# On worst-case GMRES

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joint work with

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# Outline


1. Introduction
2. Ideal GMRES
3. Worst-case GMRES
4. Ideal versus worst-case GMRES

# GMRES, Worst-case GMRES and Ideal GMRES

$\mathbf{A}x = b$, $\mathbf{A} \in \mathbb{C}^{n\times n}$ is nonsingular, $b \in \mathbb{C}^n$,

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GMRES computes $x_k \in \mathcal{K}_k(\mathbf{A}, b)$ such that $r_k \equiv b - \mathbf{A}x_k$ satisfies

$$\|r_k\| = \min_{p \in \pi_k} \|p(\mathbf{A})b\| \qquad \text{(GMRES)}$$

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$$\leq \max_{\|b\|=1} \min_{p \in \pi_k} \|p(\mathbf{A})b\| \equiv \psi_k(\mathbf{A}) \qquad \text{(worst-case GMRES)}$$

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&\leq \min_{p\in\pi_k} \|p(\mathbf{A})\| \equiv \varphi_k(\mathbf{A}) && \text{(ideal GMRES)}
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&\le 1\,.
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$$

# Questions

$$\|r_k\| \;\le\; \underbrace{\max_{\|b\|=1}\min_{p\in\pi_k}\|p(\mathbf{A})b\|}_{\psi_k(\mathbf{A})} \;\le\; \underbrace{\min_{p\in\pi_k}\|p(\mathbf{A})\|}_{\varphi_k(\mathbf{A})}$$

- **Relationship** between ideal and worst case GMRES?
- **Characterization** of solutions? Understanding?
- Existence and **uniqueness** of the solution?
- How to approximate ideal/worst-case quantities?

# Normal matrices

$$\mathbf{A} = \mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^*, \quad \mathbf{Q}^*\mathbf{Q} = \mathbf{I}\,.$$

- [Greenbaum, Gurvits '94; Joubert '94] showed:

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- Which (known) approximation problem is solved?

$$\min_{p\in\pi_k} \|p(\mathbf{A})\| \;=\; \min_{p\in\pi_k} \|\mathbf{Q}p(\mathbf{\Lambda})\mathbf{Q}^*\| \;=\; \min_{p\in\pi_k} \max_{\lambda_i} |p(\lambda_i)|\,.$$

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- How to approximate ideal/worst-case quantities?

[Greenbaum '79; Liesen, T. '04]

# Nonnormal matrices – Toh's example

$$\|r_k\| \leq \underbrace{\max_{\|b\|=1} \min_{p\in\pi_k} \|p(\mathbf{A})b\|}_{\psi_k(\mathbf{A})} \leq \underbrace{\min_{p\in\pi_k} \|p(\mathbf{A})\|}_{\varphi_k(\mathbf{A})}$$

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Consider the $4$ by $4$ matrix

$$\mathbf{A} \;=\; \begin{bmatrix} 1 & \epsilon & & \\ & -1 & \epsilon^{-1} & \\ & & 1 & \epsilon \\ & & & -1 \end{bmatrix}, \qquad \epsilon > 0\,.$$

Then, for $k = 3$,

$$0 \;\stackrel{\epsilon\to 0}{\longleftarrow}\; \psi_k(\mathbf{A}) \;<\; \varphi_k(\mathbf{A}) \;=\; \frac{4}{5}\,.$$

[Toh '97; another example in Faber, Joubert, Knill, Manteuffel '96]

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# Uniqueness

Let $\mathbf{A}$ be a nonsingular matrix. Then the $k$th ideal GMRES polynomial $p_* \in \pi_k$ that solves the problem

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Generalization to

$$\min_{p \in \mathcal{P}_m} \| \, f(\mathbf{A}) - p(\mathbf{A}) \, \|$$

can be found in [Liesen, T. '09] .

# Matrix approximation problems in spectral norm

## and characterization of Ideal GMRES

Ideal GMRES is a special case of the problem

$$\min_{\mathbf{M} \in \mathbb{A}} \| \mathbf{B} - \mathbf{M} \| = \| \mathbf{B} - \mathbf{A}_* \|$$

$\mathbf{A}_*$ is called a spectral approximation of $\mathbf{B}$ from the subspace $\mathbb{A}$.

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In our case,

$$\min_{p\in\pi_k} \|p(\mathbf{A})\| = \min_{\alpha_i\in\mathbb{C}} \left\| \mathbf{I} - \sum_{j=1}^{k} \alpha_j \mathbf{A}^j \right\| ,$$

i.e. $\mathbf{B} = \mathbf{I}$, $\mathbb{A} = \mathrm{span}\{\mathbf{A}, \ldots, \mathbf{A}^k\}$.

