

Numerical behavior of saddle point solvers

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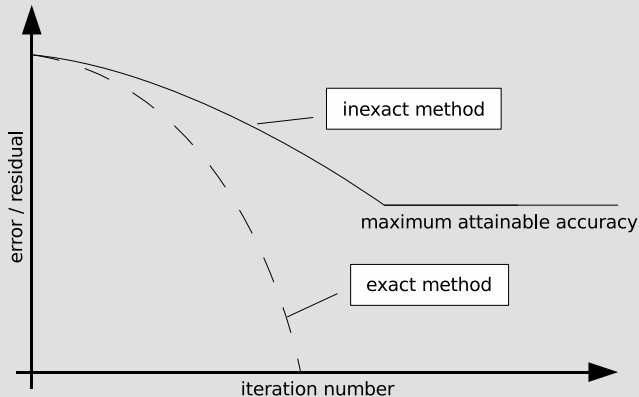
We consider a saddle point problem with the symmetric 2×2 block form

$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}.$$

- A is a square $n \times n$ nonsingular (symmetric positive definite) matrix,
- B is a rectangular $n \times m$ matrix of (full column) rank m .

Applications: mixed finite element approximations, weighted least squares, constrained optimization etc. [Benzi, Golub, and Liesen, 2005].

inexact solutions of inner systems + rounding errors
→ inexact saddle point solver



- Compute y as a solution of the Schur complement system

$$B^T A^{-1} B y = B^T A^{-1} f,$$

- compute x as a solution of

$$A x = f - B y.$$

Systems with A are solved inexactly, the computed solution $\bar{u}$ of $Au = b$ is interpreted an exact solution of a perturbed system

$$(A + \Delta A) \bar{u} = b + \Delta b, \quad \|\Delta A\| \leq \tau \|A\|, \quad \|\Delta b\| \leq \tau \|b\|, \quad \tau \kappa(A) \ll 1.$$

Iterative solution of the Schur complement system

choose y_0 , solve $Ax_0 = f - By_0$

compute α_k and $p_k^{(y)}$

$$y_{k+1} = y_k + \alpha_k p_k^{(y)}$$

$$\text{solve } Ap_k^{(x)} = -Bp_k^{(y)}$$

back-substitution:

$$\mathbf{A}: x_{k+1} = x_k + \alpha_k p_k^{(x)},$$

$$\mathbf{B}: \text{solve } Ax_{k+1} = f - By_{k+1},$$

$$\mathbf{C}: \text{solve } Au_k = f - Ax_k - By_{k+1},$$

$$x_{k+1} = x_k + u_k.$$

$$r_{k+1}^{(y)} = r_k^{(y)} - \alpha_k B^T p_k^{(x)}$$

inner
iteration

outer
iteration

The limiting (maximum attainable) accuracy is measured by the ultimate (asymptotic) values of:

- ❶ **the Schur complement residual:** $B^T A^{-1} f - B^T A^{-1} B y_k$;
- ❷ **the residuals in the saddle point system:** $f - A x_k - B y_k$ and $-B^T x_k$;
- ❸ **the forward errors:** $x - x_k$ and $y - y_k$.

Numerical example:

$$A = \text{tridiag}(1, 4, 1) \in \mathbb{R}^{100 \times 100}, \quad B = \text{rand}(100, 20), \quad f = \text{rand}(100, 1),$$

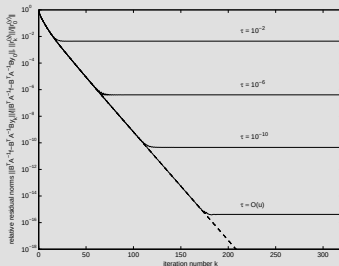
$$\kappa(A) = \|A\| \cdot \|A^{-1}\| = 7.1695 \cdot 0.4603 \approx 3.3001,$$

$$\kappa(B) = \|B\| \cdot \|B^\dagger\| = 5.9990 \cdot 0.4998 \approx 2.9983.$$

$$B^T(A + \Delta A)^{-1}B\hat{y} = B^T(A + \Delta A)^{-1}f,$$

$$\|B^T A^{-1}f - B^T A^{-1}B\hat{y}\| \leq \frac{\tau\kappa(A)}{1 - \tau\kappa(A)} \|A^{-1}\| \|B\|^2 \|\hat{y}\|.$$

