# Strongly semisimple MV-algebras and tangents

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 $L_{\infty}$  logic. (Łukasiewicz, Tarski - 1930)

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### Syntactic vs Semantic Consequence

 $L_{\infty}$  logic. (Łukasiewicz, Tarski - 1930)

### Semantics

A valuation  $v \colon \mathcal{FM} \to [0,1]$  (where  $\mathcal{FM}$  is the set of formulas on the language  $\{\to,\neg\}$ ) is a map satisfying:

- $V(\alpha \to \beta) = \min\{(1 V(\alpha)) + V(\beta), 1\}$
- $\mathbf{v}(\neg \alpha) = \mathbf{1} \mathbf{v}(\alpha)$

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### Semantics

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- $V(\alpha \to \beta) = \min\{(1 V(\alpha)) + V(\beta), 1\}$

### Calculus

Axioms	Rules
$\alpha  o (eta  o lpha)$	$\alpha, \ \alpha \to \beta \vdash \beta$
$(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$	
$((\alpha \to \beta) \to \beta) \to ((\beta \to \alpha) \to \alpha)$	
$(\neg \alpha \to \neg \beta) \to (\beta \to \alpha)$	

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Syntactic Consequence

 $\Theta \vdash_{\not k_{\infty}} \varphi$  iff there exists an  $k_{\infty}$ -proof of  $\varphi$  from  $\Theta$ .

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Syntactic Consequence

 $\Theta \vdash_{\mathbf{L}_{\infty}} \varphi$  iff there exists an  $\mathbf{L}_{\infty}$ -proof of  $\varphi$  from  $\Theta$ .

Semantic Consequence  $\Theta \models_{\mbox{$\underline{L}$}_{\infty}} \varphi \mbox{ iff for each valuation } v\colon \mathcal{FM} \to [0,1]$   $v(\Theta) = \{1\} \mbox{ implies } v(\varphi) = 1.$ 

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### Soundness:

If 
$$\Theta \vdash_{\mathbf{L}_{\infty}} \varphi$$
, then  $\Theta \models_{\mathbf{L}_{\infty}} \varphi$ .

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### Soundness:

If 
$$\Theta \vdash_{\mathbf{L}_{\infty}} \varphi$$
, then  $\Theta \models_{\mathbf{L}_{\infty}} \varphi$ .

Finite Completeness (Hay-Wójcicki):

If 
$$|\Theta| < \aleph_0$$
 and  $\Theta \models_{\begin{subarray}{c} \begin{subarray}{c} \begin{subarray}{c}$ 

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What if  $\Theta$  is not finite?

### **Theorem**

Given a set of formulas  $\Theta$ , the following are equivalent:

► For each formula  $\varphi$ ,  $\Theta \vdash_{\underline{\ell}_{\infty}} \varphi$  iff  $\Theta \models_{\underline{\ell}_{\infty}} \varphi$ .

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- ► For each formula  $\varphi$ ,  $\Theta \vdash_{\underline{\ell}_{\infty}} \varphi$  iff  $\Theta \models_{\underline{\ell}_{\infty}} \varphi$ .
- The MV-algebra presented by (Var(Θ), Θ) is semisimple (that is, its radical is {0}).

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Syntactic vs Semantic Consequence

# What if $\Theta$ is not finite?

### Theorem

Given a set of formulas  $\Theta$ , the following are equivalent:

- ▶ For each formula  $\varphi$ ,  $\Theta \vdash_{\ell_m} \varphi$  iff  $\Theta \models_{\ell_m} \varphi$ .
- ▶ The MV-algebra presented by  $(Var(\Theta), \Theta)$  is semisimple (that is, its radical is {0}).
- ▶ The MV-algebra presented by  $(Var(\Theta), \Theta)$  belongs to  $\mathbb{ISP}([0,1]_{\mathcal{MV}}).$

Syntactic vs Semantic Consequence

(Hay-Wójcicki):

If 
$$|\Theta| < \aleph_0$$
, then  $\Theta \models_{\underline{L}_{\infty}} \varphi$  iff  $\Theta \vdash_{\underline{L}_{\infty}} \varphi$ .

