A temporal semantics for Nilpotent Minimum logic

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- We conclude by presenting a completeness theorem.

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- Useful derived connectives are the following

$$\text{(negation)} \qquad \neg\varphi := \varphi \rightarrow \bot$$

$$\text{(disjunction)} \qquad \varphi \vee \psi := ((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi)$$

Syntax

(A1)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(A2)	$(\varphi \& \psi) \to \varphi$
(A3)	$(\varphi \& \psi) \to (\psi \& \varphi)$
(A4)	$(\varphi \wedge \psi) \to \varphi$
(A5)	$(\varphi \wedge \psi) \to (\psi \wedge \varphi)$
(A6)	$(\varphi \& (\varphi \to \psi)) \to (\psi \wedge \varphi)$
(A7a)	$(\varphi \to (\psi \to \chi)) \to ((\varphi \& \psi) \to \chi)$
(A7b)	$((\varphi \& \psi) \to \chi) \to (\varphi \to (\psi \to \chi))$
(A8)	$((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$
(A9)	$\bot \to \varphi$

As inference rule we have modus ponens:

(MP) $$\frac{\varphi \quad \varphi \to \psi}{\psi}$$

Syntax

Nilpotent Minimum Logic (NM), introduced in [EG01] is obtained from MTL by adding the following axioms:

(involution)	$\neg\neg\varphi \to \varphi$
(WNM)	$\neg(\varphi \& \psi) \vee ((\varphi \wedge \psi) \to (\varphi \& \psi))$

The notions of theory, syntactic consequence, proof are defined as usual.

Semantics

An MTL algebra is an algebraic structure $\langle A, *, \Rightarrow, \sqcap, \sqcup, 0, 1 \rangle$ such that

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   (Prelinearity) $$(x \Rightarrow y) \sqcup (y \Rightarrow x) = 1$$

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- An NM-algebra is an MTL-algebra that satisfies the following equations:

$$\sim\sim x = x$$
$$\sim (x * y) \sqcup ((x \sqcap y) \Rightarrow (x * y)) = 1$$

Where $\sim x$ indicates $x \Rightarrow 0$.

Semantics

As pointed in [Gis03], in *each* NM-chain it holds that:

$$x * y = \begin{cases} 0 & \text{if } x \leq n(y) \\ \min(x, y) & \text{Otherwise.} \end{cases}$$

$$x \Rightarrow y = \begin{cases} 1 & \text{if } x \leq y \\ \max(n(x), y) & \text{Otherwise.} \end{cases}$$

- Where n is a strong negation function, i.e. $n : A \rightarrow A$ is an order-reversing mapping ($x \leq y$ implies $n(x) \geq n(y)$) such that $n(0) = 1$ and $n(n(x)) = x$, for each $x \in A$. Observe that $n(x) = x \Rightarrow 0$, for each $x \in A$.

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- A negation fixpoint is an element $x \in A$ such that $n(x) = x$: note that if exists then it must be unique (otherwise n fails to be order-reversing).

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Let $\mathcal{A}$ be an NM-algebra. Each map $e\colon VAR \to A$ extends uniquely to an $\mathcal{A}$-*assignment* $v_e : FORM \to A$ in the usual ▸ inductive way

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- Let $\mathcal{A}$ be an NM-chain. We say that NM is strongly complete (respectively: finitely strongly complete, complete) with respect to $\mathcal{A}$ if for every theory Γ (respectively, for every finite theory Γ of formulas, for $\Gamma = \emptyset$) and for every formula φ we have

$$\Gamma \vdash_{\mathrm{NM}} \varphi \quad \text{iff} \quad \Gamma \models_{\mathcal{A}} \varphi$$

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Theorem 2

NM is finitely strongly complete w.r.t. every infinite NM-chain with negation fixpoint.

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Theorem 3 ([EG01, CEG+09])

NM enjoys the strong completeness with respect to $\mathcal{A}$, with $\mathcal{A} \in \left\{[0,1]_{NM}, [0,1]_{NM}^{\mathbb{Q}}\right\}$.

