Strongly semisimple MV-algebras and tangents

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Introduction
Syntactic vs Semantic Consequence

Ł∞ logic. (Łukasiewicz, Tarski - 1930)
Introduction

Syntactic vs Semantic Consequence

Ł∞ logic. (Łukasiewicz, Tarski - 1930)

Semantics

A valuation \( v : \mathcal{FM} \to [0, 1] \) (where \( \mathcal{FM} \) is the set of formulas on the language \( \{\to, \neg\} \)) is a map satisfying:

- \( v(\alpha \to \beta) = \min\{1 - v(\alpha), v(\beta), 1\} \)
- \( v(\neg \alpha) = 1 - v(\alpha) \)
Introduction
Syntactic vs Semantic Consequence

Ł∞ logic. (Łukasiewicz,Tarski - 1930)

Semantics

A valuation \( v : \mathcal{FM} \rightarrow [0, 1] \) (where \( \mathcal{FM} \) is the set of formulas on the language \( \{\rightarrow, \neg\} \)) is a map satisfying:

\[
\begin{align*}
\triangleright & \quad v(\alpha \rightarrow \beta) = \min \{ (1 - v(\alpha)) + v(\beta), 1 \} \\
\triangleright & \quad v(\neg \alpha) = 1 - v(\alpha)
\end{align*}
\]

Calculus

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Introduction

Syntactic vs Semantic Consequence

Syntactic Consequence

Θ ⊩_{Ł∞} ϕ iff there exists an Ł∞-proof of ϕ from Θ.
Introduction

Syntactic vs Semantic Consequence

Syntactic Consequence

\[ \Theta \vdash_{\mathcal{L}_\infty} \varphi \] iff there exists an \( \mathcal{L}_\infty \)-proof of \( \varphi \) from \( \Theta \).

Semantic Consequence

\[ \Theta \models_{\mathcal{L}_\infty} \varphi \] iff for each valuation \( \nu : \mathcal{F}\mathcal{M} \rightarrow [0, 1] \)

\[ \nu(\Theta) = \{1\} \] implies \( \nu(\varphi) = 1 \).
Soundness:

If $\Theta \vdash \mathcal{L}_\infty \varphi$, then $\Theta \models \mathcal{L}_\infty \varphi$. 

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**Introduction**

*Syntactic vs Semantic Consequence*

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**Other Results**
Soundness:

If $\Theta \vdash L_\infty \varphi$, then $\Theta \models L_\infty \varphi$.

Finite Completeness (Hay-Wójcicki):

If $|\Theta| < \kappa_0$ and $\Theta \models L_\infty \varphi$, then $\Theta \vdash L_\infty \varphi$. 
Introduction

Syntactic vs Semantic Consequence

What if $\Theta$ is not finite?
What if $\Theta$ is not finite?

**Theorem**

*Given a set of formulas $\Theta$, the following are equivalent:*

1. For each formula $\varphi$, $\Theta \vdash_{\mathfrak{L}_\infty} \varphi$ iff $\Theta \models_{\mathfrak{L}_\infty} \varphi$. 

2. The MV-algebra presented by $(\text{Var}(\Theta), \Theta)$ is semisimple (that is, its radical is $\{0\}$).

3. The MV-algebra presented by $(\text{Var}(\Theta), \Theta)$ belongs to $\text{ISP}([0, 1]_{\mathcal{MV}})$. 


What if $\Theta$ is not finite?

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Introduction
Syntactic vs Semantic Consequence

(Hay-Wójcicki):

If $|\Theta| < \aleph_0$, then $\Theta \models_{\mathcal{L}_\infty} \varphi$ iff $\Theta \vdash_{\mathcal{L}_\infty} \varphi$. 
Introduction
Syntactic vs Semantic Consequence

(Hay-Wójcicki):

If $|\Theta| < \aleph_0$, then $\Theta \models_{L_\infty} \varphi$ iff $\Theta \vdash_{L_\infty} \varphi$.

