A proof theoretical approach to Standard completeness

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Standard Completeness

Completeness of axiomatic systems with respect
to algebras whose lattice reduct is the real interval $[0, 1]$.

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to algebras whose lattice reduct is the real interval $[0, 1]$.

- Intended semantics for *Fuzzy logic* (Hajek 1998)
  - Conjunction interpreted as a *continuous t-norm*: and Implication interpreted as its *residuum*

Examples: standard complete logics

- UL : Logic of Left-continuous uninorms
- MTL: Logic of Left-continuous t-norms
- BL : Logic of Continuous t-norms

Introducing logics Hilbert-style

Logics are usually defined

- *discarding* axioms (enlarges the class of models)
- *adding* axioms (gives stronger logics)

from (Hilbert systems for) other logics.

Introducing logics Hilbert-style

Example : Hilbert system for FL_e

$$\alpha \to \alpha \qquad (\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$$

$$(\alpha \to (\beta \to \gamma)) \to (\beta \to (\alpha \to \gamma)) \qquad ((\alpha \cdot \beta) \to \gamma) \leftrightarrow (\alpha \to (\beta \to \gamma))$$

$$(\alpha \wedge \beta) \to \alpha \qquad (\alpha \wedge \beta) \to \beta$$

$$((\alpha \to \beta) \wedge (\alpha \to \gamma)) \to (\alpha \to (\beta \wedge \gamma)) \qquad \alpha \to (\alpha \vee \beta)$$

$$\beta \to (\alpha \vee \beta) \qquad ((\alpha \to \gamma) \wedge (\beta \to \gamma)) \to ((\alpha \vee \beta) \to \gamma)$$

$$\alpha \leftrightarrow t \to \alpha \qquad \alpha \to \top \quad \bot \to \alpha$$

$$\frac{\alpha \quad \beta}{\alpha \wedge \beta}\ (adj) \qquad \frac{\alpha \quad \alpha \to \beta}{\beta}\ (MP)$$

Introducing logics Hilbert-style

Example : Hilbert system for FL_e

$\alpha \to \alpha$	$(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$
$(\alpha \to (\beta \to \gamma)) \to (\beta \to (\alpha \to \gamma))$	$((\alpha \cdot \beta) \to \gamma) \leftrightarrow (\alpha \to (\beta \to \gamma))$
$(\alpha \wedge \beta) \to \alpha$	$(\alpha \wedge \beta) \to \beta$
$((\alpha \to \beta) \wedge (\alpha \to \gamma)) \to (\alpha \to (\beta \wedge \gamma))$	$\alpha \to (\alpha \vee \beta)$
$\beta \to (\alpha \vee \beta)$	$((\alpha \to \gamma) \wedge (\beta \to \gamma)) \to ((\alpha \vee \beta) \to \gamma)$
$\alpha \leftrightarrow t \to \alpha$	$\alpha \to \top \quad \bot \to \alpha$
$\dfrac{\alpha \quad \beta}{\alpha \wedge \beta}\ (adj)$	$\dfrac{\alpha \quad \alpha \to \beta}{\beta}\ (MP)$

UL = FL_e + $((\alpha \to \beta) \wedge t) \vee ((\beta \to \alpha) \wedge t)$ (*prelinearity*)

MTL = UL + $(f \to \alpha) \wedge (\alpha \to t)$ (*weakening*)

$\dots$

Standard Completeness: algebraic approach

Given a logic L:

1. Algebraic semantics of L (L-algebras), completeness w.r.t. countable, linearly ordered L-algebras (L-chains)
2. (Rational completeness): Embedding of countable L-chains into a dense countable L-chain.
3. Dedekind-Mac Neille style completion

Standard Completeness: algebraic approach

Given a logic L:

1. Algebraic semantics of L (L-algebras), completeness w.r.t. countable, linearly ordered L-algebras (L-chains)
2. (Rational completeness): Embedding of countable L-chains into a dense countable L-chain.
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- Step 1 and 3 well understood.
  - *Semilinear logics*: Classes of (substructural) logics complete w.r.t. chains.
- Step 2: problematic

Standard Completeness via proof theory

(Metcalfe, Montagna JSL 2007)
Given a logic L:

1. Find a suitable hypersequent calculus HL
2. Add the density rule

$$\frac{(\alpha \to p) \vee (p \to \beta) \vee \gamma}{(\alpha \to \beta) \vee \gamma} \ (density)$$