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(Hay-Wójcicki):

If 
$$|\Theta| < \aleph_0$$
, then  $\Theta \models_{\mbox{$\rlap{L}_{\infty}$}} \varphi$  iff  $\Theta \vdash_{\mbox{$\rlap{L}_{\infty}$}} \varphi$ .

If  $\Theta$  is a finite set of formulas, for each formula  $\alpha$ :

$$\Theta \cup \{\alpha\} \models_{\mathbf{L}_{\infty}} \varphi \text{ iff } \Theta \cup \{\alpha\} \vdash_{\mathbf{L}_{\infty}} \varphi.$$

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▶ The MV-algebra presented by  $(Var(\Theta), \Theta)$  is strongly semisimple.

### What if $\Theta$ is not finite?

### Theorem

For each  $\Theta$  set of formulas, the following are equivalent:

• For every  $\alpha, \varphi$ ,

$$\Theta \cup \{\alpha\} \vdash_{\boldsymbol{\ell}_{\infty}} \varphi \text{ iff } \Theta \cup \{\alpha\} \models_{\boldsymbol{\ell}_{\infty}} \varphi.$$

▶ The MV-algebra presented by  $(Var(\Theta), \Theta)$  is strongly semisimple.

An MV-algebra A is **strongly semisimple** if for every finitely generated ideal (filter) I, the MV-algebra A/I is semisimple.

# Main Goal

To present a geometric description of finitely generated strongly semisimple MV-algebras.

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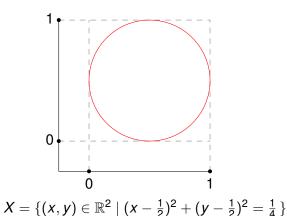
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To present a geometric description of finitely generated strongly semisimple MV-algebras.

More precisely, for each *n*-generated semisimple MV-algebra A there exists X a closed subset of  $[0, 1]^n$ , such that A is isomorphic to

 $\mathcal{M}(X) = \{f \mid_X | f : [0,1]^n \to [0,1] \text{ is a McNaughton map} \}.$ 

We will present necessary and sufficient conditions on the closed set  $X \subseteq [0,1]^n$  for  $A \cong \mathcal{M}(X)$  to be strongly semisimple.



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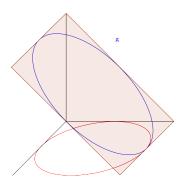
#### Key Remarks

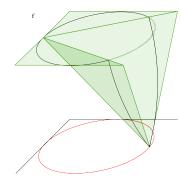
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Theorem (Busaniche, Mundici)

Let  $X \subseteq [0,1]^2$  be a closed set. The MV-algebra  $\mathcal{M}(X)$  is not strongly semisimple

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# Theorem (Busaniche, Mundici)

Let  $X \subseteq [0,1]^2$  be a closed set. The MV-algebra  $\mathcal{M}(X)$  is not strongly semisimple iff there exist a point  $x \in X$ , a sequence  $x_0, x_1, \ldots \in X$ , a unit vector  $u \in \mathbb{R}^2$ , and a real number  $\lambda > 0$ 

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- (i)  $x_i \neq x$  for all i,
- (ii)  $\lim_{i\to\infty} x_i = x$ ,

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Other Results

Theorem (Busaniche, Mundici)

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- (iv)  $\operatorname{conv}(x, x + \lambda u) \cap X = \{x\},\$

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- (i)  $x_i \neq x$  for all i,
- (ii)  $\lim_{i\to\infty} x_i = x$ ,
- (iii)  $\lim_{i\to\infty} (x_i-x)/||x_i-x||=u$ ,
- (iv)  $\operatorname{conv}(x, x + \lambda u) \cap X = \{x\}$ , and
- (v) the coordinates of x and  $x + \lambda u$  are rational.

$$X = \{(x, y) \in \mathbb{R}^2 \mid (x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 = \frac{1}{4} \}$$

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semisimple MV-algebras

Let  $\emptyset \neq X \subseteq \mathbb{R}^n$  and  $x \in \mathbb{R}^n$ .

