdash_{\mathbf{CPL}} \varphi$ iff there is some $k \geq 1$ such that $\Gamma \vdash_{\mathrm{L}_\circ^{dat}} (\bigwedge_{i=1}^{m} \circ p_i)^k \to \varphi$

Open question: do we need $k > 1$?

Conclusions

- We have investigated the possibility of defining paraconsistent logics of formal inconsistency (LFIs) based on systems of mathematical fuzzy logic by:
  (i) expanding axiomatic extensions of the fuzzy logic MTL with the characteristic consistency operators $\circ$ of LFIs
  (ii) considering their degree-preserving versions, that are paraconsistent.
- One could dually consider *inconsistency* operators $\bullet = \neg\circ$
- Together with a companion paper Ertola-Esteva-Flaminio-Godo-Noguera, these are first attempts to contribute to the study and understanding of the relationships between paraconsistency and fuzziness.