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General characterization by [Lau and Riha, 1981] and [Ziętak, 1993, 1996]
$\rightarrow$ based on the Singer's theorem [Singer, 1970].

# Singer's theorem

**Theorem** [Singer, 1970]

Let $(\mathcal{V}, \|\cdot\|)$ be a **normed vector space**, $\mathcal{X} \subset \mathcal{V}$ a $k$-dimensional linear subspace of $\mathcal{V}$, and $y \in \mathcal{V} \backslash \mathcal{X}$. $x_* \in \mathcal{X}$ is a **best approximation** of $y$ **iff** there exists $\ell$ **extremal points** $f_1, \ldots, f_\ell$ of $\Omega^*$, where $1 \leq \ell \leq k+1$ if the scalars are real and $\ell$ numbers $\omega_1, \ldots, \omega_\ell > 0$, with $\omega_1 + \cdots + \omega_\ell = 1$, such that

$$\begin{aligned} \sum_{i=1}^{\ell} \omega_i \, f_i(x) &= 0\,, \qquad \forall x \in \mathcal{X}, \\ f_i(y - x_*) &= \|y - x_*\|\,, \qquad i = 1, \ldots, \ell\,. \end{aligned}$$

Specialized for matrices by [Lau and Riha, 1981].

# Characterization of Ideal GMRES

by Faber, Joubert, Knill, Manteuffel '96

Given a polynomial $q \in \pi_k$ and $\mathbf{A}$, define the set

$$\Omega_k(q) \equiv \left\{ \begin{bmatrix} w^* \, q(\mathbf{A})^* \mathbf{A} \, w \\ \vdots \\ w^* \, q(\mathbf{A})^* \mathbf{A}^k \, w \end{bmatrix} : \; w \in \Sigma(q(\mathbf{A})), \; \|w\| = 1 \right\}$$

where $\Sigma(\mathbf{B})$ is the span of maximal right singular vectors of $\mathbf{B}$.

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**Theorem** [Faber, Joubert, Knill, Manteuffel '96]

$h \in \pi_k$ is the $k$th ideal GMRES pol. of $\mathbf{A}$ $\iff$ $\mathbf{0} \in \mathrm{cvx}(\Omega_k(h))$.

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This characterization was derived without using Singer's theorem.

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# Worst-case GMRES

For a given $k$, there exists a right hand side $b$ such that

$$\|r_k\| = \min_{p\in\pi_k} \|p(\mathbf{A})b\| = \max_{\|v\|=1} \min_{p\in\pi_k} \|p(\mathbf{A})v\|$$

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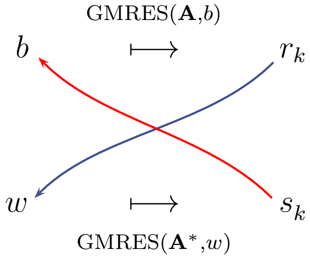
**Theorem** [Zavorin '02; Faber, T., Liesen '12]

Let $\mathbf{A} \in \mathbb{C}^{n \times n}$ be a nonsingular matrix. Then GMRES achieves the same worst-case behavior for $\mathbf{A}$ and $\mathbf{A}^*$ at every iteration.

- [Zavorin '02] $\rightarrow$ only for diagonalizable matrices.
- [Faber, T., Liesen '12] $\rightarrow$ for all nonsingular matrices.

# Cross equality for worst-case GMRES vectors

Given: $\mathbf{A} \in \mathbb{C}^{n \times n}$, $k$, a worst-case starting vector $b$

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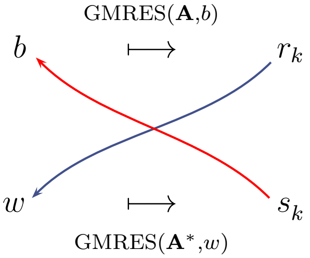


$$w \equiv \frac{r_k}{\|r_k\|}$$

It holds that

$$\|s_k\| \;=\; \|r_k\| \;=\; \psi_k(\mathbf{A}), \qquad b \;=\; \frac{s_k}{\|s_k\|}\,.$$

[Zavorin '02; Faber, T., Liesen '12]

# A new characterization of worst-case GMRES

Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be given. For $v \in \mathbb{R}^n$ and $c \in \mathbb{R}^k$ define

$$F(c, v) \;\equiv\; \frac{\|\, v - K(v)c \,\|^2}{\|\, v \,\|^2}\,,$$

where $K(v) \equiv [\mathbf{A}v, \mathbf{A}^2 v, \ldots, \mathbf{A}^k v]$. We want to characterize the solution of the problem

$$\max_{v\in\mathbb{R}^n\setminus 0}\ \min_{c\in\mathbb{R}^k} F(c, v)\,. \tag{1}$$

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**Theorem** [Faber, T., Liesen '12]

$\tilde{c} \in \mathbb{R}^k$ and $\tilde{v} \in \mathbb{R}^n$ that solve the problem (1) satisfy

$$\frac{\partial F}{\partial c}(\tilde{c}, \tilde{v}) = 0, \qquad \frac{\partial F}{\partial v}(\tilde{c}, \tilde{v}) = 0,$$

i.e., $(\tilde{c}, \tilde{v})$ is a stationary point of the function $F(c, v)$.

