On logics of formal inconsistency and fuzzy logics

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Manyval 2013, Prague 4-6 September
The major motivation behind paraconsistent logic has always been the thought that in certain circumstances we may be in a situation where our information or theory is inconsistent, and yet we are required to draw inferences in a sensible fashion.
Western Philosophy has been, in general, hostile to contradictions.

**Aristotle’s Law of Non-contradiction**

*It is impossible for the same thing to belong and not to belong at the same time to the same thing and in the same respect.*

Therefore $\varphi, \neg \varphi \models \psi$ (Classical logic is explosive)

In the presence of contradictions, Classical Logic does not allow to *draw inferences in a sensible fashion.*

**Definition**

A logic is **paraconsistent** if it is not explosive.
Non-contradiction law is finally well established in the nineteenth century in classical logic with the systems of Boole and Frege.

Paraconsistent logics arrive in the twentieth century:

- Vasil’év (1910): Aristotelian syllogistic with “S is both P and not P”.
- Orlov (1929): First axiomatization of relevant logic R.
- Jaśkowski (1948): First non-adjunctive paraconsistent logic.
  \[ \Gamma \vdash J \varphi \iff \box\Gamma \vdash S5 \box\varphi \]

Da Costa (1963): Axiomatization of a family of paraconsistent logics (C systems) and first quantified paraconsistent logic. Campinas School.

A. Avron and A. Zamansky, work also in Paraconsistency in the recent years.
Paraconsistency: basic references


  Carnielli and Marcos (2002): Logics of Formal Inconsistency (LFIs) as *paraconsistent logics that internalize the notions of consistency and inconsistency at the object-language level*. 
We are concerned with logics for reasoning with imperfect information (imprecision (e.g. vagueness), uncertainty, inconsistency, ...).

Paraconsistent fuzzy logics would be a tool to deal with inconsistent and vague information.

To the best of our knowledge, paraconsistency has not been considered in the framework of Mathematical Fuzzy Logic (MFL).
Usual (truth-preserving) fuzzy logics are explosive:

- $\varphi, \psi \vdash \varphi \land \psi$
- $\varphi \land \neg \varphi \vdash \overline{0}$
- $\overline{0} \vdash \psi$

Therefore:

- $\varphi, \neg \varphi \vdash \psi$
Given a $(\triangle)$-core fuzzy logic $L$, its **degree-preserving companion** $L \leq$ is defined as:

$$\Gamma \vdash_{L \leq} \varphi \text{ iff for every } L\text{-chain } A, \text{ every } a \in A, \text{ and every } A\text{-evaluation } v, \text{ if } a \leq v(\psi) \text{ for every } \psi \in \Gamma, \text{ then } a \leq v(\varphi).$$

- Font, Gil, Torrens, Verdú (AML, 2006): the case of Łukasiewicz logic
- Bou, Esteva, Font, Gil, Godo, Torrens, Verdú (JLC, 2009): the case of logics of bounded commutative integral residuated lattices
The theorems of $L$ and $L \leq$ coincide.

$\psi_1, \ldots, \psi_n \vdash_L \varphi$ iff $\psi_1 \land \ldots \land \psi_n \vdash_L \varphi$.

$\psi_1, \ldots, \psi_n \vdash_{L \leq} \varphi$ iff $\psi_1 \land \ldots \land \psi_n \vdash_{L \leq} \varphi$ iff $\vdash_{L \leq} \psi_1 \land \ldots \land \psi_n \rightarrow \varphi$ iff $\vdash_L \psi_1 \land \ldots \land \psi_n \rightarrow \varphi$.

$L \leq$ can be presented by the Hilbert system whose axioms are the theorems of $L$ and the following deduction rules:

- $(\land\text{-adj})$ From $\varphi$ and $\psi$, infer $\varphi \land \psi$.
- $(\text{MP}) \leq$ From $\varphi$, if $\varphi \rightarrow \psi$ is a theorem of $L$, infer $\psi$. 
Theorem

\( L \leq \) is paraconsistent iff \( L \) is not pseudo-complemented.

1. \( \varphi, \neg \varphi \vdash_{L \leq} \varphi \land \neg \varphi \)
2. \( \vdash_{L \leq} \varphi \land \neg \varphi \rightarrow 0 \) iff \( \vdash_{L} \varphi \land \neg \varphi \rightarrow 0 \) iff \( L \) is pseudo-complemented

Therefore \( L \leq \) is paraconsistent iff \( L \) is not an extension of SMTL.
Let $L$ be a logic containing a negation $\neg$, and let $\bigcirc(p)$ be a nonempty set of formulas depending exactly on the propositional variable $p$. Then $L$ is an LFI if the following holds:

(i) $\varphi, \neg \varphi \not\models \psi$ for some $\varphi$ and $\psi$, i.e., $L$ is not explosive w.r.t. $\neg$;

(ii) $\bigcirc(\varphi), \varphi \not\models \psi$ for some $\varphi$ and $\psi$;

(iii) $\bigcirc(\varphi), \neg \varphi \not\models \psi$ for some $\varphi$ and $\psi$; and

(iv) $\bigcirc(\varphi), \varphi, \neg \varphi \models \psi$ for every $\varphi$ and $\psi$.

