Strongly semisimple MV-algebras and tangents

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Introduction

Syntactic vs Semantic Consequence

$Ł_\infty$ logic. (Łukasiewicz,Tarski - 1930)

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Semantics

A valuation $v\colon \mathcal{FM} \to [0,1]$ (where $\mathcal{FM}$ is the set of formulas on the language $\{\to, \neg\}$) is a map satisfying:

- $v(\alpha \to \beta) = \min\{(1 - v(\alpha)) + v(\beta), 1\}$
- $v(\neg\alpha) = 1 - v(\alpha)$

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Calculus

Axioms	Rules
$\alpha \to (\beta \to \alpha)$ $(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$ $((\alpha \to \beta) \to \beta) \to ((\beta \to \alpha) \to \alpha)$ $(\neg\alpha \to \neg\beta) \to (\beta \to \alpha)$	$\alpha,\ \alpha \to \beta \ \vdash\ \beta$

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Syntactic Consequence

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Semantic Consequence

$\Theta \models_{\text{Ł}_\infty} \varphi$ iff for each valuation $v \colon \mathcal{FM} \to [0,1]$

$v(\Theta) = \{1\}$ implies $v(\varphi) = 1$.

Soundness:

$$\text{If } \Theta \vdash_{Ł_\infty} \varphi, \text{ then } \Theta \models_{Ł_\infty} \varphi.$$

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Soundness:

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Finite Completeness (Hay-Wójcicki):

$$\text{If } |\Theta| < \aleph_0 \text{ and } \Theta \models_{\text{Ł}_\infty} \varphi, \text{ then } \Theta \vdash_{\text{Ł}_\infty} \varphi.$$

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Syntactic vs Semantic Consequence

What if Θ is not finite?

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Theorem

Given a set of formulas Θ, the following are equivalent:

- *For each formula φ, $\Theta \vdash_{Ł_\infty} \varphi$ iff $\Theta \models_{Ł_\infty} \varphi$.*

What if Θ is not finite?

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Given a set of formulas Θ, the following are equivalent:

- *For each formula φ, $\Theta \vdash_{Ł_\infty} \varphi$ iff $\Theta \models_{Ł_\infty} \varphi$.*
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- *The MV-algebra presented by $(\mathrm{Var}(\Theta), \Theta)$ belongs to $\mathbb{ISP}([0,1]_{\mathcal{MV}})$.*

Introduction

Syntactic vs Semantic Consequence

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If Θ is a finite set of formulas, for each formula α:

$$\Theta \cup \{\alpha\} \models_{Ł_\infty} \varphi \text{ iff } \Theta \cup \{\alpha\} \vdash_{Ł_\infty} \varphi.$$

What if Θ is not finite?

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An MV-algebra A is **strongly semisimple** if for every finitely generated ideal (filter) I, the MV-algebra A/I is semisimple.

Main Goal

To present a geometric description of finitely generated strongly semisimple MV-algebras.

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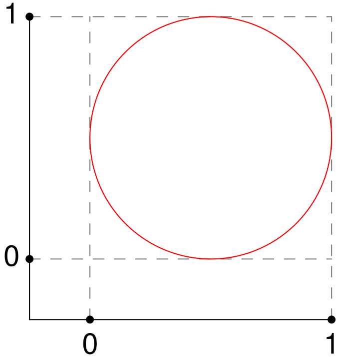
More precisely, for each n-generated semisimple MV-algebra A there exists X a closed subset of $[0,1]^n$, such that A is isomorphic to

$$\mathcal{M}(X) = \{f\upharpoonright_X \mid f\colon [0,1]^n \to [0,1] \text{ is a McNaughton map}\}.$$

We will present necessary and sufficient conditions on the closed set $X \subseteq [0,1]^n$ for $A \cong \mathcal{M}(X)$ to be strongly semisimple.

2-generated case

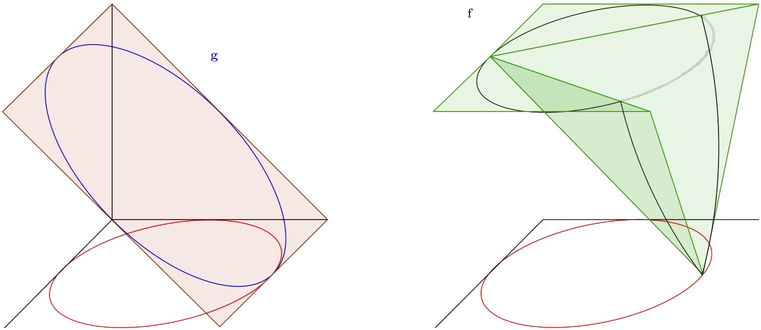
Syntactic vs Semantic Consequence



$$X = \{(x, y) \in \mathbb{R}^2 \mid (x - \tfrac{1}{2})^2 + (y - \tfrac{1}{2})^2 = \tfrac{1}{4}\}$$