   (= L + $(density)$ is rational complete)
   and prove that this rule produces no new theorems
   (Rational completeness)
3. Dedekind-Mac Neille style completion

UL: state of the art

- Differently from MTL, few algebraic proofs of standard completeness.
  - $UL + \alpha^{n-1} \leftrightarrow \alpha^n$ (San Min Wang 2012)
- Proof-theoretical:
  - (2007 Metcalfe, Montagna): UL, $UL + \alpha \leftrightarrow \alpha \cdot \alpha$

UL: state of the art

- Differently from MTL, few algebraic proofs of standard completeness.
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- Proof-theoretical:
  - (2007 Metcalfe, Montagna): UL, $UL + \alpha \leftrightarrow \alpha \cdot \alpha$

We show standard completeness for some axiomatic extensions of UL, i.e.:

- $UL + (\alpha \to \alpha \cdot \alpha)$
- $UL + (\alpha \cdot \alpha \to \alpha)$
- $UL + \alpha^k \to \alpha^n$

Our basic calculus FL_e: sequent calculus

$$\frac{}{\Rightarrow t}\ (tr) \qquad \frac{}{\alpha \Rightarrow \alpha}\ (init) \qquad \frac{}{f \Rightarrow}\ (fl)$$

$$\frac{}{\Gamma \Rightarrow \top}\ (\top) \qquad \frac{}{\Gamma, \bot \Rightarrow \Delta}\ (\bot)$$

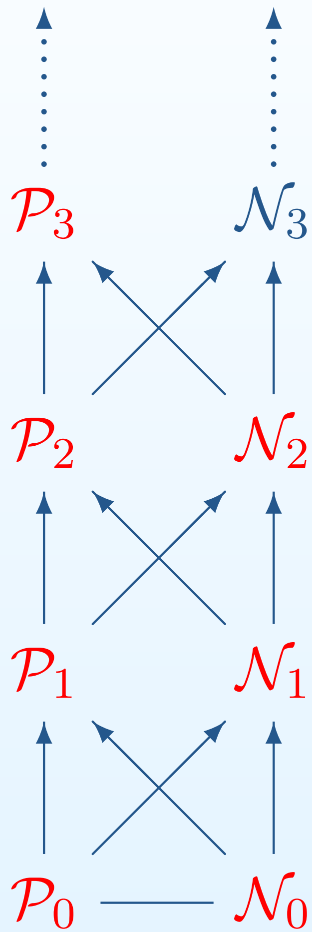
$$\frac{\Gamma \Rightarrow \alpha \quad \alpha, \Delta \Rightarrow \Pi}{\Gamma, \Delta \Rightarrow \Pi}\ (Cut) \qquad \frac{\Gamma \Rightarrow \Pi}{t, \Gamma \Rightarrow \Pi}\ (tl) \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow f}\ (fr)$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta}\ (\wedge r) \qquad \frac{\alpha_i, \Gamma \Rightarrow \Pi}{\alpha_1 \wedge \alpha_2, \Gamma \Rightarrow \Pi}\ (\wedge l) \qquad \frac{\Gamma \Rightarrow \alpha_i}{\Gamma \Rightarrow \alpha_1 \vee \alpha_2}\ (\vee r)$$

$$\frac{\alpha, \Gamma \Rightarrow \Pi \quad \beta, \Gamma \Rightarrow \Pi}{\alpha \vee \beta, \Gamma \Rightarrow \Pi}\ (\vee l) \qquad \frac{\Gamma \Rightarrow \alpha \quad \beta, \Delta \Rightarrow \Pi}{\Gamma, \alpha \rightarrow \beta, \Delta \Rightarrow \Pi}\ (\rightarrow l) \qquad \frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \rightarrow \beta}\ (\rightarrow r)$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Delta \Rightarrow \beta}{\Gamma, \Delta \Rightarrow \alpha \cdot \beta}\ (\cdot r) \qquad \frac{\alpha, \beta, \Gamma \Rightarrow \Pi}{\alpha \cdot \beta, \Gamma \Rightarrow \Pi}\ (\cdot l)$$

Calculi for (semilinear) extensions of FL_e?