$\bigcirc(p)$ is what we need to internalize the notions of consistency at the object-language level.
Having in mind the properties that a consistency operator has to verify and that core fuzzy logics are logics complete with respect to the chains, it seems reasonable to define:

**Consistency operators in non-SMTL chains**

A consistency operator over a non-SMTL chain $A$ is a unary operator $\circ : A \rightarrow A$ satisfying these minimal conditions:

- (i) $x \land \circ(x) \neq 0$ for some $x \in A$;
- (ii) $\neg x \land \circ(x) \neq 0$ for some $x \in A$;
- (iii) $x \land \neg x \land \circ(x) = 0$ for every $x \in A$.

Such an operator $\circ$ can be thought as denoting the (fuzzy) degree of ‘classicality’ (or ‘reliability’, or ‘robustness’) of $x$ with respect to the satisfaction of the law of explosion.
Proposed postulates:

(c1) If $x \land \neg x \neq 0$ then $\circ(x) = 0$;

(c2) If $x \in \{0, 1\}$ then $\circ(x) = 1$;

(c3) If $\neg x = 0$ and $x \leq y$ then $\circ(x) \leq \circ(y)$. 
Definition

Let $L$ be a non-$SMTL$ logic. $L_\circ$ is the expansion of $L$ in a language which incorporates a new unary connective $\circ$ with the following axioms:

(A1) $\neg (\varphi \land \neg \varphi \land \circ \varphi)$
(A2) $\circ \bar{1}$
(A3) $\circ \bar{0}$

and the following inference rules:

(sCng) $\frac{(\varphi \leftrightarrow \psi) \lor \delta}{(\circ \varphi \leftrightarrow \circ \psi) \lor \delta}$
(Coh) $\frac{\neg \neg \varphi \land (\varphi \rightarrow \psi) \lor \delta}{(\circ \varphi \rightarrow \circ \psi) \lor \delta}$
Some properties of logics $L_\circ$

- Chain-completeness: the logic $L_\circ$ is strongly complete with respect to the class of $L_\circ$-chains
- Conservativeness: $L_\circ$ is a conservative expansion of $L$
- Real completeness preservation: a logic $L_\circ$ is complete over $[0, 1]$-chains for deductions from a finite (resp. arbitrary) set of premises iff it is so the logic $L$. 

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Some interesting extensions / expansions

Recall the general form of $\circ$ operators in $L$ chains:

\[ (x) \text{ remains undetermined in the interval } \mathbb{I}_{\neg} = \{x < 1 \mid \neg(x) = 0\}. \]

Next we consider some particular logics depending on $\circ$ in this interval.
1) the case $\mathbb{I}_\neg = \emptyset$: the logic $L_{\neg\neg}^{-}$

The logic $L_{\neg\neg}^{-}$ is defined as the extension of $L$ by adding the following rule:

$$
\frac{-\neg \varphi}{\varphi}
$$

Then define the logic $L_{\circ \neg\neg}^{-}$ as the expansion $L_\circ$ with the rule $(\neg \neg)$.

Observe that over chains, $\circ(x) = 1$ if $x \in \{0, 1\}$ and 0 otherwise.

Relation with Baaz-Monteiro’s $\Delta$ operator:

- $\circ(\varphi) = \Delta(\varphi \lor \neg \varphi)$ and $\Delta(\varphi) = \circ(\varphi) \land \varphi$.
- $L_{\circ \neg\neg}^{-}$ “equivalent” to $(L_\Delta)^{-}$
2) the case of crisp $\circ$ operators

<table>
<thead>
<tr>
<th>$L^c_0$</th>
<th>$L_0 + (c) \circ \varphi \lor \neg \circ \varphi$</th>
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<tbody>
<tr>
<td>$L^\text{min}_0$</td>
<td>$L_0 + (A4) \varphi \lor \neg \varphi \lor \neg \circ \varphi$</td>
</tr>
<tr>
<td>$L^\text{max}_0$</td>
<td>$L_0 + (\neg \neg \circ) \frac{\neg \circ \varphi \lor \delta}{\circ \varphi \lor \delta}$</td>
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A family of Fuzzy LFIs

Our ultimate goal is the axiomatization of the expansions of paraconsistent logics $L^\leq$ with a consistency operator $\circ$.