A Bouligand-Severi tangent of X at x is a unit vector  $u \in \mathbb{R}^n$  such that X contains a sequence  $x_1, x_2, \ldots$  with the following properties:

- (i)  $x_i \neq x$  for all i;
- (ii)  $\lim_{i\to\infty} x_i = x$ ; and
- (iii)  $\lim_{i\to\infty} (x_i x)/||x_i x|| = u$ .

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Other Results

Definition (Bouligand, Severi)

Let  $\emptyset \neq X \subseteq \mathbb{R}^n$  and  $x \in \mathbb{R}^n$ .

A Bouligand-Severi tangent of X at x is a unit vector  $u \in \mathbb{R}^n$  such that X contains a sequence  $x_1, x_2, \ldots$  with the following properties:

- (i)  $x_i \neq x$  for all i;
- (ii)  $\lim_{i\to\infty} x_i = x$ ; and
- (iii)  $\lim_{i\to\infty} (x_i-x)/||x_i-x||=u$ .

The tangent u is said to be **outgoing** if there exists  $\lambda > 0$  such that

$$\operatorname{conv}(x, x + \lambda u) \cap X = \{x\}.$$

# Finitely generated case

**Key Remarks** 

Importance of the 2-generated case:

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# Finitely generated case

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Importance of the 2-generated case:

An MV-algebra *A* is strongly semisimple iff every 2-generated subalgebra of *A* is strongly semisimple.

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An MV-algebra *A* is strongly semisimple iff every 2-generated subalgebra of *A* is strongly semisimple.

We need

- n-dimensional generalisation of Bouligand-Severi tangents.
- the right definition of "rational" outgoingness.

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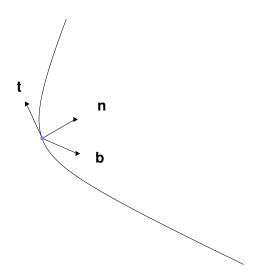
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Let  $\gamma \colon [a,b] \to \mathbb{R}^n$  be a  $\mathbb{C}^k$   $(k \le n)$  function such that for all a < t < b, the k-tuple of vectors

$$(\gamma'(t), \gamma''(t), \ldots, \gamma^{(k)}(t))$$

forms a linear independent set in  $\mathbb{R}^n$ .

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$$(\gamma'(t), \gamma''(t), \ldots, \gamma^{(k)}(t))$$

forms a linear independent set in  $\mathbb{R}^n$ . The Gram-Schmidt orthonormalization process yields an orthonormal k-tuple

$$(v_1(t),\ldots,v_k(t)),$$

called the **Frenet** k-frame of  $\gamma$  at  $\gamma(t)$ .

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#### **Definition**

A k-tuple  $u = (u_1, \ldots, u_k)$  of pairwise orthogonal unit vectors in  $\mathbb{R}^n$  is said to be a k-tangent of X at X if: there is a sequence of points  $x_1, x_2, \ldots \in X$  such that

 $\blacktriangleright \lim_{i\to\infty} x_i = x;$ 

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1 
$$\lim_{i\to\infty} (x_i-x)/||x_i-x||=u_1$$
,

$$2 \lim_{i \to \infty} \frac{x_i - x - p_{\mathbb{R}u_1}(x_i - x)}{||x_i - x - p_{\mathbb{R}u_1}(x_i - x)||} = u_2,$$

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$$k \lim_{i \to \infty} \frac{x_i - x - p_{\mathbb{R}u_1 + \dots + \mathbb{R}u_{k-1}}(x_i - x)}{||x_i - x - p_{\mathbb{R}u_1 + \dots + \mathbb{R}u_{k-1}}(x_i - x)||} = u_k.$$