(Ciabattoni, Galatos, Terui 2008).
Sets $\mathcal{P}_n, \mathcal{N}_n$ of formulas defined by:

$\mathcal{P}_0, \mathcal{N}_0 :=$ Atomic formulas

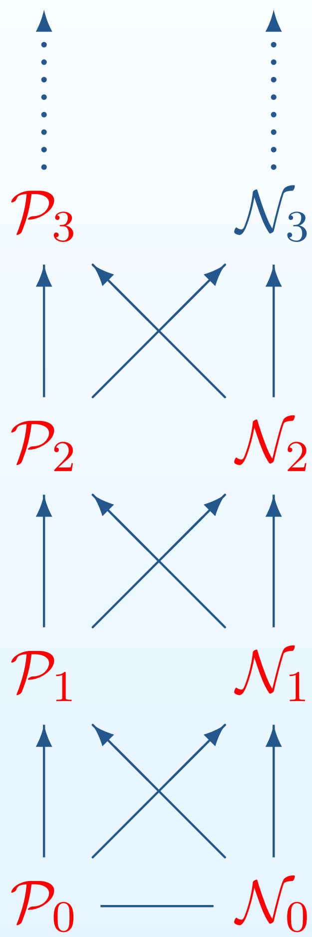
$$\mathcal{P}_{n+1} := \mathcal{N}_n \mid \mathcal{P}_{n+1} \cdot \mathcal{P}_{n+1} \mid \mathcal{P}_{n+1} \vee \mathcal{P}_{n+1} \mid 1 \mid \bot$$

$$\mathcal{N}_{n+1} := \mathcal{P}_n \mid \mathcal{P}_{n+1} \rightarrow \mathcal{N}_{n+1} \mid \mathcal{N}_{n+1} \wedge \mathcal{N}_{n+1} \mid 0 \mid \top$$

Examples:

- To the class $\mathcal{N}_2$ belong :
  $\alpha \rightarrow \alpha \cdot \alpha \qquad \alpha \cdot \alpha \rightarrow \alpha$
- To the class $\mathcal{P}_3$ belong :
  $\neg\alpha \vee \neg\neg\alpha \qquad ((\alpha \rightarrow \beta) \wedge t) \vee ((\beta \rightarrow \alpha) \wedge t)$

Calculi for (semilinear) extensions of FL_e?



Algorithm to convert axioms into “good” rules, preserving cut-elimination.

- Axioms in $\mathcal{N}_2 \Rightarrow$ Sequent structural rules
- Axioms in (subclass of) $\mathcal{P}_3 \Rightarrow$ Hypersequent structural rules
- ...?

Hypersequent calculus HUL for UL

(Avron '89): Hypersequent

$$\Gamma_1 \Rightarrow \Pi_1 \mid \ldots \mid \Gamma_n \Rightarrow \Pi_n$$

where for all $i = 1, \ldots n$, $\Gamma_i \Rightarrow \Pi_i$ is an ordinary sequent
$\mid$ is intended to denote a meta-level disjunction.

Hypersequent calculus HUL for UL

Embedd sequent rules for FL_e into hypersequents

$$\frac{}{G|\Rightarrow t}\ (tr) \qquad \frac{}{G|\alpha \Rightarrow \alpha}\ (init) \qquad \frac{}{G|f \Rightarrow}\ (fl)$$

$$\frac{}{G|\Gamma \Rightarrow \top}\ (\top) \qquad \frac{}{G|\Gamma, \bot \Rightarrow \Delta}\ (\bot)$$

$$\frac{G|\Gamma \Rightarrow \alpha \quad G|\alpha, \Delta \Rightarrow \Pi}{G|\Gamma, \Delta \Rightarrow \Pi}\ (Cut) \qquad \frac{G|\Gamma \Rightarrow \Pi}{G|t, \Gamma \Rightarrow \Pi}\ (tl) \qquad \frac{G|\Gamma \Rightarrow}{G|\Gamma \Rightarrow f}\ (fr)$$

$$\frac{G|\Gamma \Rightarrow \alpha \quad G|\Gamma \Rightarrow \beta}{G|\Gamma \Rightarrow \alpha \wedge \beta}\ (\wedge r) \qquad \frac{G|\alpha_i, \Gamma \Rightarrow \Pi}{G|\alpha_1 \wedge \alpha_2, \Gamma \Rightarrow \Pi}\ (\wedge l) \qquad \frac{G|\Gamma \Rightarrow \alpha_i}{G|\Gamma \Rightarrow \alpha_1 \vee \alpha_2}\ (\vee r)$$