**Axiomatization of $L^\leq$**

It is obtained by taking the same axioms of $L_\circ$ and adding the following inference rules:

- **(Adj-\land)** from $\varphi$ and $\psi$ deduce $\varphi \land \psi$
- **(MP-r)** if $\vdash_{L_\circ} \varphi \rightarrow \psi$, then from $\varphi$ derive $\psi$
- **(Cong-r)** if $\vdash_{L_\circ} (\varphi \leftrightarrow \psi) \lor \delta$ then derive $(\circ \varphi \leftrightarrow \circ \psi) \lor \delta$
- **(Coh-r)** if $\vdash_{L_\circ} (\neg \neg \varphi \land (\varphi \rightarrow \psi)) \lor \delta$ then derive $(\circ \varphi \rightarrow \circ \psi) \lor \delta$

Similarly, when we replace $L_\circ$ by any of the above considered expansions / extensions.
A family of Fuzzy LFI\textsc{s}

<table>
<thead>
<tr>
<th>Logic</th>
<th>Inference rules</th>
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<tbody>
<tr>
<td>$L_\leq$</td>
<td>rules of $L_\leq$ + (Cong-r) $\frac{\vdash_{L_\leq} (\varphi \leftrightarrow \psi) \lor \delta}{(\varphi \leftrightarrow \varphi) \lor \delta}$ (Coh-r) $\frac{\vdash_{L_\leq} (\neg \neg \varphi \land (\varphi \rightarrow \psi)) \lor \delta}{(\varphi \rightarrow \varphi) \lor \delta}$</td>
</tr>
<tr>
<td>$(L_{\neg\neg})_\leq$</td>
<td>rules of $L_\leq$ + (\neg\neg-r) $\frac{\vdash_{L_{\neg\neg}} \neg \neg \varphi}{\varphi}$</td>
</tr>
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<td>rules of $L_\leq$ + (\neg\neg_{\text{o}}-r) $\frac{\vdash_{L^\text{max}_\leq} \neg \neg \varphi \lor \delta}{\varphi \lor \delta}$</td>
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In the context of LFIs, it is a desirable property to recover classical reasoning by means of the consistency connective $\circ$:

\[(\text{DAT}) \quad \Gamma \vdash_{\text{CPL}} \varphi \iff \circ(\Theta), \Gamma \vdash_{\text{L}} \varphi.\]

where $\Theta$, $\Gamma$ and $\varphi$ are in the language of CPL. This is known as \textit{Derivability Adjustment Theorem} (DAT).

When the operator $\circ$ \textit{suitably propagates} through connectives of a LFI logic $L$ the DAT reduces to this simplified form:

\[
(\text{PDAT}) \quad \Gamma \vdash_{\text{CPL}} \varphi \iff \\{\circ p_1, \ldots, \circ p_n\} \cup \Gamma \vdash_{\text{L}} \varphi
\]

where $\{p_1, \ldots, p_n\}$ is the set of propositional variables occurring in $\Gamma \cup \{\varphi\}$. 

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Is there a DAT for the LFI logics $L_\leq$?

Consider this (simplified form) of the translation:

$$(PDAT^*) \vdash_{\text{CPL}} \varphi \iff \{\circ p_1, \ldots, \circ p_n\} \vdash_{L_\leq} \varphi$$

$$(\text{iff} \quad \vdash_{L_\circ} \left( \bigwedge_{i=1}^{n} \circ p_i \right) \rightarrow \varphi)$$

Unfortunately, this does not hold in general:

$\vdash_{\text{CPL}} p \lor \neg p$ but, in general, $\not\vdash_{L_\leq} \circ p \rightarrow (p \lor \neg p)$

Define $L_{\circ}^{\text{dat}}$ as the extension of $L_\circ$ with the axiom $\circ \varphi \rightarrow (\varphi \lor \neg \varphi)$

A DAT property for $L_\leq$

$\Gamma \vdash_{\text{CPL}} \varphi$ iff there is some $k \geq 1$ such that $\Gamma \vdash_{L_\circ^{\text{dat}}} (\bigwedge_{i=1}^{m} \circ p_i)^k \rightarrow \varphi$

Open question: do we need $k > 1$?
Conclusions

- We have investigated the possibility of defining paraconsistent logics of formal inconsistency (LFIs) based on systems of mathematical fuzzy logic by:
  (i) expanding axiomatic extensions of the fuzzy logic MTL with the characteristic consistency operators \( \circ \) of LFIs
  (ii) considering their degree-preserving versions, that are paraconsistent.
- One could dually consider *inconsistency* operators \( \bullet = \neg \circ \)
- Together with a companion paper Ertola-Esteva-Flaminio-Godo-Noguera, these are first attempts to contribute to the study and understanding of the relationships between paraconsistency and fuzziness.