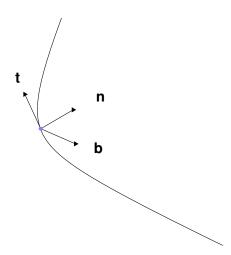
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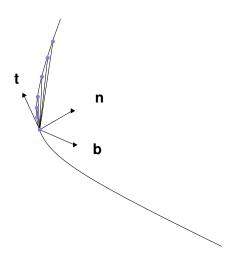
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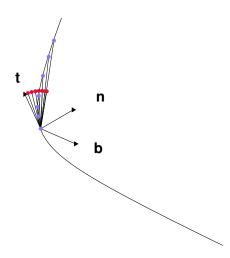
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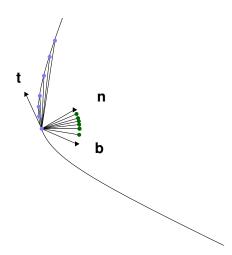
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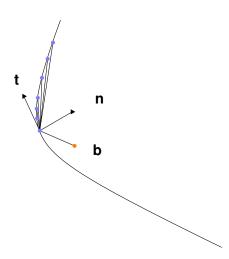
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### Theorem (LMC, Mundici)

Suppose  $\gamma \colon [a,b] \to \mathbb{R}^n$  is a  $\mathbb{C}^{k+1}$  function and  $a < t_0 < b$  is such that  $\gamma'(t_0), \gamma''(t_0), \ldots, \gamma^{(k)}(t_0)$  are linearly independent and let  $v = (v_1, \ldots, v_k)$  be its Frenet k-frame at  $\gamma(t_0)$ .

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## Theorem (LMC, Mundici)

Suppose  $\gamma \colon [a,b] \to \mathbb{R}^n$  is a  $\mathbb{C}^{k+1}$  function and  $a < t_0 < b$  is such that  $\gamma'(t_0), \gamma''(t_0), \ldots, \gamma^{(k)}(t_0)$  are linearly independent and let  $v = (v_1, \ldots, v_k)$  be its Frenet k-frame at  $\gamma(t_0)$ .

Then the set  $\gamma([t_0 - \epsilon, t_0 + \epsilon]) \subseteq \mathbb{R}^n$  has exactly two k-tangents at  $\gamma(t_0)$ ,

$$v \text{ and } (-v_1, v_2, -v_3, \ldots).$$

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#### Theorem (Busaniche, Mundici)

Let  $X \subseteq [0,1]^2$  be a closed set. The MV-algebra  $\mathcal{M}(X)$  is not strongly semisimple iff there exist a point  $x \in X$ , and a unit vector  $u \in \mathbb{R}^2$  such that

- (i) u is a Bouligand-Severi tangent of X at x; and
- (ii) there exists a real number  $\lambda > 0$

$$\operatorname{conv}(x, x + \lambda u) \cap X = \{x\}$$

and the coordinates of x and  $x + \lambda u$  are rational.

A tangent *u* outgoing if there is a and a rational simplex *S*, such that

$$conv(x, x + \lambda u)$$

of 
$$X \subseteq \mathbb{R}^n$$
 at  $x$  is rationally  $\lambda \in \mathbb{R}_{>0}$  ,

$$)=S$$

and

$$\{x\} = S \cap X$$
.

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A k-tangent  $u=(u_1,\ldots,u_k)$  of  $X\subseteq\mathbb{R}^n$  at x is rationally outgoing if there is a  $\lambda\in\mathbb{R}_{>0}$ , and a rational simplex S, such that

$$conv(x, x + \lambda u)$$

$$)=\mathcal{S}$$

and

$$\{x\} = S \cap X.$$

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A k-tangent  $u = (u_1, \dots, u_k)$  of  $X \subseteq \mathbb{R}^n$  at x is rationally **outgoing** if there is a k-tuple  $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{R}_{>0}^k$ , and a rational simplex S, such that

$$conv(x, x + \lambda u)$$

)=S

and

$$\{x\} = S \cap X$$
.

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Rationally outgoing tangent

A k-tangent  $u=(u_1,\ldots,u_k)$  of  $X\subseteq\mathbb{R}^n$  at x is rationally outgoing if there is a k-tuple  $\lambda=(\lambda_1,\ldots,\lambda_k)\in\mathbb{R}_{>0}^k$ , and a rational simplex S, such that

$$\operatorname{conv}(x, x + \lambda_1 u_1, \dots, x + \lambda_1 u_1 + \dots + \lambda_k u_k) = S$$