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- Over these three values we can define the semantics associated to a negation and an implication operations:

$\neg_3$	
0	1
$\frac{1}{2}$	$\frac{1}{2}$
1	0

$\rightarrow_3$	0	$\frac{1}{2}$	1
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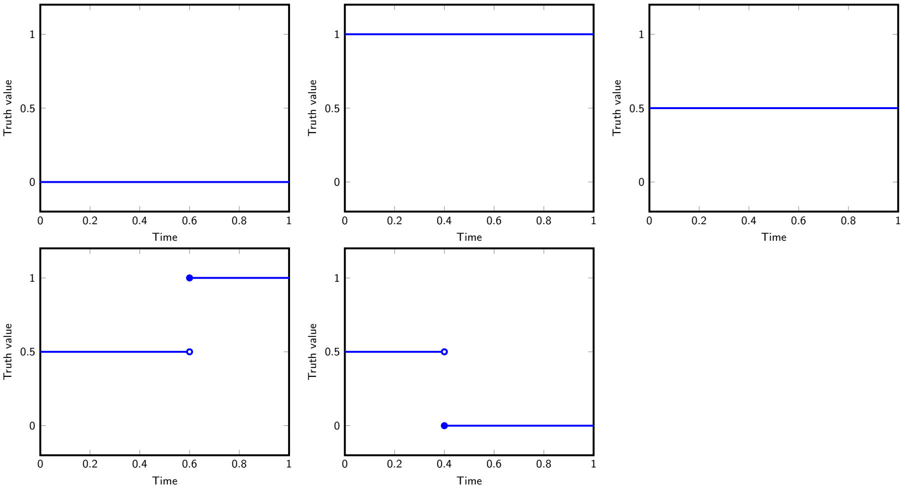
- In the proposed semantics a temporal assignment (over a temporal flow $\langle T, \leq \rangle$) is a function $v : \ FORM \times T \rightarrow \{0, \frac{1}{2}, 1\}$.
- However, not arbitrary assignments are admitted: in our semantics $v(\varphi, \cdot)$ must behaves as follows:

Temporal semantics



Condition 2.1

We restrict to the following types of temporal assignments $v : FORM \times T \to \{0, \frac{1}{2}, 1\}$, *for every* $\varphi \in FORM$:

1. $v(\varphi, \cdot)$ *is constant, to* 0, $\frac{1}{2}$ *or* 1.
   In this case we say that, respectively $v(\varphi, \cdot) \approx 0$, $v(\varphi, \cdot) \approx \frac{1}{2}$, $v(\varphi, \cdot) \approx 1$.
2. *There is a* $t \in T$, *with* $t \neq \min(T)$ *(if* T *has a minimum) such that*
   $$v(\varphi, t') = 0 \text{ for every } t' \geq t, \text{ and } v(\varphi, t'') = \frac{1}{2}, \text{ for every } t'' < t.$$
   In this case we say that $v(\varphi, \cdot) \approx t_0$.
3. *There is a* $t \in T$, *with* $t \neq \min(T)$ *(if* T *has a minimum) such that*
   $$v(\varphi, t') = 1 \text{ for every } t' \geq t, \text{ and } v(\varphi, t'') = \frac{1}{2}, \text{ for every } t'' < t.$$
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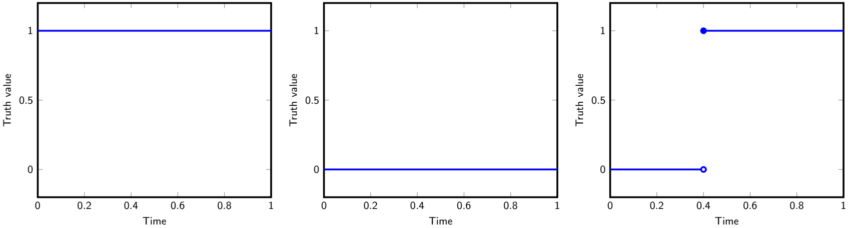
Temporal semantics, a comparison with the case of Gödel logic



Now we introduce the definition of temporal assignment (first on the variables, and then we will extend it over formulas):

Definition 4

A temporal assignment over variables (associated to a temporal flow $\langle T, \leq \rangle$) is a function $v: \ VAR \times T \to \{0, \frac{1}{2}, 1\}$ such that one of the following holds, for every $x \in VAR$:

- $v(x, \cdot)$ is constant.
- There is a $t \in T$, with $t \neq \min(T)$ (if T has a minimum) such that

$$v(\varphi, t') = 0 \text{ for every } t' \geq t, \text{ and } v(\varphi, t'') = \frac{1}{2}, \text{ for every } t'' < t.$$