If $\Theta$ is a finite set of formulas, for each formula $\alpha$:

$\Theta \cup \{\alpha\} \models_{L_\infty} \varphi$ iff $\Theta \cup \{\alpha\} \vdash_{L_\infty} \varphi$. 
What if $\Theta$ is not finite?
Introduction
Syntactic vs Semantic Consequence

What if $\Theta$ is not finite?

Theorem
For each $\Theta$ set of formulas, the following are equivalent:

- For every $\alpha, \varphi$,

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Strongly semisimple MV-algebras and tangents
L.M. Cabrer
What if $\Theta$ is not finite?

**Theorem**

*For each $\Theta$ set of formulas, the following are equivalent:*

- For every $\alpha, \varphi$,

  $$\Theta \cup \{\alpha\} \vdash_{\mathcal{L}_\infty} \varphi \iff \Theta \cup \{\alpha\} \models_{\mathcal{L}_\infty} \varphi.$$  

- The MV-algebra presented by $(\text{Var}(\Theta), \Theta)$ is **strongly semisimple**.

---

**Definition**

An MV-algebra $A$ is **strongly semisimple** if for every finitely generated ideal (filter) $I$, the MV-algebra $A/I$ is semisimple.
Introduction
Syntactic vs Semantic Consequence

What if $\Theta$ is not finite?

Theorem
For each $\Theta$ set of formulas, the following are equivalent:

▶ For every $\alpha, \varphi$,

$$\Theta \cup \{\alpha\} \vdash_{\mathbf{L}_\infty} \varphi \text{ iff } \Theta \cup \{\alpha\} \models_{\mathbf{L}_\infty} \varphi.$$ 

▶ The MV-algebra presented by $(\text{Var}(\Theta), \Theta)$ is strongly semisimple.

An MV-algebra $A$ is strongly semisimple if for every finitely generated ideal (filter) $I$, the MV-algebra $A/I$ is semisimple.
Main Goal

To present a geometric description of finitely generated strongly semisimple MV-algebras.
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To present a geometric description of finitely generated strongly semisimple MV-algebras.

More precisely, for each $n$-generated semisimple MV-algebra $A$ there exists $X$ a closed subset of $[0, 1]^n$, such that $A$ is isomorphic to

$$\mathcal{M}(X) = \{ f|_X \mid f : [0, 1]^n \to [0, 1] \text{ is a McNaughton map} \}.$$

We will present necessary and sufficient conditions on the closed set $X \subseteq [0, 1]^n$ for $A \cong \mathcal{M}(X)$ to be strongly semisimple.
2-generated case

Syntactic vs Semantic Consequence

\[ X = \{(x, y) \in \mathbb{R}^2 \mid (x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 = \frac{1}{4}\} \]
2-generated case
Syntactic vs Semantic Consequence
2-generated case

Theorem (Busaniche, Mundici)

Let $X \subseteq [0, 1]^2$ be a closed set. The MV-algebra $\mathcal{M}(X)$ is not strongly semisimple
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Let $X \subseteq [0, 1]^2$ be a closed set. The MV-algebra $\mathcal{M}(X)$ is not strongly semisimple iff there exist a point $x \in X$, a sequence $x_0, x_1, \ldots \in X$, a unit vector $u \in \mathbb{R}^2$, and a real number $\lambda > 0$
2-generated case

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(i) $x_i \neq x$ for all $i$,
(ii) $\lim_{i \to \infty} x_i = x$, 
(iii) $\lim_{i \to \infty} (x_i - x) / ||x_i - x|| = u$, 
(iv) $\text{conv}(x, x + \lambda u) \cap X = \{x\}$, 
(v) the coordinates of $x$ and $x + \lambda u$ are rational.
2-generated case

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Let $X \subseteq [0, 1]^2$ be a closed set. The MV-algebra $\mathcal{M}(X)$ is not strongly semisimple iff there exist a point $x \in X$, a sequence $x_0, x_1, \ldots \in X$, a unit vector $u \in \mathbb{R}^2$, and a real number $\lambda > 0$ such that

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(ii) $\lim_{i \to \infty} x_i = x$,

(iii) $\lim_{i \to \infty} (x_i - x) / ||x_i - x|| = u$, where $||x||$ denotes the Euclidean norm of $x$.
2-generated case