and

$$\{x\} = S \cap X.$$

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A k-tangent  $u = (u_1, \dots, u_k)$  of  $X \subseteq \mathbb{R}^n$  at x is rationally **outgoing** if there is a *k*-tuple  $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{R}_{\geq 0}^k$ . and a rational simplex S, such that

$$\operatorname{conv}(x, x + \lambda_1 u_1, \dots, x + \lambda_1 u_1 + \dots + \lambda_k u_k) \subseteq S$$

and

$$\{x\} = S \cap X.$$

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Rationally outgoing tangent

A k-tangent  $u=(u_1,\ldots,u_k)$  of  $X\subseteq\mathbb{R}^n$  at x is rationally outgoing if there is a k-tuple  $\lambda=(\lambda_1,\ldots,\lambda_k)\in\mathbb{R}_{>0}{}^k$ , and a rational simplex S, together with a face  $F\subseteq S$  such that

$$\operatorname{conv}(x, x + \lambda_1 u_1, \dots, x + \lambda_1 u_1 + \dots + \lambda_k u_k) \subseteq S$$

and

$$\{x\} = S \cap X.$$

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A k-tangent  $u=(u_1,\ldots,u_k)$  of  $X\subseteq\mathbb{R}^n$  at x is rationally outgoing if there is a k-tuple  $\lambda=(\lambda_1,\ldots,\lambda_k)\in\mathbb{R}_{>0}{}^k$ , and a rational simplex S, together with a face  $F\subseteq S$  such that

$$\operatorname{conv}(x, x + \lambda_1 u_1, \dots, x + \lambda_1 u_1 + \dots + \lambda_k u_k) \subseteq S,$$

$$\operatorname{conv}(x, x + \lambda_1 u_1, \dots, x + \lambda_1 u_1 + \dots + \lambda_k u_k) \not\subseteq F$$

and

$$\{x\} = S \cap X$$
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A k-tangent  $u=(u_1,\ldots,u_k)$  of  $X\subseteq\mathbb{R}^n$  at x is rationally outgoing if there is a k-tuple  $\lambda=(\lambda_1,\ldots,\lambda_k)\in\mathbb{R}_{>0}{}^k$ , and a rational simplex S, together with a face  $F\subseteq S$  such that

$$\operatorname{conv}(x,x+\lambda_1u_1,\ldots,x+\lambda_1u_1+\cdots+\lambda_ku_k)\subseteq\mathcal{S},$$

$$\operatorname{conv}(x, x + \lambda_1 u_1, \dots, x + \lambda_1 u_1 + \dots + \lambda_k u_k) \not\subseteq F$$

and

$$F \cap X = S \cap X$$
.

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For any closed  $X \subseteq [0,1]^n$  the following conditions are equivalent:

- (i) The MV-algebra  $\mathcal{M}(X)$  is strongly semisimple.
- (ii) For no k = 1, ..., n 1, X has a rationally outgoing k-tangent.

Let u be a rationally outgoing k-tangent of X at x and let S be a rational k-simplex together with a proper face  $F \subseteq S$  and reals  $\lambda_1, \ldots, \lambda_k \in \mathbb{R}^k_{>0}$  such that

(a) 
$$\operatorname{conv}(x, x + \lambda_1 u_1, \dots, x + \lambda_1 u_1 + \dots + \lambda_k u_k) \subseteq S$$
,

(b) 
$$\operatorname{conv}(x, x + \lambda_1 u_1, \dots, x + \lambda_1 u_1 + \dots + \lambda_k u_k) \not\subseteq F$$
, and

(c) 
$$S \cap X = F \cap X$$
.

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- (a)  $\operatorname{conv}(x, x + \lambda_1 u_1, \dots, x + \lambda_1 u_1 + \dots + \lambda_k u_k) \subseteq S$ ,
- (b)  $\operatorname{conv}(x, x + \lambda_1 u_1, \dots, x + \lambda_1 u_1 + \dots + \lambda_k u_k) \not\subseteq F$ , and
- (c)  $S \cap X = F \cap X$ .

Let  $f, g \in \mathcal{M}([0,1]^n)$  be maps such that

$$f(v) = 0$$
 iff  $v \in F$  and  $g(v) = 0$  iff  $v \in S$ .