2-generated case

Syntactic vs Semantic Consequence



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Theorem (Busaniche, Mundici)

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(i) $x_i \neq x$ *for all* i,

(ii) $\lim_{i\to\infty} x_i = x$,

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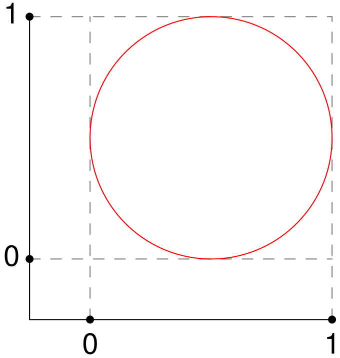
(iv) $\mathrm{conv}(x, x+\lambda u) \cap X = \{x\}$, *and*

(v) *the coordinates of* x *and* $x + \lambda u$ *are rational.*

2-generated case

Syntactic vs Semantic Consequence



$$X = \{(x, y) \in \mathbb{R}^2 \mid (x - \tfrac{1}{2})^2 + (y - \tfrac{1}{2})^2 = \tfrac{1}{4}\}$$

Definition (Bouligand, Severi)

Let $\emptyset \neq X \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}^n$.
A **Bouligand-Severi tangent of X at x** is a unit vector $u \in \mathbb{R}^n$ such that X contains a sequence $x_1, x_2, \ldots$ with the following properties:

(i) $x_i \neq x$ for all i;

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The tangent u is said to be **outgoing** if there exists $\lambda > 0$ such that

$$\mathrm{conv}(x, x + \lambda u) \cap X = \{x\}.$$

Finitely generated case

Key Remarks

- Importance of the 2-generated case:

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  An MV-algebra A is strongly semisimple iff every 2-generated subalgebra of A is strongly semisimple.

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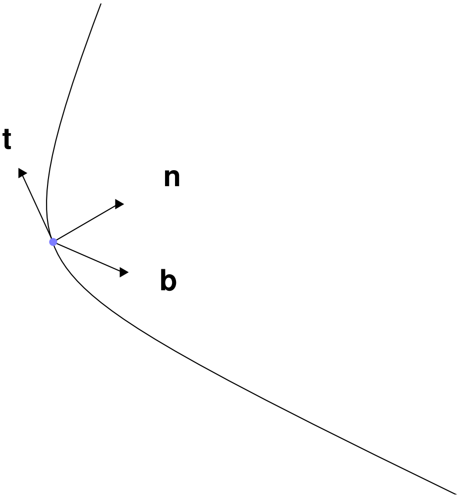
We need

- n-dimensional generalisation of Bouligand-Severi tangents.
- the right definition of "rational" outgoingness.

Finitely generated case

Bouligand-Severi tangents of higher dimension



Finitely generated case

Bouligand-Severi tangents of higher dimension

Let $\gamma\colon [a, b] \to \mathbb{R}^n$ be a C^k ($k \leq n$) function such that for all $a < t < b$, the k-tuple of vectors

$$\big(\,\gamma'(t), \gamma''(t), \ldots, \gamma^{(k)}(t)\,\big)$$

forms a linear independent set in $\mathbb{R}^n$.

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forms a linear independent set in $\mathbb{R}^n$.
The Gram-Schmidt orthonormalization process yields an orthonormal k-tuple

$$\left(v_1(t), \ldots, v_k(t)\right),$$

called the **Frenet k-frame** of γ at $\gamma(t)$.

Finitely generated case

Bouligand-Severi tangents of higher dimension

Definition

A k-tuple $u = (u_1, \ldots, u_k)$ of pairwise orthogonal unit vectors in $\mathbb{R}^n$ is said to be a ***k*-tangent of *X* at *x*** if: there is a sequence of points $x_1, x_2, \ldots \in X$ such that

- $\lim_{i \to \infty} x_i = x$;

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  2. $\lim_{i\to\infty} \dfrac{x_i - x - p_{\mathbb{R}u_1}(x_i - x)}{||x_i - x - p_{\mathbb{R}u_1}(x_i - x)||} = u_2$,

