Lexicographic MV-algebras. Through a generalization of Di Nola-Lettieri functors

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Outline

Goal 1:

When an MV-algebra can be represented as a lexicographic product between an ℓu-group and an ℓ-group?

Goal 2:

What other MV-algebras are equivalent with ℓ-groups?

Goal 3:

Are there "states" on MV-algebra which save infinitesimals?

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  - G an ℓ-group
  - A an MV-algebra s.t.

$$A \simeq \Gamma(\mathbb{Z} \times_{lex} G, \langle 1, 0 \rangle)$$

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  - A a local MV-algebra with retractive radical

LEXICOGRAPHIC PRODUCT OF ℓ-GROUPS

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$H \times_{lex} G$ is an ℓ-group

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H is an o-group and G is an ℓ-group

- It is easy to see:

$(H \times_{lex} G, \langle u, 0 \rangle)$ is an ℓu-group

iff

(H, u) is an ou-group and G is an ℓ-group

OUR GOAL 1

- (H, u) an *ou*-group
- G an ℓ-group
- A an MV-algebra s.t.

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Our goal 1

- (H, u) an *ou*-group
- G an ℓ-group
- A an MV-algebra s.t.

$$A \simeq \Gamma(H \times_{lex} G, \langle u, 0 \rangle)$$

- What kind of MV-algebra is A?

Ideals of MV-algebras

We recall:

Definition.
An ideal I of an MV-algebra A is called *prime* if it is proper and

$$a \odot b^* \in I \text{ or } b \odot a^* \in I$$

for any $a, b \in A$.

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Proposition.
If A is an MV-algebra and I is a proper ideal of A, then t.f.a.e.:

1. *I is a prime ideal,*
2. *A/I is an MV-chain.*

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An ideal I of an MV-algebra A is called *retractive* if there exists a morphism $\delta_I : A/I \to A$ such that

$$\pi_I \circ \delta_I = id_{A/I}.$$

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Remark.
If I is retractive, then A/I is isomorphic with a subalgebra of A.

Ideals of MV-algebras

Definition.

An ideal I of an MV-algebra A is called *strict* if

$$\pi_I(a) < \pi_I(b) \text{ implies } a < b,$$

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- $\pi_I(a) < \pi_I(b)$ means that $a \odot b^* \in I$ and $b \odot a^* \notin I$.
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Example.

In any local MV-algebra, the radical is strict.

A FIRST REMARK

Proposition.

Let A be an MV-algebra such that

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where (H, u) is an ou-group and G is a nontrivial ℓ-group.

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There exists an ideal I of A such that the following are satisfied:

1. $I \neq \{0\}$
2. *I is strict*
3. *I is retractive*
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5. $\tau \leq a \leq \tau^*$, *for any* $\tau \in I$ *and any* $a \in A \setminus \langle I \rangle$.

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Sketch of the proof.

The desired ideal is $I = \{ \langle 0, k\rangle \mid k \in G_+\}$. □

LEXICOGRAPHIC IDEALS OF MV-ALGEBRAS

Definition.

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We denote by $LexId(A)$ the set of all lexicographic ideals of A.

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Lemma.

If I is a lexicographic ideal of an MV-algebra A, then $I \subseteq Rad(A)$.

A "NEW" CLASS OF MV-ALGEBRAS

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For any $I \in LexId(A)$, we consider

$$S_I = \delta_I(A/I).$$

S_I is an MV-subalgebra of A and it is isomorphic with A/I.

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Example.
Any perfect MV-algebra is lexicographic.

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- lexicographic ideals of $C\mathbb{Z}$:
  - $I_1 = \{mc^2 \mid m \in \mathbb{Z}_+ \cup \{0\}\}$
  - $I_2 = \{nc + mc^2 \mid n, m \in \mathbb{Z}_+ \cup \{0\}\} = Rad(C\mathbb{Z})$

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- $C\mathbb{Z}$ is a lexicographic MV-algebra

Properties of Lexicographic MV-algebras

Proposition.
The class of lexicographic MV-algebras does not form a variety.

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The class of lexicographic MV-algebras does not form a variety.

Sketch of the proof.