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Let $X \subseteq [0, 1]^2$ be a closed set. The MV-algebra $\mathcal{M}(X)$ is not strongly semisimple iff there exist a point $x \in X$, a sequence $x_0, x_1, \ldots \in X$, a unit vector $u \in \mathbb{R}^2$, and a real number $\lambda > 0$ such that

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(iii) $\lim_{i \to \infty} \frac{(x_i - x)}{||x_i - x||} = u$,

(iv) $\text{conv}(x, x + \lambda u) \cap X = \{x\}$,
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2-generated case

Syntactic vs Semantic Consequence

\[ X = \{(x, y) \in \mathbb{R}^2 \mid (x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 = \frac{1}{4}\} \]
Definition (Bouligand, Severi)

Let $\emptyset \neq X \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}^n$. A **Bouligand-Severi tangent of $X$ at $x$** is a unit vector $u \in \mathbb{R}^n$ such that $X$ contains a sequence $x_1, x_2, \ldots$ with the following properties:

(i) $x_i \neq x$ for all $i$;

(ii) $\lim_{i \to \infty} x_i = x$; and

(iii) $\lim_{i \to \infty} (x_i - x)/\|x_i - x\| = u$.
2-generated case
Bouligand-Severi Tangents

Definition (Bouligand, Severi)

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(ii) $\lim_{i \to \infty} x_i = x$; and

(iii) $\lim_{i \to \infty} (x_i - x)/||x_i - x|| = u$.

The tangent $u$ is said to be **outgoing** if there exists $\lambda > 0$ such that

$$\text{conv}(x, x + \lambda u) \cap X = \{x\}.$$
Finitely generated case

Key Remarks

- Importance of the 2-generated case:
Finitely generated case

Importance of the 2-generated case:

An MV-algebra $A$ is strongly semisimple iff every 2-generated subalgebra of $A$ is strongly semisimple.
Finitely generated case

Key Remarks

- Importance of the 2-generated case:
  An MV-algebra $A$ is strongly semisimple iff every 2-generated subalgebra of $A$ is strongly semisimple.

We need

- $n$-dimensional generalisation of Bouligand-Severi tangents.
- the right definition of “rational” outgoingness.
Finitely generated case

Bouligand-Severi tangents of higher dimension
Finitely generated case

Bouligand-Severi tangents of higher dimension

Let \( \gamma: [a, b] \to \mathbb{R}^n \) be a \( C^k \) \( (k \leq n) \) function such that for all \( a < t < b \), the \( k \)-tuple of vectors

\[
( \gamma'(t), \gamma''(t), \ldots, \gamma^{(k)}(t) )
\]

forms a linear independent set in \( \mathbb{R}^n \).
Let $\gamma: [a, b] \to \mathbb{R}^n$ be a $C^k$ ($k \leq n$) function such that for all $a < t < b$, the $k$-tuple of vectors
\[
(\gamma'(t), \gamma''(t), \ldots, \gamma^{(k)}(t))
\]
forms a linear independent set in $\mathbb{R}^n$. The Gram-Schmidt orthonormalization process yields an orthonormal $k$-tuple
\[
(\nu_1(t), \ldots, \nu_k(t)),
\]
called the Frenet $k$-frame of $\gamma$ at $\gamma(t)$. 

Finitely generated case
Bouligand-Severi tangents of higher dimension
Finitely generated case

Bouligand-Severi tangents of higher dimension

Definition

A $k$-tuple $u = (u_1, \ldots, u_k)$ of pairwise orthogonal unit vectors in $\mathbb{R}^n$ is said to be a $k$-tangent of $X$ at $x$ if:

1. there is a sequence of points $x_1, x_2, \ldots \in X$ such that
   \[ \lim_{i \to \infty} x_i = x; \]
2. \[ \lim_{i \to \infty} (x_i - x) / ||x_i - x|| = u_1, \]
3. \[ \lim_{i \to \infty} (x_i - x - pR u_1 (x_i - x)) / ||x_i - x - pR u_1 (x_i - x)|| = u_2, \]
4. \[ \vdots \]
5. \[ \lim_{i \to \infty} (x_i - x - pR u_1 + \ldots + R u_k - 1 (x_i - x)) / ||x_i - x - pR u_1 + \ldots + R u_k - 1 (x_i - x)|| = u_k. \]
Finitely generated case
Bouligand-Severi tangents of higher dimension