$$\frac{G|\alpha, \Gamma \Rightarrow \Pi \quad G|\beta, \Gamma \Rightarrow \Pi}{G|\alpha \vee \beta, \Gamma \Rightarrow \Pi}\ (\vee l) \qquad \frac{G|\Gamma \Rightarrow \alpha \quad G|\beta, \Delta \Rightarrow \Pi}{G|\Gamma, \alpha \to \beta, \Delta \Rightarrow \Pi}\ (\to l) \qquad \frac{G|\alpha, \Gamma \Rightarrow \beta}{G|\Gamma \Rightarrow \alpha \to \beta}\ (\to r)$$

$$\frac{G|\Gamma \Rightarrow \alpha \quad G|\Delta \Rightarrow \beta}{G|\Gamma, \Delta \Rightarrow \alpha \cdot \beta}\ (\cdot r) \qquad \frac{G|\alpha, \beta, \Gamma \Rightarrow \Pi}{G|\alpha \cdot \beta, \Gamma \Rightarrow \Pi}\ (\cdot l)$$

Hypersequent calculus HUL for UL

We add:

- Suitable rules to manipulate the additional layer of structure.

$$\frac{G}{G \mid \Gamma \Rightarrow \alpha}\ (\mathrm{ew}) \qquad \frac{G \mid \Gamma \Rightarrow \alpha \mid \Gamma \Rightarrow \alpha}{G \mid \Gamma \Rightarrow \alpha}\ (\mathrm{ec})$$

Hypersequent calculus HUL for UL

We add:

- Suitable rules to manipulate the additional layer of structure.

$$\frac{G}{G \,|\, \Gamma \Rightarrow \alpha} \text{ (ew)} \qquad \frac{G \,|\, \Gamma \Rightarrow \alpha \,|\, \Gamma \Rightarrow \alpha}{G \,|\, \Gamma \Rightarrow \alpha} \text{ (ec)}$$

- A hypersequent structural rule corresponding to prelinearity :

$$((\alpha \to \beta) \wedge t) \vee ((\beta \to \alpha) \wedge t)$$

The axiom is in class $\mathcal{P}_3$, using the algorithm we get:

$$\frac{G \,|\, \Gamma_1, \Delta_1 \Rightarrow \Pi_1 \quad G \,|\, \Gamma_2, \Delta_2 \Rightarrow \Pi_2}{G \,|\, \Gamma_1, \Gamma_2 \Rightarrow \Pi_1 \,|\, \Delta_1, \Delta_2 \Rightarrow \Pi_2} \text{ (com)}$$

Correspondence axioms - rules

Class	Axiom	Rule
$\mathcal{N}_2$	$(\alpha \to t) \wedge (f \to \alpha)$	$\dfrac{G \mid \Pi \Rightarrow \Psi}{G \mid \Pi, \alpha \Rightarrow \Psi}\ (wl) \qquad \dfrac{G \mid \Pi \Rightarrow}{G \mid \Pi \Rightarrow \alpha}\ (wr)$
	$\alpha \to \alpha \cdot \alpha$	$\dfrac{G_1 \vert \Pi, \Gamma, \Gamma \Rightarrow \Psi}{G_1 \vert \Pi, \Gamma \Rightarrow \Psi}\ (c)$
	$\alpha \cdot \alpha \to \alpha$	$\dfrac{G_1 \vert \Pi, \Gamma_1 \Rightarrow \Psi \quad G_1 \vert \Pi, \Gamma_2 \Rightarrow \Psi}{G_1 \vert \Pi, \Gamma_1, \Gamma_2, \Rightarrow \Psi}\ (mgl)$
	$\alpha^k \to \alpha^n$	$\dfrac{G \vert \Pi, \Gamma_1^n \Rightarrow \Psi \quad \ldots \quad G_1 \vert \Pi, \Gamma_k^n \Rightarrow \Psi}{G_1 \vert \Pi, \Gamma_1, \ldots \Gamma_k \Rightarrow \Psi}\ (knot_k^n)$
$\mathcal{P}_2$	$\alpha \vee \neg\alpha$	$\dfrac{G \mid \Pi, \Gamma \Rightarrow \Psi}{G \mid \Gamma \Rightarrow \ \mid \Pi \Rightarrow \Psi}$ *(em)*
$\mathcal{P}_3$	$\neg\alpha \vee \neg\neg\alpha$	$\dfrac{G \mid \Gamma_1, \Gamma_2 \Rightarrow}{G \mid \Gamma_1 \Rightarrow \ \mid \Gamma_2 \Rightarrow}$ *(lq)*