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Temporal semantics

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Remark 2.1

We will consider only $\rightarrow, \neg$*, as connectives. This is because, as pointed out in [EGCN03], in Nilpotent Minimum logic the disjunction* $\vee$ *is definable from* $\neg, \rightarrow$*.*

Definition 5

Let v be a temporal assignment over variables, associated to some temporal flow T. Its extension $v' : FORM \times T \to \{0, \frac{1}{2}, 1\}$ to formulas is defined, inductively, in the following way, for every $\varphi \in FORM$, and $t \in T$:

$$v'(\varphi, t) := \begin{cases} v(x,t) & \text{if } \varphi = x \\ 0 & \text{if } \varphi = \bot \\ \neg_3 v'(\psi, t) & \text{if } \varphi = \neg\psi \\ v'_d(\psi \to \chi, t) & \text{if } \varphi = \psi \to \chi. \end{cases}$$

Where $\psi, \chi \in FORM$, $x \in VAR$ and

$$v'_d(\psi \to \chi, t) := \begin{cases} v'(\psi, t) \to_3 v'(\chi, t) & \text{if } v'(\psi, t) \to_3 v'(\chi, t) = v'(\psi, t') \to_3 v'(\chi, t'), \\ & \text{for every } t' \geq t \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

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- The function v_d associates to an assignment v its "definitive behavior": this function is necessary to restrict ourself on the assignments of Condition 2.1, as the following proposition shows.

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The temporal assignments previously defined satisfy Condition 2.1.

Temporal semantics

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The temporal assignments previously defined satisfy Condition 2.1.

Definition 7 (consequence)

Let $\langle T, \leq \rangle$ be a temporal assignment, Γ a theory, and φ a formula. With

$$\Gamma \models_T \varphi$$

we mean that for every temporal assignment w such that $w(\psi, t) = 1$, for every $\psi \in \Gamma, t \in T$ it holds that $w(\varphi, t) = 1$, for every $t \in T$.

Theorem 8 (Completeness theorem)

Let $\langle T, \leq \rangle$ be a temporal flow. Then for each formula φ and finite theory Γ

$$\Gamma \vdash_{NM} \varphi \quad \textit{iff} \quad \Gamma \models_T \varphi.$$

Completeness

Theorem 8 (Completeness theorem)

Let $\langle T, \leq \rangle$ be a temporal flow. Then for each formula φ and finite theory Γ

$$\Gamma \vdash_{NM} \varphi \quad \textit{iff} \quad \Gamma \models_T \varphi.$$

Example 2.1

Let $\langle T, \leq \rangle = \langle \mathbb{N}, \geq_{\mathbb{N}} \rangle$: it follows that $\mathcal{A}_T \simeq NM_\infty$. We have that, for each formula φ and each finite theory Γ:

$$\Gamma \vdash_{NM} \varphi \quad \textit{iff} \quad \Gamma \models_{\mathcal{A}_T} \varphi.$$

Consider now $\langle T, \leq \rangle \in \{\langle \mathbb{R}, \leq_{\mathbb{R}} \rangle, \langle \mathbb{Q}, \leq_{\mathbb{Q}} \rangle\}$: it follows that $\mathcal{A}_T \simeq [0,1]_{NM}$ or $\mathcal{A}_T \simeq [0,1]_{NM}^{\mathbb{Q}}$. It can be show that for each formula φ and theory Γ:

$$\Gamma \vdash_{NM} \varphi \quad \textit{iff} \quad \Gamma \models_T \varphi.$$

- The basic idea is to show that every temporal flow $\langle T, \leq \rangle$ can induce an algebraic structure $\langle T', \leq' \rangle$ that is isomorphic to an infinite NM-chain with negation fixpoint.

Completeness proof, main concepts

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- Viceversa, from the lattice reduct of an NM-chain with negation fixpoint $\mathcal{A}$, we can obtain an algebraic structure of the form $\langle T', \leq' \rangle$, isomorphic to $\mathcal{A}$ associated to some temporal flow $\langle T, \leq \rangle$.