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- there is a sequence of points $x_1, x_2, \ldots \in X$ such that

  $\lim_{i \to \infty} x_i = x$;

  $x_i - x \notin \mathbb{R}u_1 + \cdots + \mathbb{R}u_{k-1}$;
Finitely generated case
Bouligand-Severi tangents of higher dimension

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  - $1 \lim_{i \to \infty} (x_i - x)/\|x_i - x\| = u_1$, 

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- $\lim_{i \to \infty} (x_i - x)/\|x_i - x\| = u_1$.
Finitely generated case

Bouligand-Severi tangents of higher dimension

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   \[
   \lim_{i \to \infty} x_i = x;
   \]
   \[
   x_i - x \not\in \mathbb{R} u_1 + \cdots + \mathbb{R} u_{k-1};
   \]
   \[
   1 \lim_{i \to \infty} (x_i - x)/\|x_i - x\| = u_1,
   \]
   \[
   2 \lim_{i \to \infty} \frac{x_i - x - p_{\mathbb{R}u_1}(x_i - x)}{\|x_i - x - p_{\mathbb{R}u_1}(x_i - x)\|} = u_2,
   \]
Finitely generated case
Bouligand-Severi tangents of higher dimension

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A $k$-tuple $u = (u_1, \ldots, u_k)$ of pairwise orthogonal unit vectors in $\mathbb{R}^n$ is said to be a $k$-tangent of $X$ at $x$ if:

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3. $\lim_{i \to \infty} (x_i - x)/\|x_i - x\| = u_1$,
4. $\lim_{i \to \infty} \frac{x_i - x - p_{\mathbb{R}u_1}(x_i - x)}{\|x_i - x - p_{\mathbb{R}u_1}(x_i - x)\|} = u_2$,
5. \ldots
Definition

A \( k \)-tuple \( u = (u_1, \ldots, u_k) \) of pairwise orthogonal unit vectors in \( \mathbb{R}^n \) is said to be a **\( k \)-tangent of \( X \) at \( x \)** if:

1. there is a sequence of points \( x_1, x_2, \ldots \in X \) such that:
   - \( \lim_{i \to \infty} x_i = x \);
   - \( x_i - x \notin \mathbb{R}u_1 + \cdots + \mathbb{R}u_{k-1} \);
   - \( \lim_{i \to \infty} (x_i - x)/||x_i - x|| = u_1 \),
   - \( \lim_{i \to \infty} \frac{x_i - x - p_{\mathbb{R}u_1}(x_i - x)}{||x_i - x - p_{\mathbb{R}u_1}(x_i - x)||} = u_2 \),
   - \( \cdots \)
   - \( \lim_{i \to \infty} \frac{x_i - x - p_{\mathbb{R}u_1 + \cdots + \mathbb{R}u_{k-1}}(x_i - x)}{||x_i - x - p_{\mathbb{R}u_1 + \cdots + \mathbb{R}u_{k-1}}(x_i - x)||} = u_k \).
Finitely generated case

Bouligand-Severi tangents of higher dimension

Key Remarks

Bouligand-Severi tangents of higher dimension
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Bouligand-Severi tangents of higher dimension

Theorem (LMC, Mundici)

Suppose $\gamma : [a, b] \rightarrow \mathbb{R}^n$ is a $C^{k+1}$ function and $a < t_0 < b$ is such that $\gamma'(t_0), \gamma''(t_0), \ldots, \gamma^{(k)}(t_0)$ are linearly independent and let $v = (v_1, \ldots, v_k)$ be its Frenet $k$-frame at $\gamma(t_0)$. Then the set $\gamma([t_0 - \epsilon, t_0 + \epsilon]) \subseteq \mathbb{R}^n$ has exactly two $k$-tangents at $\gamma(t_0)$, $v$ and $(-v_1, v_2, -v_3, \ldots)$. 