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(c) proves that  $f \upharpoonright_X$  belongs to a maximal ideal of  $\mathcal{M}(X)$  iff  $g \upharpoonright_X$  does.

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Main Result

Sketch of the proof

(c) proves that  $f \upharpoonright_X$  belongs to a maximal ideal of  $\mathcal{M}(X)$  iff  $g \upharpoonright_X$  does.

(a) and (b) imply that  $f \upharpoonright_X$  does not belong to the ideal generated by  $g \upharpoonright_X$ .

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## Main Result

(i)⇒(ii)

#### Lemma

Let  $P \subseteq [0,1]^n$  be a polyhedron such that  $X \subseteq P$ .

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Let  $P \subseteq [0,1]^n$  be a polyhedron such that  $X \subseteq P$ . Since  $(u_1, \ldots, u_l)$  is a k-tangent of X at x,

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Let  $P \subseteq [0,1]^n$  be a polyhedron such that  $X \subseteq P$ . Since  $(u_1,\ldots,u_l)$  is a k-tangent of X at x, there exist  $\delta_1,\ldots,\delta_k \in \mathbb{R}_{>0}$  with

$$\operatorname{conv}(x, x + \delta_1 u_1, \dots, x + \delta_1 u_1 + \dots + \delta_k u_k) \subseteq P.$$

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Let  $f,g\in\mathcal{M}([0,1]^n)$  be such that  $f\upharpoonright_X$  does not belong to the ideal generated by  $g\upharpoonright_X$  and that  $f\upharpoonright_X$  belongs to a maximal ideal of  $\mathcal{M}(X)$  iff  $g\upharpoonright_X$  does.

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Let  $f, g \in \mathcal{M}([0,1]^n)$  be such that  $f \upharpoonright_X$  does not belong to the ideal generated by  $g \upharpoonright_X$  and that  $f \upharpoonright_X$  belongs to a maximal ideal of  $\mathcal{M}(X)$  iff  $g \upharpoonright_X$  does.

Let the map  $\eta: X \to [0,1]^2$  be defined by

$$\eta = (f \upharpoonright_X, g \upharpoonright_X).$$

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Let the map  $\eta \colon X \to [0,1]^2$  be defined by

$$\eta = (f \upharpoonright_X, g \upharpoonright_X).$$

Then  $\mathcal{M}(\eta(X))$  is isomorphic to the subalgebra of  $\mathcal{M}(X)$  generated by  $g \upharpoonright_X$  and  $f \upharpoonright_X$ .

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Let the map  $\eta \colon X \to [0,1]^2$  be defined by

$$\eta = (f \upharpoonright_X, g \upharpoonright_X).$$

Then  $\mathcal{M}(\eta(X))$  is isomorphic to the subalgebra of  $\mathcal{M}(X)$  generated by  $g \upharpoonright_X$  and  $f \upharpoonright_X$ . Then  $\eta(X)$  has a rationally outgoing 1-tangent.

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If  $\rho \colon [0,1]^n \to [0,1]^2$  is a map defined by

$$\rho(v) = (f(v), g(v))$$

for 
$$f,g\in\mathcal{M}([0,1]^n)$$

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If  $\rho \colon [0,1]^n \to [0,1]^2$  is a map defined by

$$\rho(v) = (f(v), g(v))$$

for  $f, g \in \mathcal{M}([0,1]^n)$  and  $\rho(X)$  has a rationally outgoing 1-tangent,

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If  $\rho \colon [0,1]^n \to [0,1]^2$  is a map defined by

$$\rho(v) = (f(v), g(v))$$

for  $f, g \in \mathcal{M}([0,1]^n)$  and  $\rho(X)$  has a rationally outgoing 1-tangent, then for some  $k \in \{1, \ldots, n-1\}$ , X has a rationally outgoing k-tangent.

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- Byproduct: Extension of the definition of Frenet frames to closed sets.
- Geometric description of prime filters of a finitely generated semisimple MV-algebra.

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- Byproduct: Extension of the definition of Frenet frames to closed sets.
- Geometric description of prime filters of a finitely generated semisimple MV-algebra.
- Strongly semisimple Riesz spaces.

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Thank you for your attention!

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