$$\| -B^T A^{-1}f + B^T A^{-1}B y_k - r_k^{(y)} \| \leq \frac{O(\tau)\kappa(A)}{1 - \tau\kappa(A)} \|A^{-1}\| \|B\| (\|f\| + \|B\| Y_k).$$



$$-B^T A^{-1} f + B^T A^{-1} B y_k = -B^T x_k - B^T A^{-1} (f - A x_k - B y_k)$$

$$\|f - A x_k - B y_k\| \leq \frac{O(\alpha_1) \kappa(A)}{1 - \tau \kappa(A)} (\|f\| + \|B\| Y_k),$$

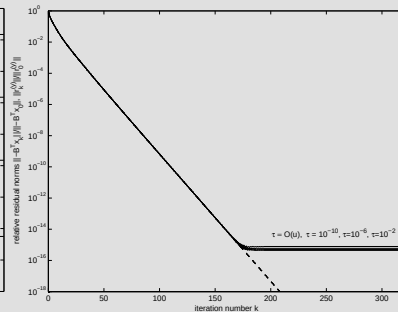
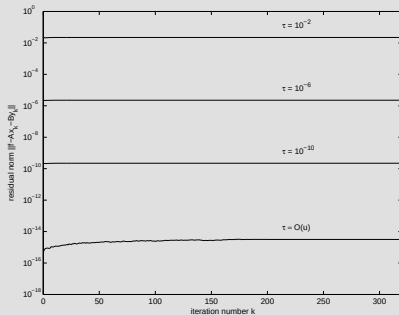
$$\| -B^T x_k - r_k^{(y)} \| \leq \frac{O(\alpha_2) \kappa(A)}{1 - \tau \kappa(A)} \|A^{-1}\| \|B\| (\|f\| + \|B\| Y_k),$$

$$Y_k \equiv \max\{\|y_i\| \mid i = 0, 1, \dots, k\}.$$

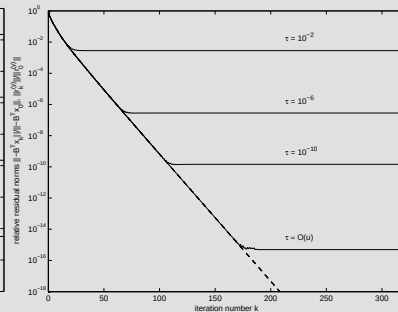
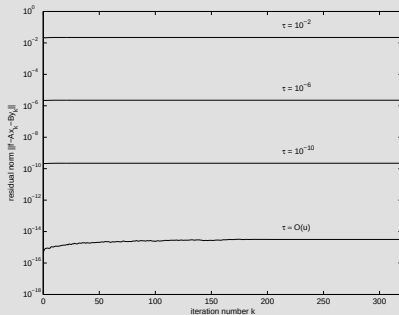
Back-substitution scheme		α_1	α_2
A:	Generic update $x_{k+1} = x_k + \alpha_k p_k^{(x)}$	τ	u
B:	Direct substitution $x_{k+1} = A^{-1}(f - B y_{k+1})$	τ	τ
C:	Corrected dir. subst. $x_{k+1} = x_k + A^{-1}(f - A x_k - B y_{k+1})$	u	τ

} additional system with A

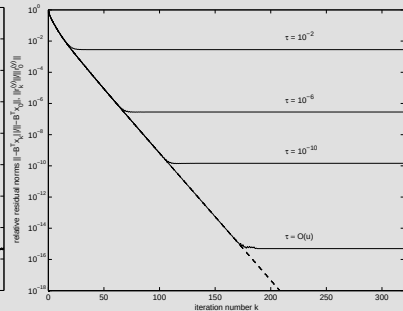
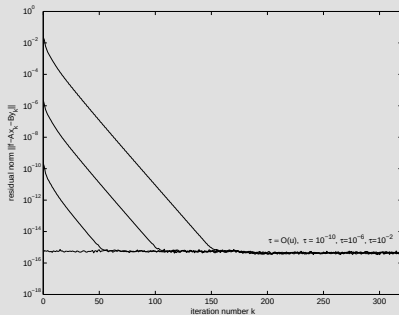
Generic update: $x_{k+1} = x_k + \alpha_k p_k^{(x)}$