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- The main point is that we can find a bijection between the assignments over $\langle T, \leq\rangle$ and $\langle T', \leq'\rangle$, preserving the validity. That is, for every theory Γ, and formula φ

$$\Gamma \models_{\langle T, \leq\rangle} \varphi \qquad \text{iff} \qquad \Gamma \models_{\langle T', \leq'\rangle} \varphi.$$

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- Since NM enjoys the finite strong completeness w.r.t. every NM-chain with negation fixpoint, we obtain the completeness theorem w.r.t. our temporal semantics.
- The fact that NM enjoys the strong completeness w.r.t. the temporal flows $\langle \mathbb{R}, \leq_{\mathbb{R}} \rangle$ and $\langle \mathbb{Q}, \leq_{\mathbb{Q}} \rangle$ is essentially due to the strong completeness theorem of NM w.r.t. $[0,1]_{\mathrm{NM}}$, and $[0,1]^{\mathbb{Q}}_{\mathrm{NM}}$.

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APPENDIX

Definition 9

Let $\mathcal{A}$ be an NM-algebra. Each map $e\colon\ VAR \to A$ extends uniquely to an $\mathcal{A}$-*assignment* $v_e:\ FORM \to A$, by the following inductive prescriptions:

- $v_e(\perp) = 0$
- $v_e(\varphi \to \psi) = v_e(\varphi) \Rightarrow v_e(\psi)$
- $v_e(\varphi \& \psi) = v_e(\varphi) * v_e(\psi)$
- $v_e(\varphi \wedge \psi) = v_e(\varphi) \sqcap v_e(\psi)$

back

Completeness

Let $\mathcal{A}$ be an NM-chain. We say that NM is strongly complete (respectively: finitely strongly complete, complete) with respect to $\mathcal{A}$ if for every theory Γ (respectively, for every finite theory Γ of formulas, for $\Gamma = \emptyset$) and for every formula φ we have

$$\Gamma \vdash_{\mathrm{NM}} \varphi \quad \text{iff} \quad \Gamma \models_{\mathcal{A}} \varphi$$

Theorem 10 ([EG01, Gis03])

Let $\mathcal{A}$ be an infinite NM-chain with negation fixpoint. Then NM is complete w.r.t. $\mathcal{A}$.

This result can be improved:

Theorem 11

Let $\mathcal{A}$ be an infinite NM-chain with negation fixpoint. Then NM is finitely strongly complete w.r.t. $\mathcal{A}$.

back

Note that, by analyzing the question from a different perspective, a temporal assignment is a function that associates to every formula a certain sequence (indexed by the instants of time) of truth-values:

Definition 12

Given a temporal assignment v (over a temporal flow $\langle T, \leq \rangle$), one can define a function $\cdot^v$ from the set of formulas into the set of sequences of $\{0, \frac{1}{2}, 1\}^T$ by

$$\varphi^v := v(\varphi, \cdot).$$

We set $\mathcal{T}_T = \{\varphi^v : \varphi$ is a formula and v is a temporal assignment over $\langle T, \leq \rangle\}$.

Since we are interested in the definitive behavior of a temporal assignment, we now define an operator that "capture" the behavior of an assignment.

Definition 13

Let φ, v be a formula and a temporal assignment over a temporal flow $\langle T, \leq \rangle$, and let $T' = T \cup \{-\infty\}$. The definitive behavior operator $d : \mathcal{T}_T \to T' \times \left\{0, \frac{1}{2}, 1\right\}$ is defined as follows:

The fact that d is a well defined map is assured by Theorem 6.

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- $d(\varphi^v) = \langle -\infty, 1 \rangle$ if $\varphi^v \approx 1$.

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- $d(\varphi^v) = \langle t, 1 \rangle$ if $\varphi^v \approx t_1$.

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The fact that d is a well defined map is assured by Theorem 6.

Remark 4.1

The first component of the pairs $\langle t, i \rangle$ indicates the instant of time in which the function φ^v assumes the "stable" value: this last one (0, $\frac{1}{2}$ or 1) is specified in the second component. This justify the fact that $\langle -\infty, i \rangle$ indicates that the function assumes always the value i.

Completeness I

Definition 14

Let $T' = T \cup \{-\infty\}$. We define a total order relation $\leq_{T''}$, over $T'' = T' \times \{0,1\} \cup \{\langle -\infty, \frac{1}{2} \rangle\}$, as follows:

- for each $t, t' \in T$, with $t < t'$,
  $\langle -\infty, 0 \rangle <_{T''} \langle t, 0 \rangle <_{T''} \langle t', 0 \rangle <_{T''} \langle -\infty, \frac{1}{2} \rangle <_{T''} \langle t', 1 \rangle <_{T''} \langle t, 1 \rangle <_{T''} \langle -\infty, 1 \rangle$.