Correspondence axioms - rules

Class	Axiom	Rule
$\mathcal{N}_2$	$(\alpha \to t) \wedge (f \to \alpha)$	$\dfrac{G \mid \Pi \Rightarrow \Psi}{G \mid \Pi, \alpha \Rightarrow \Psi}\ (wl) \quad \dfrac{G \mid \Pi \Rightarrow}{G \mid \Pi \Rightarrow \alpha}\ (wr)$
	$\alpha \to \alpha \cdot \alpha$	$\dfrac{G_1 \vert \Pi, \Gamma, \Gamma \Rightarrow \Psi}{G_1 \vert \Pi, \Gamma \Rightarrow \Psi}\ (c)$
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	$\alpha^k \to \alpha^n$	$\dfrac{G \vert \Pi, \Gamma_1^n \Rightarrow \Psi \quad \ldots \quad G_1 \vert \Pi, \Gamma_k^n \Rightarrow \Psi}{G_1 \vert \Pi, \Gamma_1, \ldots \Gamma_k \Rightarrow \Psi}\ (knot_k^n)$

- We will focus on extensions of UL with axioms in $\mathcal{N}_2 \Longrightarrow$ Extensions of HUL with sequent structural rules.

Recall: Standard Completeness via proof theory

Given a logic L:

1. A suitable hypersequent calculus HL
2. Density elimination
3. Dedekind-Mac Neille style completion

Density elimination

- Density rule in hypersequent calculus :

$$\frac{G \mid \Lambda \Rightarrow p \mid \Sigma, p \Rightarrow \Delta}{G \mid \Lambda, \Sigma \Rightarrow \Delta} \ (density)$$

  where p does not occur in the conclusion (*eigenvariable*). Similar to cut elimination

$$\frac{G \mid \Lambda \Rightarrow \alpha \quad G \mid \Sigma, \alpha \Rightarrow \Delta}{G \mid \Lambda, \Sigma \Rightarrow \Delta} \ (cut)$$

- Proof by induction on the length of derivations

Density elimination

(Ciabattoni, Metcalfe TCS 2008)

Given a density-free derivation, ending in

$$\dfrac{\begin{array}{c}\vdots\; d' \\ G \mid \Lambda \Rightarrow p \mid \Sigma, p \Rightarrow \Delta\end{array}}{G \mid \Lambda, \Sigma \Rightarrow \Delta}\ \text{(density)}$$

Density elimination

(Ciabattoni, Metcalfe TCS 2008)

$$\dfrac{\begin{array}{c}\vdots\, d' \\ G \mid \Lambda, \Sigma \Rightarrow \Delta \mid \Sigma, \Lambda \Rightarrow \Delta\end{array}}{G \mid \Lambda, \Sigma \Rightarrow \Delta}\ (\mathrm{EC})$$

- Asymmetric substitution: p is replaced
  - With $\Sigma \Rightarrow \Delta$ when occurring on the right
  - With Λ when occurring on the left
- Problem: An axiom $p \Rightarrow p$ would be converted into $\Lambda, \Sigma \Rightarrow \Delta$...not an axiom anymore!

- **Theorem.**(Ciabattoni, Metcalfe 2008) Each calculus extending HUL with *premise-balanced rules* admits density elimination.
  - Idea: substitute components

$$\Pi, p^k \Rightarrow p \longrightarrow \Pi, \Lambda^{k-1} \Rightarrow t$$

(The axiom $p \Rightarrow p$ becomes the axiom $\Rightarrow t$).

- **Theorem.**(Ciabattoni, Metcalfe 2008) Each calculus extending HUL with *premise-balanced rules* admits density elimination.