Direct substitution: $x_{k+1} = A^{-1}(f - By_{k+1})$



Corrected direct substitution: $x_{k+1} = x_k + A^{-1}(f - Ax_k - By_{k+1})$

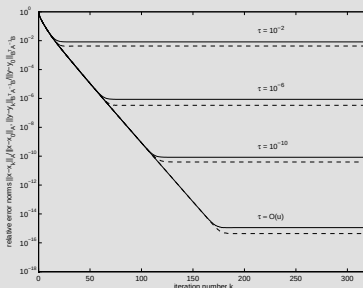


Forward error of computed approximate solution

$$\|x - x_k\| \leq \gamma_1 \|f - Ax_k - By_k\| + \gamma_2 \| -B^T x_k \|,$$

$$\|y - y_k\| \leq \gamma_2 \|f - Ax_k - By_k\| + \gamma_3 \| -B^T x_k \|,$$

$$\gamma_1 = \sigma_{\min}^{-1}(A), \quad \gamma_2 = \sigma_{\min}^{-1}(B), \quad \gamma_3 = \sigma_{\min}^{-1}(B^T A^{-1} B).$$



- compute $x \in N(B^T)$ as a solution of the projected system

$$(I - \Pi)A(I - \Pi)x = (I - \Pi)f,$$

- compute y as a solution of the least squares problem

$$By \approx f - Ax,$$

Π is the orthogonal projector onto $R(B)$.

The least squares with B are solved inexactly, i.e. the computed solution $\bar{v}$ of $Bv \approx c$ is an exact solution of a perturbed least squares problem

$$(B + \Delta B)\bar{v} \approx c + \Delta c, \quad \|\Delta B\| \leq \tau \|B\|, \quad \|\Delta c\| \leq \tau \|c\|, \quad \tau \kappa(B) \ll 1.$$

Iterative solution of the null-space projected system

choose x_0 , solve $By_0 \approx f - Ax_0$

compute α_k and $p_k^{(x)} \in N(B^T)$

$$x_{k+1} = x_k + \alpha_k p_k^{(x)}$$

$$\left| \begin{array}{l} \text{solve } Bp_k^{(y)} \approx r_k^{(x)} - \alpha_k Ap_k^{(x)} \\ \text{back-substitution:} \\ \mathbf{A: } y_{k+1} = y_k + p_k^{(y)}, \\ \mathbf{B: } \text{solve } By_{k+1} \approx f - Ax_{k+1}, \\ \mathbf{C: } \text{solve } Bv_k \approx f - Ax_{k+1} - By_k, \\ y_{k+1} = y_k + v_k. \end{array} \right.$$

$$r_{k+1}^{(x)} = r_k^{(x)} - \alpha_k Ap_k^{(x)} - Bp_k^{(y)}$$

} inner
iteration

} outer
iteration

$$\|f - Ax_k - By_k - r_k^{(x)}\| \leq \frac{O(\alpha_3)\kappa(B)}{1 - \tau\kappa(B)} (\|f\| + \|A\|X_k),$$

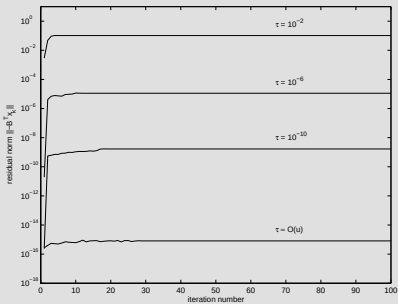
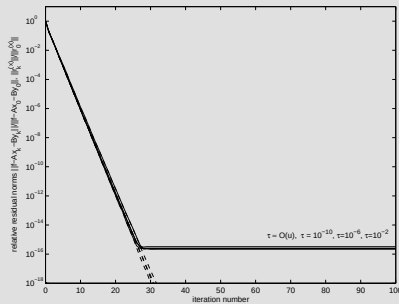
$$\| -B^T x_k \| \leq \frac{O(\tau)\kappa(B)}{1 - \tau\kappa(B)} \|B\|X_k,$$