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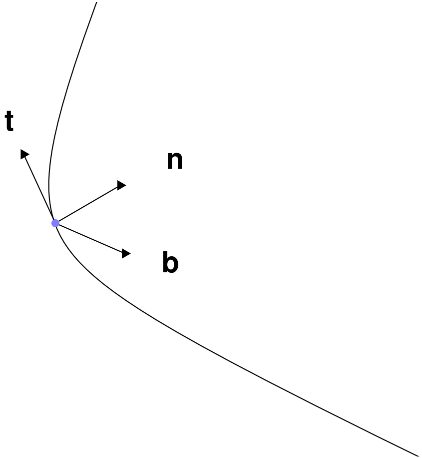

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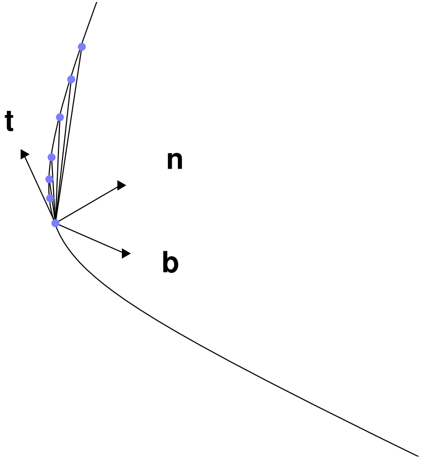

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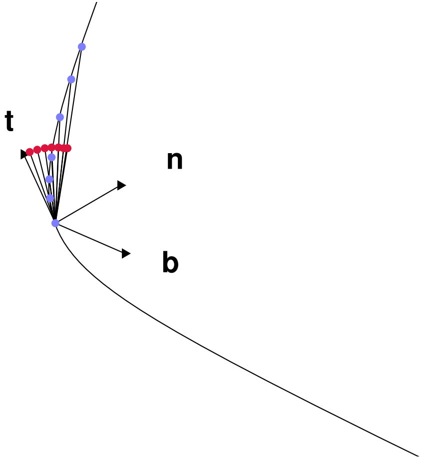

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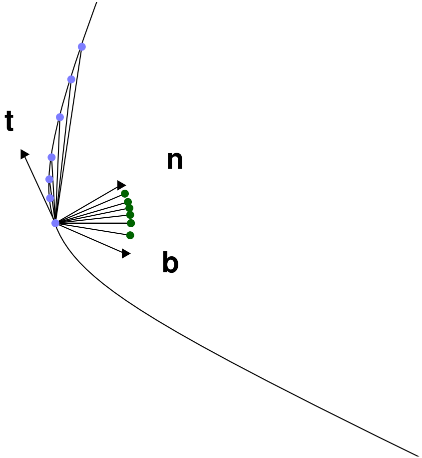

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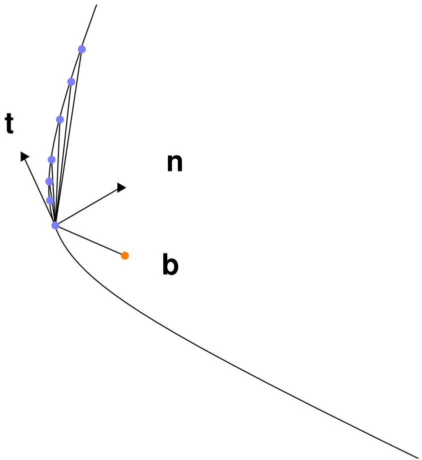

Finitely generated case

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Theorem (LMC, Mundici)

Suppose $\gamma \colon [a, b] \to \mathbb{R}^n$ *is a* C^{k+1} *function and* $a < t_0 < b$ *is such that* $\gamma'(t_0), \gamma''(t_0), \ldots, \gamma^{(k)}(t_0)$ *are linearly independent and let* $v = (v_1, \ldots, v_k)$ *be its Frenet* k*-frame at* $\gamma(t_0)$*.*

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Then the set $\gamma([t_0 - \epsilon, t_0 + \epsilon]) \subseteq \mathbb{R}^n$ *has exactly two* k*-tangents at* $\gamma(t_0)$*,*

$$v \text{ and } (-v_1, v_2, -v_3, \ldots).$$

Finitely generated case

Rationally outgoing tangent

Theorem (Busaniche, Mundici)

Let $X \subseteq [0,1]^2$ be a closed set. The MV-algebra $\mathcal{M}(X)$ is not strongly semisimple iff there exist a point $x \in X$, and a unit vector $u \in \mathbb{R}^2$ such that

(i) *u is a Bouligand-Severi tangent of X at x; and*

(ii) *there exists a real number $\lambda > 0$*

$$\mathrm{conv}(x, x + \lambda u) \cap X = \{x\}$$

and the coordinates of x and $x + \lambda u$ are rational.