- Chang MV-algebra C is a lexicographic MV-algebra
- $C \times C$ is not a lexicographic MV-algebra

$\square$

Characterization Theorem

Theorem.
Let A be an MV-algebra. T.f.a.e.:

1. *A is a lexicographic MV-algebra,*
2. *there exist an ou-group (H, u) and a nontrivial ℓ-group G s.t.*

$$A \simeq \Gamma(H \times_{lex} G, \langle u, 0 \rangle).$$

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- let $I \in LexId(A)$
- take the *ou*-group $(H, u) \simeq \Gamma^{-1}(A/I)$
- take the nontrivial ℓ-group $G \simeq \Delta^{-1}(\langle I \rangle)$

Characterization Theorem

Sketch of the proof.

- let $I \in LexId(A)$
- take the *ou*-group $(H, u) \simeq \Gamma^{-1}(A/I)$
- take the nontrivial ℓ-group $G \simeq \Delta^{-1}(\langle I \rangle)$
- there exists a morphism $\delta_I : A/I \to A$ such that $\pi_I \circ \delta_I = id_{A/I}$
- for any $a \in A$, we set

$$s_a = \delta_I(\pi_I(a)) \qquad \epsilon_a = a \odot s_a^* \qquad \tau_a = a^* \odot s_a$$

- we have $s_a \in S_I$ and $\tau_a, \epsilon_a \in I$ such that

$$a = (s_a \odot \tau_a^*) \oplus \epsilon_a = (s_a \oplus \epsilon_a) \odot \tau_a^*$$

Characterization Theorem

Sketch of the proof (cont).

- we have the following isomorphisms:

$$\zeta_I : S_I \to \Gamma(H, u) \qquad \eta_I : I \to G_+$$

- We define

$$f_I : A \to \Gamma(H \times_{lex} G, \langle u, 0 \rangle)$$

$$f_I(a) = \langle \zeta_I(s_a), \eta_I(\epsilon_a) - \eta_I(\tau_a) \rangle$$

- f_I is an isomorphism of MV-algebras

$\square$

Consequences

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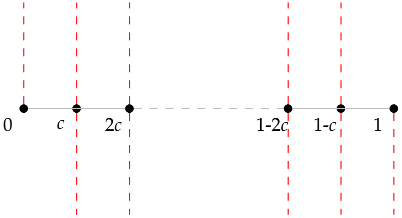
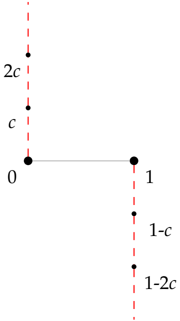
Example.

We obtain two different representations:



Representation from I_1



Representation from I_2

Consequences

Corollary.

Let A be a lexicographic MV-algebra. T.f.a.e.:

1. *there exist a subgroup of the reals $(\mathbb{R}', 1)$ and a nontrivial ℓ-group G such that*

$$A \simeq \Gamma(\mathbb{R}' \times_{lex} G, \langle 1, 0 \rangle),$$

2. $Rad(A) \in LexId(A)$.

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Remark.

- Di Nola, Lettieri (1996)
  - $\mathbb{R}'$ a subgroup of $\mathbb{R}$ and G an ℓ-group such that

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  - A a local MV-algebra with retractive radical

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- $Rad(A) \notin LexId(A)$

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Let A be a lexicographic MV-algebra. We have two cases:

- $Rad(A)$ is retractive
  - local MV-algebras with retractive ideals
  - Di Nola, Lettieri (1996)
- $Rad(A)$ is not retractive?

Outline

Goal 2:

What other MV-algebras are equivalent with ℓ-groups?

Di Nola-Lettieri functor Δ

- the category of ℓ-groups $\mathcal{AG}$
- the category of perfect MV-algebras $\mathcal{MV}_{perf}$
- Di Nola, Lettieri (1994) defined the functor

$$\Delta\colon \mathcal{AG} \to \mathcal{MV}_{perf}$$

  - $\Delta(G) = \Gamma(\mathbb{Z} \times_{lex} G, \langle 1, 0\rangle)$
  - $\Delta(g) = \langle id_{\mathbb{Z}}, g\rangle$

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Theorem. (Di Nola, Lettieri)
The categories $\mathcal{AG}$ and $\mathcal{MV}_{perf}$ are equivalent.

A generalization of Di Nola-Lettieri functor

- we fix an MV-chain L

A generalization of Di Nola-Lettieri functor

- we fix an MV-chain L
- we define the category $\mathcal{L}ex\mathcal{MV}_L$
  - *objects:*
    lexicographic MV-algebras A for which there exists $I \in LexId(A)$ s.t.
    $$A/I \simeq L$$
  - *arrows:*
    MV-algebras morphisms

A GENERALIZATION OF DI NOLA-LETTIERI FUNCTOR

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A generalization of Di Nola-Lettieri functor

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$$\Delta_L : \mathcal{AG} \to \mathcal{LexMV}_L$$

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Theorem.
For any MV-chain L, the categories $\mathcal{AG}$ and $\mathcal{LexMV}_L$ are equivalent.