Premise-balanced rules are rules which do not change the number of metavariables occurrences...none of the structural rules we consider are such

Class	Axiom	Rule
$\mathcal{N}_2$	$\alpha \to \alpha \cdot \alpha$	$\dfrac{G_1 \mid \Pi, \Gamma, \Gamma \Rightarrow \Psi}{G_1 \mid \Pi, \Gamma \Rightarrow \Psi}\ (c)$
	$\alpha \cdot \alpha \to \alpha$	$\dfrac{G_1 \mid \Pi, \Gamma_1 \Rightarrow \Psi \quad G_1 \mid \Pi, \Gamma_2 \Rightarrow \Psi}{G_1 \mid \Pi, \Gamma_1, \Gamma_2, \Rightarrow \Psi}\ (mgl)$
	$\alpha^k \to \alpha^n$	$\dfrac{G \mid \Pi, \Gamma_1^n \Rightarrow \Psi \quad \ldots \quad G_1 \mid \Pi, \Gamma_k^n \Rightarrow \Psi}{G_1 \mid \Pi, \Gamma_1, \ldots \Gamma_k \Rightarrow \Psi}\ (knot_k^n)$

Example

- Idea: substitute components

$$\Pi, p^k \Rightarrow p \longrightarrow \Pi, \Lambda^{k-1} \Rightarrow t$$

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Consider the application of a non balanced rule:

$$\frac{\Pi, p^3 \Rightarrow p \quad \Pi, p^3 \Rightarrow p}{\Pi, p^2 \Rightarrow p} \ (knot_2^3)$$

Can we get:

$$\frac{\Pi, \Lambda^2 \Rightarrow t \quad \Pi, \Lambda^2 \Rightarrow t}{\Pi, \Lambda \Rightarrow t} \ ?$$

A first result

- We find a class of nonbalanced structural rules for which density elimination works with the same substitution.
  - Includes $(knot_k^n)$ for $n, k \neq 1$.

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- We find a class of nonbalanced structural rules for which density elimination works with the same substitution.
  - Includes $(knot_k^n)$ for $n, k \neq 1$.

$$\frac{\Pi, p^3 \Rightarrow p \quad \Pi, p^3 \Rightarrow p}{\Pi, p^2 \Rightarrow p} \; (knot_2^3)$$

can be restructured into a derivation

$$\begin{array}{c} \Pi, \Lambda^2 \Rightarrow t \quad \Pi, \Lambda^2 \Rightarrow t \\ \vdots \\ \Pi, \Lambda \Rightarrow t \end{array}$$

Contraction and mingle

- The method does not work for rules of kind $(knot_1^n)$ and $(knot_k^1)$. In HUL all these rules turn out to be equivalent to contraction and mingle, respectively.

Contraction and mingle

- The method does not work for rules of kind $(knot_1^n)$ and $(knot_k^1)$. In HUL all these rules turn out to be equivalent to contraction and mingle, respectively.

We can show anyway that $HUL + (c)$ and $HUL + (mgl)$ admit density elimination.

$$\frac{G_1|\Pi, \Gamma, \Gamma \Rightarrow \Psi}{G_1|\Pi, \Gamma \Rightarrow \Psi}\ (c)$$

$$\frac{G_1|\Pi, \Gamma_1 \Rightarrow \Psi \quad G_1|\Pi, \Gamma_2 \Rightarrow \Psi}{G_1|\Pi, \Gamma_1, \Gamma_2, \Rightarrow \Psi}\ (mgl)$$

A new approach: Proof by cases

Consider a density-free derivation, ending in

$$\begin{array}{c} \vdots\, d' \\ G \mid \Lambda \Rightarrow {\color{red}p} \mid \Sigma, {\color{red}p} \Rightarrow \Delta \end{array}$$

A new approach: Proof by cases

We instantiate p with t, obtaining

$$\begin{array}{c} \vdots\, d' \\ G \mid \Lambda \Rightarrow {\color{red}t} \mid \Sigma, {\color{red}t} \Rightarrow \Delta \end{array}$$

A new approach: Proof by cases

$$\dfrac{\vdots\, d'}{G \mid \Lambda \Rightarrow t \mid \Sigma, t \Rightarrow \Delta}$$

We find density free proofs of:

$$\begin{array}{c} G|\Lambda \Rightarrow t \\ \vdots\, d_1 \\ G|\Lambda, \Sigma \Rightarrow \Delta \end{array} \qquad \begin{array}{c} G|\Sigma, t \Rightarrow \Delta \\ \vdots\, d_2 \\ G|\Lambda, \Sigma \Rightarrow \Delta \end{array}$$