Finitely generated case

Rationally outgoing tangent

Definition

A tangent u of $X \subseteq \mathbb{R}^n$ at x **is rationally outgoing** if there is a $\lambda \in \mathbb{R}_{>0}$, and a rational simplex S, such that

$$\mathrm{conv}(x, x + \lambda u) = S$$

and

$$\{x\} = S \cap X.$$

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Definition

A k-tangent $u = (u_1, \ldots, u_k)$ of $X \subseteq \mathbb{R}^n$ at x **is rationally outgoing** if there is a $\lambda \in \mathbb{R}_{>0}$,
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A k-tangent $u = (u_1, \ldots, u_k)$ of $X \subseteq \mathbb{R}^n$ at x **is rationally outgoing** if there is a k-tuple $\lambda = (\lambda_1, \ldots, \lambda_k) \in {\mathbb{R}_{>0}}^k$, and a rational simplex S, together with a face $F \subseteq S$ such that

$$\mathrm{conv}(x, x + \lambda_1 u_1, \ldots, x + \lambda_1 u_1 + \cdots + \lambda_k u_k) \subseteq S$$

and

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$$\mathrm{conv}(x, x + \lambda_1 u_1, \ldots, x + \lambda_1 u_1 + \cdots + \lambda_k u_k) \not\subseteq F$$

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and

$$F \cap X = S \cap X.$$

Main Result

Theorem

For any closed $X \subseteq [0,1]^n$ the following conditions are equivalent:

(i) *The MV-algebra $\mathcal{M}(X)$ is strongly semisimple.*

(ii) *For no $k = 1, \ldots, n-1$, X has a rationally outgoing k-tangent.*

Main Result

(i)⇒(ii)

Let u be a rationally outgoing k-tangent of X at x and let S be a rational k-simplex together with a proper face $F \subseteq S$ and reals $\lambda_1, \dots, \lambda_k \in \mathbb{R}^k_{>0}$ such that

(a) $\mathrm{conv}(x, x + \lambda_1 u_1, \dots, x + \lambda_1 u_1 + \cdots + \lambda_k u_k) \subseteq S$,

(b) $\mathrm{conv}(x, x + \lambda_1 u_1, \dots, x + \lambda_1 u_1 + \cdots + \lambda_k u_k) \not\subseteq F$, and

(c) $S \cap X = F \cap X$.

Main Result

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Let $f, g \in \mathcal{M}([0,1]^n)$ be maps such that

$$f(v) = 0 \text{ iff } v \in F \quad \text{and} \quad g(v) = 0 \text{ iff } v \in S.$$

Main Result

(i)$\Rightarrow$(ii)

(c) proves that $f\restriction_X$ belongs to a maximal ideal of $\mathcal{M}(X)$ iff $g\restriction_X$ does.

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(a) and (b) imply that $f\restriction_X$ does not belong to the ideal generated by $g\restriction_X$.

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Lemma

Let $P \subseteq [0,1]^n$ be a polyhedron such that $X \subseteq P$. Since $(u_1, \ldots, u_l)$ is a k-tangent of X at x, there exist $\delta_1, \ldots, \delta_k \in \mathbb{R}_{>0}$ with

$$\mathrm{conv}(x, x + \delta_1 u_1, \ldots, x + \delta_1 u_1 + \cdots + \delta_k u_k) \subseteq P.$$

Main Result

(ii)$\Rightarrow$(i)

Let $f, g \in \mathcal{M}([0,1]^n)$ be such that $f\restriction_X$ does not belong to the ideal generated by $g\restriction_X$ and that $f\restriction_X$ belongs to a maximal ideal of $\mathcal{M}(X)$ iff $g\restriction_X$ does.

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Then $\mathcal{M}(\eta(X))$ is isomorphic to the subalgebra of $\mathcal{M}(X)$ generated by $g\restriction_X$ and $f\restriction_X$. Then $\eta(X)$ has a rationally outgoing 1-tangent.

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If $\rho\colon [0,1]^n \to [0,1]^2$ *is a map defined by*

$$\rho(v) = (f(v), g(v))$$

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Lemma

If $\rho\colon [0,1]^n \to [0,1]^2$ is a map defined by

$$\rho(v) = (f(v), g(v))$$

for $f, g \in \mathcal{M}([0,1]^n)$ and $\rho(X)$ has a rationally outgoing 1-tangent, then for some $k \in \{1, \ldots, n-1\}$, X has a rationally outgoing k-tangent.

Other Results

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- Byproduct: Extension of the definition of Frenet frames to closed sets.

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- Geometric description of prime filters of a finitely generated semisimple MV-algebra.
- Strongly semisimple Riesz spaces.

Thank you for your attention!

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