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Theorem.
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Remark.
Di Nola-Lettieri functor Δ is exactly the functor Δ_L when $L = L_2$.

Outline

Goal 3:

Are there "states" on MV-algebra which save infinitesimals?

States on non-semisimple MV-algebras

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- states on MV-algebras do not preserve positive infinitesimals
- each perfect MV-algebra admits only one trivial state
- Chang MV-algebra C has only one state s which is trivial:

$$s(x) = \begin{cases} 0, & \text{if } x \in Rad(C) \\ 1, & \text{if } x \in Rad(C)^* \end{cases}$$

The codomain for lexicographic states

Example.

- Consider an infinitesimal $\varepsilon \in {}^*[0,1]$ such that $\varepsilon \neq 0$

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- $Rad(\mathscr{L}(\mathbb{R})) = \{r\varepsilon \mid r \in \mathbb{R}_+\}$
- $Rad(\mathscr{L}(\mathbb{R}))$ is the unique lexicographic ideal of $\mathscr{L}(\mathbb{R})$

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- $Rad(\mathscr{L}(\mathbb{R}))$ is the unique lexicographic ideal of $\mathscr{L}(\mathbb{R})$
- We obtain that
$$\mathscr{L}(\mathbb{R}) \simeq \Gamma(\mathbb{R} \times_{lex} \mathbb{R}, \langle 1, 0 \rangle)$$

I-LEXICOGRAPHIC STATES

Definition.
Let A be a lexicographic MV-algebra and let $I \in LexId(A)$.
An *I-lexicographic state* of A is a map

$$\mathbf{s}_I : A \to \mathscr{L}(\mathbb{R})$$

($\mathbf{s}_I$1) $\mathbf{s}_I(1) = 1$

($\mathbf{s}_I$2) for all $a, b \in A$ s.t. $a \odot b = 0$, $\mathbf{s}_I(a \oplus b) = \mathbf{s}_I(a) + \mathbf{s}_I(b)$

($\mathbf{s}_I$3) $\mathbf{s}_I \restriction_{S_I}$ is a state of S_I.

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($\mathbf{s}_I$3) $\mathbf{s}_I \restriction_{S_I}$ is a state of S_I.

A I-lexicographic state $\mathbf{s}_I$ is *faithful* if $\mathbf{s}_I(a) = 0$ implies $a = 0$.

Lexicographic states

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Let A be a lexicographic MV-algebra. A map

$$\mathbf{s} : A \to \mathscr{L}(\mathbb{R})$$

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for every $I \in LexId(A)$.

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We denote by $Lex\mathscr{S}(A)$ the set of all lexicographic states of A.

Characterization Theorem

Theorem.
Let A be a lexicographic MV-algebra and $\mathbf{s} : A \to \mathscr{L}(\mathbb{R})$ be a map. T.f.a.e.:

1. $\mathbf{s}$ *is a lexicographic state of A,*
2. *for every ou-group (H, u) and every nontrivial ℓ-group G s.t.*

$$A \simeq \Gamma(H \times_{lex} G, \langle u, 0 \rangle),$$

there exist an unique ℓu-state $h : (H, u) \to (\mathbb{R}, 1)$ and an unique ℓ-state $\gamma : G \to \mathbb{R}$ s.t.

$$\mathbf{s}(\langle x, y \rangle) = h(x) + \gamma(y)\varepsilon,$$

for every $\langle x, y \rangle \in A$.

Consequences

Proposition.

Let A be a countable lexicographic MV-algebra.
If $Rad(A) \in LexId(A)$, then t.f.a.e.:

1. *A admits a faithful $Rad(A)$-lexicographic state,*
2. *$\langle Rad(A) \rangle$ has a faithful ℓ-state.*

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Let A be a countable lexicographic MV-algebra.
If $Rad(A) \in LexId(A)$, then t.f.a.e.:

1. *A admits a faithful $Rad(A)$-lexicographic state*,
2. *$\langle Rad(A) \rangle$ has a faithful ℓ-state*.

Corollary.
Chang MV-algebra C admits a faithful lexicographic state.

Thank you!