# Consequences of the new characterization

Let $b$, $\|b\| = 1$, be a worst-case starting vector and

$$r_k = p_k(\mathbf{A})b, \qquad \|r_k\| = \psi_k(\mathbf{A})$$

the corresponding GMRES residual vector.

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**Theorem** [Faber, T., Liesen '12]

- $b$ is a right singular vector of $p_k(\mathbf{A})$, i.e.

$$\psi_k(\mathbf{A})^2\, b \;=\; p_k(\mathbf{A}^T)p_k(\mathbf{A})\, b\,.$$

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- $p_k$ is also a worst-case GMRES polynomial for $\mathbf{A}^T$.

# Worst-case GMRES polynomials need not be unique

**Theorem** [Faber, T., Liesen '12]

A worst-case GMRES polynomial for the Toh matrix

$$\mathbf{A} = \begin{bmatrix} 1 & \epsilon & & \\ & -1 & \epsilon^{-1} & \\ & & 1 & \epsilon \\ & & & -1 \end{bmatrix}, \qquad \epsilon > 0,$$

and the step $k = 3$ is **not unique**.

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and the step $k = 3$ is **not unique**.

**Example**: If $\varepsilon = 0.1$, then both

$$-39.9^{-1}(z - 1.181)(z + 0.939)(z + 35.96)$$

and

$$39.9^{-1}(z + 1.181)(z - 0.939)(z - 35.96)$$

are worst-case GMRES polynomials of $\mathbf{A}$.

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- If $\Omega_k(p_*)$ is **convex** then $\psi_k(\mathbf{A}) = \varphi_k(\mathbf{A})$ .

[Faber, Joubert, Knill, Manteuffel '96]

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  [Faber, Joubert, Knill, Manteuffel '96]
- $\psi_k(\mathbf{A}) = \varphi_k(\mathbf{A})$ **iff**

$$\max_{v \in \mathbb{R}^n \setminus 0} \min_{c \in \mathbb{R}^k} F(c, v) = \min_{c \in \mathbb{R}^k} \max_{v \in \mathbb{R}^n \setminus 0} F(c, v) .$$

Existence of a **saddle point** of $F(c, v)$? [Faber, T., Liesen '12]

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6. $\psi_k(\mathbf{A}) = \varphi_k(\mathbf{A})$ **iff** $F(c, v)$ has a **saddle point**.

# Related papers


- V. Faber, P. Tichý and J. Liesen, [Ideal and Worst-case GMRES: Characterization and examples, in preparation, (2012?).]
- J. Liesen and P. Tichý, [On best approximations of polynomials in matrices in the matrix 2-norm, SIMAX, 31 (2009), pp. 853–863.]
- K. C. Toh, [ GMRES vs. ideal GMRES, SIMAX, 18 (1997), pp. 30–36.]
- V. Faber, W. Joubert, E. Knill, and T. Manteuffel, [Minimal residual method stronger than polynomial preconditioning, SIMAX, 17 (1996), pp. 707–729.]
- A. Greenbaum and L. N. Trefethen, [GMRES/CR and Arnoldi/Lanczos as matrix approximation problems, SISC, 15 (1994), pp. 359–368.]

# Related papers


- V. Faber, P. Tichý and J. Liesen, [Ideal and Worst-case GMRES: Characterization and examples, in preparation, (2012?).]
- J. Liesen and P. Tichý, [On best approximations of polynomials in matrices in the matrix 2-norm, SIMAX, 31 (2009), pp. 853–863.]
- K. C. Toh, [ GMRES vs. ideal GMRES, SIMAX, 18 (1997), pp. 30–36.]
- V. Faber, W. Joubert, E. Knill, and T. Manteuffel, [Minimal residual method stronger than polynomial preconditioning, SIMAX, 17 (1996), pp. 707–729.]
- A. Greenbaum and L. N. Trefethen, [GMRES/CR and Arnoldi/Lanczos as matrix approximation problems, SISC, 15 (1994), pp. 359–368.]


**Thank you for your attention!**