The importance of Structure in Algebraic Preconditioners (Level-based Algebraic Preconditioning)

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Outline

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Preconditioned iterative methods

Solving large, sparse SPD systems by iterative methods

$$Ax = b$$

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Algebraic preconditioning as a transformation

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In particular: Incomplete decompositions

- As usual, should be cheap, fast to compute, implying fast converging preconditioned iterative method
- sparse enough
- providing just sufficient approximation of the algebraic problem if this makes computations faster
- Our target is robustness

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1. Show the importance of structure of the matrix and its decomposition in algebraic preconditioners.
2. Present the effect separately from the other possible improvements (no compensations, no diagonal changes etc.).
3. Propose a new way to level-based strategies in incomplete decompositions.
4. The techniques are a basis of the HSL code MI22 which is being developed.

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- Dropping entries with "smaller magnitudes" (absolutely/relatively) (Zlatev et al. (1978), Munksgaard (1980), Axelsson (1972, 1983 et al. etc.)
- But: if only magnitudes of entries are used - structural information may be lost

Brief notes on development of algebraic preconditioners: II.

- Plassman, Jones (1995): no structure, just the memory predictability, see also Freund, Nachtigal, (1990). Similarly Lin, Moré with extended memory. ILUT by Saad, (1994).

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- Allowing fill up to a maximum length ℓ of any fill path (Watts III, (1981)).
- Practically: A fill entry is permitted provided $level(i,j) \le \ell$.

$$level(i,j) = \min_{1 \le l \le \min\{i,j\}} \{level(i,l) + level(l,j) + 1\}$$

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- Structure of levels helps but it has its strong drawbacks as well.

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- Our MI22 preconditioner is a new way to use level-based information, memory prediction and dropping at the same time.

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First component of our approach: new setting of levels

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- Set $level(i,j)$ for individual entries: For $\ell < ngroup$: $level(i,j) = (\ell - 1) * (l/ngroup) + 1$ where l $(1 \leq l \leq ngroup0)$ is the index of the group a_{ij} belongs to, and slightly differently otherwise.

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- During the $IC(\ell)$ decomposition, entries of the factor L that correspond to nonzero entries of A are assigned the level $level(i,j)$.
- Each potential fill entry l_{ij} is assigned a level

$$level(i,j) = \min_{1 \le l \le \min\{i,j\}} \{level(i,l) + level(l,j) + 1\}.$$

A fill entry is permitted provided $level(i,j) \le k$.

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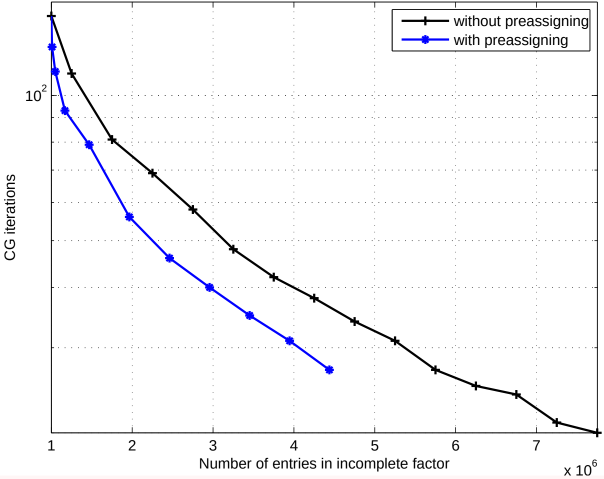

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- (-)The improvements are often small. We intend to construct a robust strategy which is used as a default value.
- Open problem: determine more sophisticated rules to preassign levels.

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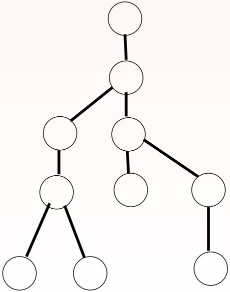

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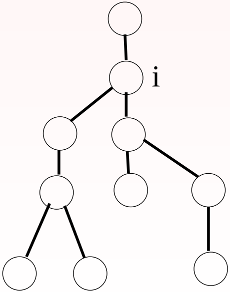

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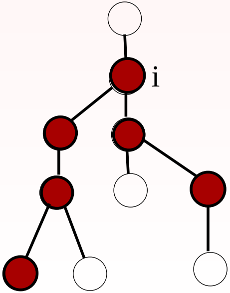

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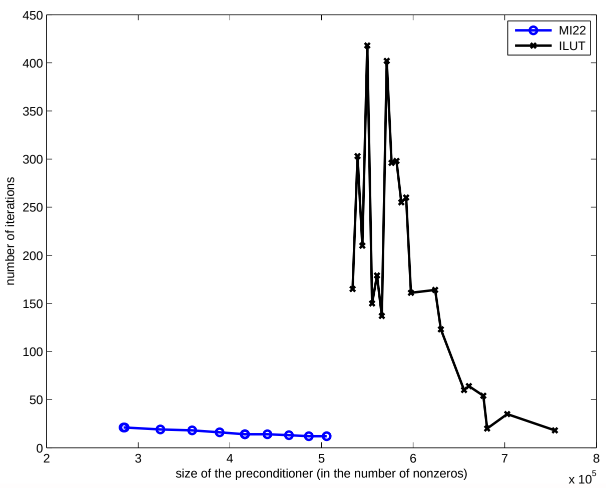
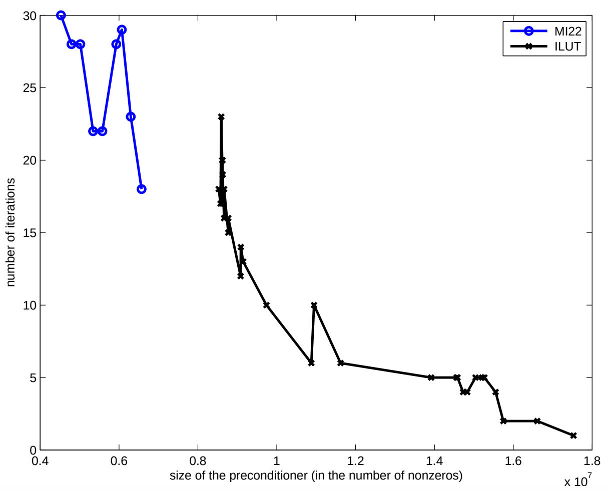
Level-based × value-based: example 1