$$X_k \equiv \max\{\|x_i\| \mid i = 0, 1, \dots, k\}.$$

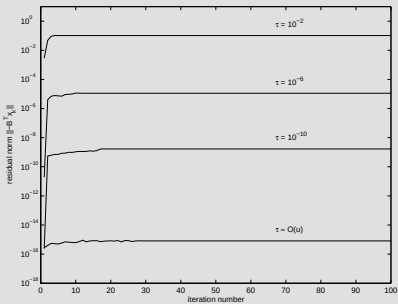
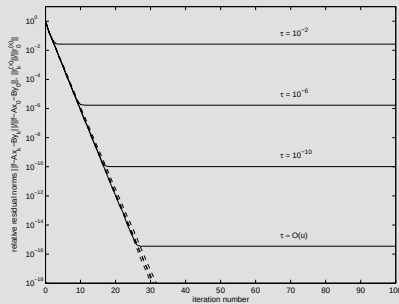
Back-substitution scheme	α_3
A: Generic update $y_{k+1} = y_k + p_k^{(y)}$	u
B: Direct substitution $y_{k+1} = B^\dagger(f - Ax_{k+1})$	τ
C: Corrected dir. subst. $y_{k+1} = y_k + B^\dagger(f - Ax_{k+1} - By_k)$	u

} additional least square with B

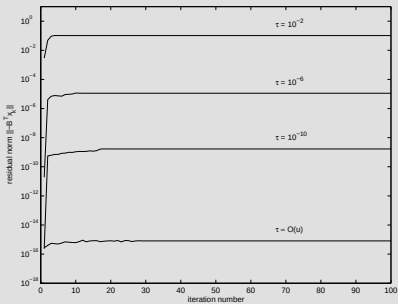
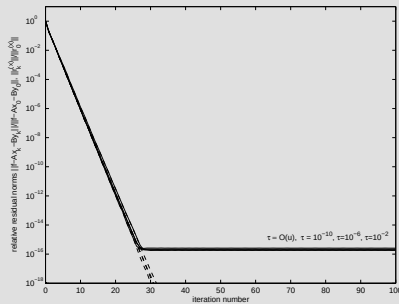
Generic update: $y_{k+1} = y_k + p_k^{(y)}$



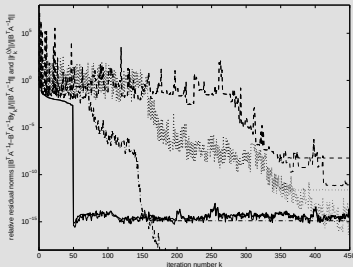
Direct substitution: $y_{k+1} = B^\dagger(f - Ax_{k+1})$



Corrected direct substitution: $y_{k+1} = y_k + B^\dagger(f - Ax_{k+1} - By_k)$



- All bounds of the limiting accuracy depend on the maximum norm of computed iterates, cf. [Greenbaum, 1997].
- The accuracy measured by the residuals of the saddle point problem depends on the choice of the back-substitution scheme [J, R, 2008].
- Care must be taken when solving nonsymmetric systems [J, R, 2008b].



- The residuals in the outer iteration process and the forward errors of computed approximations are proportional to the backward error in solution of inner systems.

Thank you for your attention.

`http://www.cs.cas.cz/~miro`

P. Jiránek and M. Rozložník. Maximum attainable accuracy of inexact saddle point solvers. *SIAM J. Matrix Anal. Appl.*, 29(4):1297–1321, 2008.

P. Jiránek and M. Rozložník. Limiting accuracy of segregated solution methods for nonsymmetric saddle point problems. *J. Comput. Appl. Math.*, to appear.

- M. Benzi, G. H. Golub, and J. Liesen. Numerical solution of saddle point problems. *Acta Numer.*, 14:1–137, 2005.
- A. Greenbaum. Estimating the attainable accuracy of recursively computed residual methods. *SIAM J. Matrix Anal. Appl.*, 18(3):535–551, 1997.
- P. Jiránek and M. Rozložník. Maximum attainable accuracy of inexact saddle point solvers. *SIAM J. Matrix Anal. Appl.*, 29(4):1297–1321, 2008a.
- P. Jiránek and M. Rozložník. Limiting accuracy of segregated solution methods for nonsymmetric saddle point problems. *J. Comput. Appl. Math.*, 215: 28–37, 2008b.