Now:

Completeness II

Definition 15

For each temporal assignment v (over a temporal flow $\langle T, \leq \rangle$) the function $s^v : FORM \to T''$ has the following behavior:

- $s^v(x_i) = d(x_i^v)$.
- $s^v(\bot) = \langle -\infty, 0 \rangle$.
- If $s^v(\varphi) = \langle a, n \rangle$ and $s^v(\psi) = \langle b, n' \rangle$, then

$$s^v(\neg\varphi) = \langle a, 1 - n \rangle$$

$$s^v(\varphi \to \psi) = \begin{cases} \langle -\infty, 1 \rangle & \text{If } \langle a, n \rangle \leq_{T''} \langle b, n' \rangle \\ s^v(\neg\varphi) \curlyvee s^v(\psi) & \text{Otherwise} \end{cases}$$

Where $\curlyvee$ denotes the maximum over $\leq_{T''}$.

It is immediate to check that

Proposition 4.1

For each formula φ, and temporal assignment v it holds that:

$$s^v(\neg\neg\varphi) = s^v(\varphi).$$

The following theorem shows that Theorem 5 and Theorem 15 are equivalent, from the point of view of the “definitive behavior” of an assignment.

Theorem 16

Let v be a temporal assignment. For every formula φ it holds that

$$s^v(\varphi) = d(\varphi^v).$$

In this section we show that the temporal semantics previously introduced is complete w.r.t. the logic NM.

Proposition 5.1

Given a temporal flow $\langle T, \leq \rangle$ *there is an NM-chain* $\mathcal{A}_T$*, with negation fixpoint* $\langle -\infty, \frac{1}{2} \rangle$*, whose lattice reduct is* $\langle T'', \leq_{T''} \rangle$*.*

Proposition 5.2

Let $\langle T, \leq \rangle$ *be a temporal flow and* φ *be a formula. For every temporal assignment* v *there is an* $\mathcal{A}_T$*-assignment* v' *such that* $s^v(\varphi) = v'(\varphi)$*; conversely, for every* $\mathcal{A}_T$*-assignment* w *there is a temporal assignment* w' *over* $\langle T, \leq \rangle$ *such that* $s^{w'}(\varphi) = w(\varphi)$*.*

Theorem 17

Let $\langle T, \leq \rangle$ be a temporal flow. Then, for every formula φ and theory Γ it holds that

$$\Gamma \models_T \varphi \quad \textit{iff} \quad \Gamma \models_{\mathcal{A}_T} \varphi,$$

where $\Gamma \models_T \varphi$ means that for every temporal assignment v such that $s^v(\psi) = \langle -\infty, 1 \rangle$ for every $\psi \in \Gamma$, it holds that $s^v(\varphi) = \langle -\infty, 1 \rangle$.

We finally obtain

Theorem 18 (Completeness theorem)

Let $\langle T, \leq \rangle$ be a temporal flow. Then for each formula φ and finite theory Γ

$$\Gamma \vdash_{NM} \varphi \quad \textit{iff} \quad \Gamma \models_T \varphi.$$

Now we conclude by giving some examples of temporal flows connected to interesting NM-chains:

Example 5.1

Let $\langle T, \leq \rangle = \langle \mathbb{N}, \geq_{\mathbb{N}} \rangle$*: it follows that* $\mathcal{A}_T \simeq NM_\infty$*. Thanks to* **??** *we have that, for each formula* φ *and each finite theory* Γ*:*

$$\Gamma \vdash_{NM} \varphi \quad \textit{iff} \quad \Gamma \models_{\mathcal{A}_T} \varphi.$$

Consider now $\langle T, \leq \rangle \in \{\langle \mathbb{R}, \leq_{\mathbb{R}} \rangle, \langle \mathbb{Q}, \leq_{\mathbb{Q}} \rangle\}$*: it follows that* $\mathcal{A}_T \simeq [0,1]_{NM}$ *or* $\mathcal{A}_T \simeq [0,1]_{NM}^{\mathbb{Q}}$*. From Theorems 3 and 17, and with an argument similar to the one given in the proof of* **??** *we have that for each formula* φ *and theory* Γ*:*

$$\Gamma \vdash_{NM} \varphi \quad \textit{iff} \quad \Gamma \models_{T} \varphi.$$