S1RMT3M1, cylindrical shell problem, n=5489

Second component of our approach: keeping structure

Level-based × value-based: example 2

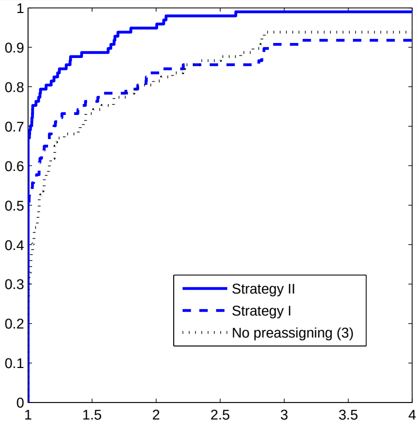


NASASRB, structural mechanics, n=54870

Second component of our approach: keeping structure

97 problems; efficiency profiles (Dolan, Moré, 2001) for 3 levels

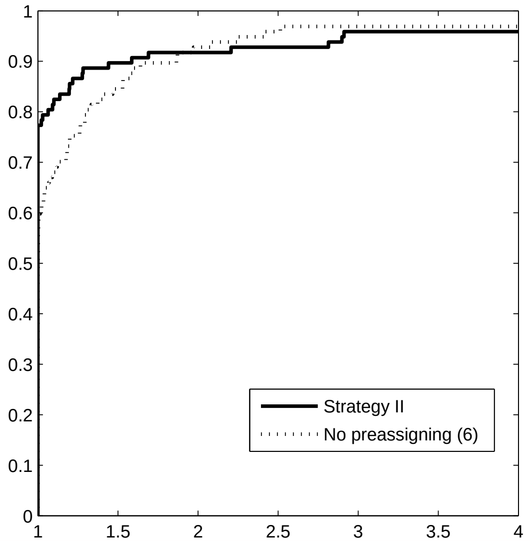
efficiency=size $\times$ iterations; fractions $p(\alpha)$ for which a solver is within a factor of α of the best solver.



Strategy I.: stress on sparsity; Strategy II.: denser and faster option

Second component of our approach: keeping structure

Efficiency profiles for 6 levels.



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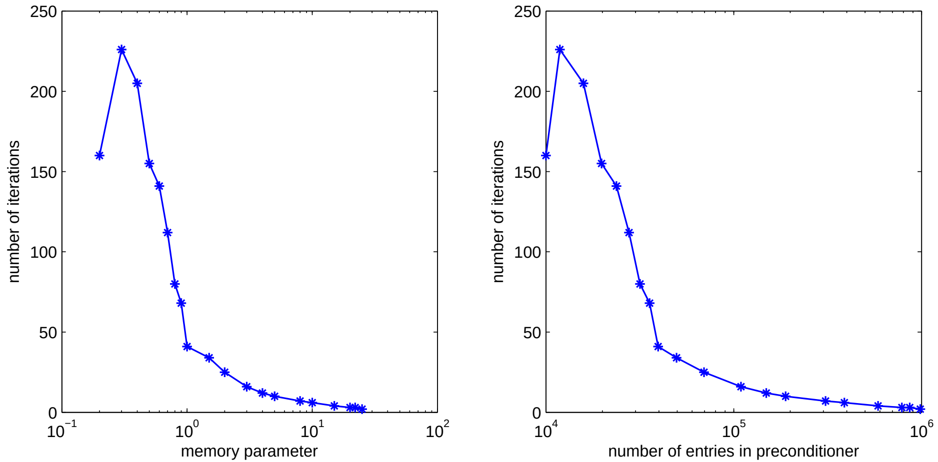
MI22: scaling the preconditioner for simple (2D Poisson) problem



MI22 with levels versus $IC(\tau)$(also via MI22)

TUBE1, cylindrical shell, n=21498

struc	drop=0.0		drop=10^{-7}	
level	size	its	size	its
5	1250952	†	1227570	†
6	1660827	429	1618808	423
7	1807337	405	1756733	408
8	2178312	272	2104496	281
9	2368289	260	2280081	267
10	3026431	184	2873613	185
11	3968731	426	3656826	335
12	4874629	†	4398086	†
13	5849563	†	5178688	†
14	6840871	664	5938543	647
15	7838623	262	6680235	215

IC(τ)	size	its
55	280626	†
50	1458024	†
45	2076970	†
40	2252687	†
1e-3	16139618	†
1e-4	9001342	†
5e-5	9649083	471
2e-5	9610841	87
1e-5	10050227	18
5e-6	10741254	6
1e-6	12451396	2
0	21802746	1

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But: Reorderings may minimize the effect.

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- Statistic $s_{ij} \geq 0$ (like CPU time) for the test set $\aleph$.
- Define for $j \in \aleph$:

$$k(s_{ij}, min_j s_{ij}, \alpha) = \begin{matrix} 1 & \text{if} \;\; s_{ij} \leq \alpha \; min_j s_{ij} \\ 0 & otherwise \end{matrix}$$

- displaystyle $p_i(\alpha) = \frac{\sum_j k(s_{ij}, min_j s_{ij}, \alpha)}{|\aleph|}$ for $\alpha \geq 1$.

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