A new approach: Proof by cases

$$\begin{array}{c} \vdots\, d' \\ G \mid \Lambda \Rightarrow \color{red}{t} \mid \Sigma, \color{red}{t} \Rightarrow \Delta \end{array}$$

We find density free proofs of:

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Finally combining :

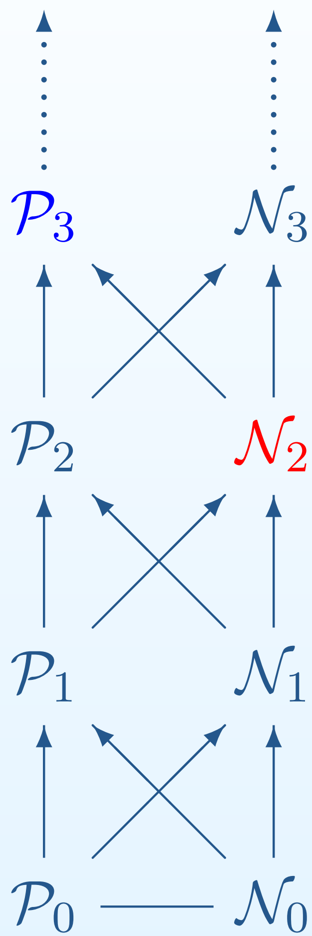
$$\dfrac{\begin{array}{c} \vdots\, d' \\ G|\Lambda \Rightarrow t|\Sigma, t \Rightarrow \Delta \\ \vdots\, d_1 \\ G|\Lambda, \Sigma \Rightarrow \Delta|\Sigma, t \Rightarrow \Delta \\ \vdots\, d_2 \\ G|\Lambda, \Sigma \Rightarrow \Delta|\Lambda, \Sigma \Rightarrow \Delta \end{array}}{G|\Lambda, \Sigma \Rightarrow \Delta}\,(EC)$$

Recall: Standard Completeness via proof theory

Given a logic L:

1. A suitable hypersequent calculus HL
2. Density elimination
3. Dedekind-Mac Neille style completion

Step 3: closure under order-theoretic completions



(Ciabattoni, Terui, Galatos 2011) Axioms on FL $\leftrightarrow$ equations over residuated lattices

- A subclass of equations in class $\mathcal{N}_2$ are preserved by Dedekind-MacNeille completion. All the axioms we considered are in this class.
- A subclass of equations in class $\mathcal{P}_3$ are preserved by Dedekind-MacNeille completion, when applied to subdirectly irreducible algebras

Standard completeness for extensions of UL

- Standard completeness for extensions of UL with axioms belonging to a subclass of $\mathcal{N}_2$. In particular:
  - $UL + \alpha^k \rightarrow \alpha^n$ standard complete, for any n, k (includes mingle and contraction axioms). (Baldi 2013 - submitted for publication), (2013 Baldi, Ciabattoni - work in progress)

Standard completeness for extensions of MTL

- $MTL = UL + (f \to \alpha) \wedge (\alpha \to t)$
- Hypersequent calculus $HMTL = HUL + (wl) + (wr)$

Standard completeness for extensions of MTL

- Density Elimination holds for $HMTL$ extended with any structural sequent rule
  - Any axiomatic extension of MTL with axioms within $\mathcal{N}_2$ is standard complete (2008 Ciabattoni, Metcalfe).
- Density elimination holds for extensions of $HMTL$ with structural hypersequent rules which do not "mix too much" the components (convergent rules)
  - Any axiomatic extension of MTL with axioms within a subclass of $\mathcal{P}_3$ is standard complete (2012 Baldi, Ciabattoni, Spendier) .

Examples of convergent rules

- Axioms in $\mathcal{P}_3$ extending MTL
  - (wnm) :

$$\neg(\alpha \cdot \beta) \vee (\alpha \wedge \beta) \rightarrow (\alpha \cdot \beta)$$