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Rationally outgoing tangent

Main Result
Sketch of the proof

Other Results
Finitely generated case
Bouligand-Severi tangents of higher dimension

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Suppose \( \gamma: [a, b] \to \mathbb{R}^n \) is a \( C^{k+1} \) function and \( a < t_0 < b \) is such that \( \gamma'(t_0), \gamma''(t_0), \ldots, \gamma^{(k)}(t_0) \) are linearly independent and let \( v = (v_1, \ldots, v_k) \) be its Frenet k-frame at \( \gamma(t_0) \).

Then the set \( \gamma([t_0 - \epsilon, t_0 + \epsilon]) \subseteq \mathbb{R}^n \) has exactly two \( k \)-tangents at \( \gamma(t_0) \),

\[ v \text{ and } (-v_1, v_2, -v_3, \ldots). \]
Finitely generated case
Rationally outgoing tangent

Theorem (Busaniche, Mundici)
Let $X \subseteq [0, 1]^2$ be a closed set. The MV-algebra $\mathcal{M}(X)$ is not strongly semisimple iff there exist a point $x \in X$, and a unit vector $u \in \mathbb{R}^2$ such that

(i) $u$ is a Bouligand-Severi tangent of $X$ at $x$; and
(ii) there exists a real number $\lambda > 0$

$$\text{conv}(x, x + \lambda u) \cap X = \{x\}$$

and the coordinates of $x$ and $x + \lambda u$ are rational.
Finitely generated case
Rationally outgoing tangent

Definition
A tangent \( u \) of \( X \subseteq \mathbb{R}^n \) at \( x \) is rationally outgoing if there is a \( \lambda \in \mathbb{R}_{>0} \), and a rational simplex \( S \), such that

\[
\text{conv}(x, x + \lambda u) = S
\]

and

\[
\{x\} = S \cap X.
\]
Definition

A $k$-tangent $u = (u_1, \ldots, u_k)$ of $X \subseteq \mathbb{R}^n$ at $x$ is **rationally outgoing** if there is a $\lambda \in \mathbb{R}_{>0}$, and a rational simplex $S$, such that

$$ \text{conv}(x, x + \lambda u) = S $$

and

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Finitely generated case
Rationally outgoing tangent

**Definition**
A $k$-tangent $u = (u_1, \ldots, u_k)$ of $X \subseteq \mathbb{R}^n$ at $x$ is **rationally outgoing** if there is a $k$-tuple $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{R}_{>0}^k$, and a rational simplex $S$, such that

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Finitely generated case
Rationally outgoing tangent

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\text{conv}(x, x + \lambda_1 u_1, \ldots, x + \lambda_1 u_1 + \cdots + \lambda_k u_k) = S
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Finitely generated case
Rationally outgoing tangent

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A $k$-tangent $u = (u_1, \ldots, u_k)$ of $X \subseteq \mathbb{R}^n$ at $x$ is rationally outgoing if there is a $k$-tuple $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{R}_{>0}^k$, and a rational simplex $S$, such that

$$\text{conv}(x, x + \lambda_1 u_1, \ldots, x + \lambda_1 u_1 + \cdots + \lambda_k u_k) \subseteq S$$

and

$$\{x\} = S \cap X.$$
Finitely generated case
Rationally outgoing tangent

Definition
A $k$-tangent $u = (u_1, \ldots, u_k)$ of $X \subseteq \mathbb{R}^n$ at $x$ is rationally outgoing if there is a $k$-tuple $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{R}_{>0}^k$, and a rational simplex $S$, together with a face $F \subseteq S$ such that
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\text{conv}(x, x + \lambda_1 u_1, \ldots, x + \lambda_1 u_1 + \cdots + \lambda_k u_k) \subseteq S
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and
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Finitely generated case
Rationally outgoing tangent