  - (lq) :

$$\neg\alpha \vee \neg\neg\alpha$$

Examples of convergent rules

- Axioms in $\mathcal{P}_3$ extending MTL
  - (wnm) :
$$\neg(\alpha \cdot \beta) \vee (\alpha \wedge \beta) \to (\alpha \cdot \beta)$$
  - (lq) :
$$\neg\alpha \vee \neg\neg\alpha$$
- Corresponding convergent rules
  - $$\frac{\begin{array}{cc} G \mid \Gamma_2, \Gamma_1, \Delta_1 \Rightarrow \Pi_1 & G \mid \Gamma_1, \Gamma_3, \Delta_1 \Rightarrow \Pi_1 \\ G \mid \Gamma_1, \Gamma_1, \Delta_1 \Rightarrow \Pi_1 & G \mid \Gamma_2, \Gamma_3, \Delta_1 \Rightarrow \Pi_1 \end{array}}{G \mid \Gamma_2, \Gamma_3 \Rightarrow \ \mid \Gamma_1, \Delta_1 \Rightarrow \Pi_1} \ (wnm)$$
  - $$\frac{G \mid \Gamma_1, \Gamma_2 \Rightarrow}{G \mid \Gamma_1 \Rightarrow \ \mid \Gamma_2 \Rightarrow} \ (lq)$$

Our results

- Standard completeness for extensions of UL with axioms belonging to a subclass of $\mathcal{N}_2$. In particular:
  - $UL + \alpha^k \to \alpha^n$ standard complete, for any n, k (includes mingle and contraction axioms). (Baldi 2013 - submitted for publication), (2013 Baldi, Ciabattoni - work in progress)
- Standard completeness for extensions MTL:
  - Any axiomatic extension of MTL with axioms within a subclass of $\mathcal{P}_3$ is standard complete (2012 Baldi, Ciabattoni, Spendier) .

Work in progress

- A general characterization of density elimination, hence standard completeness, for:
    - Extensions of MTL with axioms up to the class $\mathcal{P}_3$ in the substructural hierarchy.
    - Extensions of UL with axioms up to the class $\mathcal{N}_2$ in the substructural hierarchy.
    - Extension of noncommutative variants of MTL and UL.
    - Logics with involutive negation. Long standing open problem: IUL
- Treatment of axioms beyond the class $\mathcal{P}_3$ (Display calculus?)

Appendix A: A class of structural rules

Let HL be HUL extended with any structural sequent rule

$$\frac{G_1|\Pi_1, \Psi_1 \Rightarrow \Delta_1 \quad \ldots \Pi_1, \Psi_m \Rightarrow \Delta_1}{G_1|\Pi_1, \Gamma_1, \ldots \Gamma_k \Rightarrow \Delta_1}\ (r)$$

HL admits density elimination if (r) satisfies the following:

- Each Ψ_i is a multiset $\{\Gamma_{i_1}, \ldots, \Gamma_{i_{n_i}}\}$ with $i_1 \ldots i_{n_i}$ varying over $\{1, \ldots k\}$
- Either the minimum among the n_i is bigger than k or the maximum is smaller than k
- For any Γ_i there is at least one Ψ_j where Γ_i does not appear.
- For any Γ_i there is at least one Ψ_j where Γ_i appears more then once .

Appendix B: Convergent rules

Definition. Let (r) be a hypersequent structural rule with $G|S_i$, $i \in \{1, ..m\}$ premises, $C_1|...|C_q$ conclusion.

(0-pivot) $G|S_i$ is a 0-*pivot* if there is an $s \in \{1, \ldots, q\}$ such that $R(S_i) = R(C_s)$ and metavariables in $L(S_i)$ are contained in $L(C_s)$.

(n-pivot) $G|S_j$ is an n-*pivot for* $G|S_i$ with respect to $[\Delta_k/_{\Gamma_k}]_{k\in\{1,\ldots,n\}}$, with $\Gamma_k \in L(S_i)$ and $\Delta_k \in L(S_j)$, if the following conditions hold:

- $G|S_j$ is a 0-*pivot*
- $R(S_i) = R(S_j)$,
- $L(S_j) = L(\ S_i[^{\Delta_k}/_{\Gamma_k}]^l_{k\in\{1,\ldots,n\}}\ )$,
- If $n > 1$, $G|S_j$ is a $(n-1)$-*pivot* for n premises $G|S_{j_p}$, $p = 1, \ldots, n$, with respect to $[^{\Delta_k}/_{\Gamma_k}]_{k\in\{1,\ldots,n\}\setminus\{p\}}$.

Definition. A completed hypersequent rule (r) is *convergent* if for each premise $G|S_i$ one of the following conditions holds:

- $R(S_i) = \emptyset$,
- $G|S_i$ is a 0-*pivot*
- there is a premise $G|S_j$ which is an n-*pivot* for $G|S_i$, with $n > 0$.