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$$\text{conv}(x, x + \lambda_1 u_1, \ldots, x + \lambda_1 u_1 + \cdots + \lambda_k u_k) \subseteq S,$$

$$\text{conv}(x, x + \lambda_1 u_1, \ldots, x + \lambda_1 u_1 + \cdots + \lambda_k u_k) \not\subseteq F$$

and

$$\{x\} = S \cap X.$$
Finitely generated case
Rationally outgoing tangent

Definition
A \( k \)-tangent \( u = (u_1, \ldots, u_k) \) of \( X \subseteq \mathbb{R}^n \) at \( x \) is **rationally outgoing** if there is a \( k \)-tuple \( \lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{R}_{>0}^k \), and a rational simplex \( S \), together with a face \( F \subseteq S \) such that

\[
\text{conv}(x, x + \lambda_1 u_1, \ldots, x + \lambda_1 u_1 + \cdots + \lambda_k u_k) \subseteq S,
\]

\[
\text{conv}(x, x + \lambda_1 u_1, \ldots, x + \lambda_1 u_1 + \cdots + \lambda_k u_k) \not\subseteq F
\]

and

\[
F \cap X = S \cap X.
\]
Main Result

Theorem

For any closed $X \subseteq [0, 1]^n$ the following conditions are equivalent:

(i) The MV-algebra $\mathcal{M}(X)$ is strongly semisimple.

(ii) For no $k = 1, \ldots, n - 1$, $X$ has a rationally outgoing $k$-tangent.
Main Result

(i)⇒(ii)

Let $u$ be a rationally outgoing $k$-tangent of $X$ at $x$ and let $S$ be a rational $k$-simplex together with a proper face $F \subseteq S$ and reals $\lambda_1, \ldots, \lambda_k \in \mathbb{R}_0^k$ such that

(a) $\text{conv}(x, x + \lambda_1 u_1, \ldots, x + \lambda_1 u_1 + \cdots + \lambda_k u_k) \subseteq S$,
(b) $\text{conv}(x, x + \lambda_1 u_1, \ldots, x + \lambda_1 u_1 + \cdots + \lambda_k u_k) \not\subseteq F$, and
(c) $S \cap X = F \cap X$. 
Main Result

(i) \Rightarrow (ii)

Let $u$ be a rationally outgoing $k$-tangent of $X$ at $x$ and let $S$ be a rational $k$-simplex together with a proper face $F \subset S$ and reals $\lambda_1, \ldots, \lambda_k \in \mathbb{R}^k_{>0}$ such that

(a) $\text{conv}(x, x + \lambda_1 u_1, \ldots, x + \lambda_1 u_1 + \cdots + \lambda_k u_k) \subseteq S$,
(b) $\text{conv}(x, x + \lambda_1 u_1, \ldots, x + \lambda_1 u_1 + \cdots + \lambda_k u_k) \nsubseteq F$, and
(c) $S \cap X = F \cap X$.

Let $f, g \in \mathcal{M}([0, 1]^n)$ be maps such that

$$f(v) = 0 \text{ iff } v \in F \quad \text{and} \quad g(v) = 0 \text{ iff } v \in S.$$
Main Result

(i)⇒(ii)

(c) proves that $f \upharpoonright_X$ belongs to a maximal ideal of $\mathcal{M}(X)$ iff $g \upharpoonright_X$ does.
Main Result

(i) ⇒ (ii)

(c) proves that $f|_X$ belongs to a maximal ideal of $\mathcal{M}(X)$ iff $g|_X$ does.

(a) and (b) imply that $f|_X$ does not belong to the ideal generated by $g|_X$. 
Main Result

(i) \implies (ii)

Lemma

Let \( P \subseteq [0, 1]^n \) be a polyhedron such that \( X \subseteq P \).
Main Result

(i) ⇒ (ii)

Lemma

Let $P \subseteq [0, 1]^n$ be a polyhedron such that $X \subseteq P$. Since $(u_1, \ldots, u_l)$ is a $k$-tangent of $X$ at $x$, 

...
Main Result

(i) $\Rightarrow$ (ii)

Lemma

Let $P \subseteq [0, 1]^n$ be a polyhedron such that $X \subseteq P$. Since $(u_1, \ldots, u_l)$ is a $k$-tangent of $X$ at $x$, there exist $\delta_1, \ldots, \delta_k \in \mathbb{R}_{>0}$ with

$$\text{conv}(x, x + \delta_1 u_1, \ldots, x + \delta_1 u_1 + \cdots + \delta_k u_k) \subseteq P.$$
Main Result

(ii) ⇒ (i)

Let \( f, g \in M([0, 1]^n) \) be such that \( f \upharpoonright_X \) does not belong to the ideal generated by \( g \upharpoonright_X \) and that \( f \upharpoonright_X \) belongs to a maximal ideal of \( M(X) \) iff \( g \upharpoonright_X \) does.
Main Result

(ii) $\Rightarrow$ (i)

Let $f, g \in \mathcal{M}([0, 1]^n)$ be such that $f \upharpoonright_X$ does not belong to the ideal generated by $g \upharpoonright_X$ and that $f \upharpoonright_X$ belongs to a maximal ideal of $\mathcal{M}(X)$ iff $g \upharpoonright_X$ does.

Let the map $\eta : X \to [0, 1]^2$ be defined by

$$\eta = (f \upharpoonright_X, g \upharpoonright_X).$$
Main Result

(ii) ⇒ (i)

Let \( f, g \in \mathcal{M}([0, 1]^n) \) be such that \( f \upharpoonright_X \) does not belong to the ideal generated by \( g \upharpoonright_X \) and that \( f \upharpoonright_X \) belongs to a maximal ideal of \( \mathcal{M}(X) \) iff \( g \upharpoonright_X \) does.

Let the map \( \eta : X \rightarrow [0, 1]^2 \) be defined by

\[
\eta = (f \upharpoonright_X, g \upharpoonright_X).
\]

Then \( \mathcal{M}(\eta(X)) \) is isomorphic to the subalgebra of \( \mathcal{M}(X) \) generated by \( g \upharpoonright_X \) and \( f \upharpoonright_X \).
Main Result
(ii) ⇒ (i)

Let \( f, g \in \mathcal{M}([0, 1]^n) \) be such that \( f \upharpoonright_X \) does not belong to the ideal generated by \( g \upharpoonright_X \) and that \( f \upharpoonright_X \) belongs to a maximal ideal of \( \mathcal{M}(X) \) iff \( g \upharpoonright_X \) does.

Let the map \( \eta : X \to [0, 1]^2 \) be defined by

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\eta = (f \upharpoonright_X, g \upharpoonright_X).
\]

Then \( \mathcal{M}(\eta(X)) \) is isomorphic to the subalgebra of \( \mathcal{M}(X) \) generated by \( g \upharpoonright_X \) and \( f \upharpoonright_X \). Then \( \eta(X) \) has a rationally outgoing 1-tangent.
Main Result

(ii) ⇒ (i)

Lemma

If $\rho : [0, 1]^n \to [0, 1]^2$ is a map defined by

$$\rho(v) = (f(v), g(v))$$

for $f, g \in \mathcal{M}([0, 1]^n)$
Main Result
(ii)⇒(i)

Lemma
If \( \rho: [0, 1]^n \to [0, 1]^2 \) is a map defined by
\[
\rho(v) = (f(v), g(v))
\]
for \( f, g \in \mathcal{M}([0, 1]^n) \) and \( \rho(X) \) has a rationally outgoing 1-tangent,
Main Result

(ii) $\Rightarrow$ (i)

Lemma

If $\rho : [0, 1]^n \to [0, 1]^2$ is a map defined by

$$\rho(v) = (f(v), g(v))$$

for $f, g \in \mathcal{M}([0, 1]^n)$ and $\rho(X)$ has a rationally outgoing 1-tangent, then for some $k \in \{1, \ldots, n - 1\}$, $X$ has a rationally outgoing $k$-tangent.
Strongly semisimple MV-algebras and tangents

L.M. Cabrer

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Rationally outgoing tangent

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Other Results

Other Results

Byproduct: Extension of the definition of Frenet frames to closed sets.

Geometric description of prime filters of a finitely generated semisimple MV-algebra.

Strongly semisimple Riesz spaces.
Other Results

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- Byproduct: Extension of the definition of Frenet frames to closed sets.
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Strongly semisimple MV-algebras and tangents

Thank you for your attention!

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