On prescribing restarted GMRES

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1 Introduction

This extended abstract addresses the question whether the sequence of residual norms generated by the GMRES method [6] when it is restarted, can consist of arbitrary non-increasing positive values and whether this is possible with a matrix having prescribed eigenvalues. For *full* GMRES, Greenbaum and Strakoš [3] proved that if a residual norm convergence curve is generated by GMRES, it can also be obtained with a matrix having prescribed spectrum. Greenbaum, Pták and Strakoš [4] complemented this with showing that *any* nonincreasing sequence of residual norms can be generated by GMRES. Finally, in Arioli, Pták and Strakoš [1] a complete parametrization was given of all pairs $\{A, b\}$ such that A has prescribed spectrum and such that a prescribed residual norm convergence curve is generated.

The case where GMRES is restarted was considered by Vecharynski and Langou [7]. They investigated the special situation when during every restart cycle, all residual norms stagnate except for the very last iteration inside the cycle. In this iteration it is assumed that the residual norm is strictly decreasing. Let us denote the very last residual norm at the end of the kth cycle with $\|r_k\|$ and let us consider GMRES restarted every m iterations, which we denote with GMRES(m). In [7] it was shown that there exists a matrix and a right hand side such that the residual norms

$$\|r_0\|, \|r_1\|, \ldots, \|r_k\|$$

generated by GMRES(m) applied to the corresponding linear system can take any values provided that each residual norm is strictly smaller than the previous residual norm. The authors also showed this is possible with an arbitrary spectrum for A. They restricted themselves to the case where the product km is less than the dimension of the linear system.

2 Topics of the talk

The talk will address the question whether the above mentioned results for GMRES(m) can be extended to a more general case. The stagnation during every restart cycle assumed in [7] is very unlikely and in practice the residual norms *inside* cycles will decrease and from time to time they might stagnate. As a first issue, we will investigate the influence of stagnating residual norms on neighboring residual norms (both inside the cycle and inside neighboring cycles). This will enable us to know, if we prescribe some stagnating iterations, the values that neighboring residual norms can take. As a second issue, we give a parametrization of the matrices and right hand sides that generate GMRES(m) convergence curves with stagnation at prescribed iterations and where the matrix has a prescribed spectrum. This parametrization will also reveal some information on the convergence curve that would have been generated if we had *not* restarted GMRES.

Our parametrization is based on a parametrization of the matrices and right hand sides that generate a prescribed non-increasing convergence curve when full GMRES is applied. It differs from the parametrization in [1]. GMRES residual norms are invariant under unitary transformation of the system matrix and right hand side. More precisely, the residual norms generated by $Ax = b$ are the same as the residual norms generated by $(UAU^*)x = Ub$ for any unitary U. In the parametrization of [1], this unitary invariance is incorporated by the unitary matrix whose columns generate the Krylov subspace $A\mathcal{K}_n(A, b)$, where n is the dimension of the linear system. The results in [2, Section 3] lead easily to a different parametrization where the unitary invariance is incorporated by the unitary matrix whose columns generate the Krylov subspace $\mathcal{K}_n(A, b)$. This parametrization is more suited for our purpose, because we will focus on the Hessenberg matrices generated in every restart cycle with the Arnoldi process, which builds orthogonal bases for Krylov subspaces of the form $\mathcal{K}_m(A, r_k)$. The parametrization is a follows.

Theorem 2.1 *Assume we are given n positive numbers* $(\lambda_1, \ldots, \lambda_n)$ *and n positive real numbers*

$$f(0) \geq f(1) \geq \cdots \geq f(n-1) > 0.$$

If A is a matrix of order n and b a nonzero n-dimensional vector, then the following assertions are equivalent:

1. *The GMRES method applied to A and right-hand side b with zero initial guess yields residuals* $r^{(k)}$, $k = 0, \ldots, n-1$ *such that*

$$\|r^{(k)}\| = f(k), \quad k = 0, \ldots, n-1,$$

and A has eigenvalues $\lambda_1, \ldots, \lambda_n$.

2. *The matrix A is of the form*

$$A = V \begin{bmatrix} & g^T & \\ 0 & & R \end{bmatrix}^{-1} C^{(n)} \begin{bmatrix} & g^T & \\ 0 & & R \end{bmatrix} V^*$$

and $b = f(0)Ve_1$, *where V is a unitary matrix,* $C^{(n)}$ *is the companion matrix of the polynomial with roots* $\lambda_1, \ldots, \lambda_n$, *R is nonsingular upper triangular of size* $n-1$ *and*

$$g_1 = \frac{1}{f(0)}, \qquad g_k = \frac{\sqrt{f(k-2)^2 - f(k-1)^2}}{f(k-2)f(k-1)}, \qquad k = 2, \ldots, n.$$

P r o o f (outline): From [2, Corollary 3.7] we immediately get a parametrization of the matrices and right hand sides that generate a prescribed non-increasing convergence curve when full GMRES is applied. The vector $R_h^{-T}\hat{h}$ in that parametrization depends on the prescribed convergence curve only and this dependency can be written out more explicitly as

$$\hat{h}^T R_h^{-1} e_k = f(0)^2 \frac{\sqrt{f(k-1)^2 - f(k)^2}}{f(k-1)f(k)}, \qquad k = 1, \ldots, n-1.$$

This follows from exploiting the formula

$$\hat{h}^T R_h^{-1} e_k = \frac{f(0)^2 \eta_k}{\sqrt{\eta_{k+1}^2 + \ldots \eta_n^2}\sqrt{\eta_k^2 + \ldots \eta_n^2}},$$

where $\eta_k = \sqrt{f(k-1)^2 - f(k)^2}$ in [5]. □

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References

[1] M. Arioli, V. Pták and Z. Strakoš, *Krylov Sequences of maximal length and convergence of GMRES*, B.I.T., 38(1998), pp. 636–643.

[2] J. Duintjer Tebbens and G. Meurant, *Any Ritz value behavior is possible for Arnoldi and for GMRES with any convergence curve*, submitted(2011).

[3] A. Greenbaum and Z. Strakoš, *Matrices that generate the same Krylov Residual Spaces*, in Recent Advances in Iterative Methods, G.H. Golub, A. Greenbaum and M. Luskin, eds, 50(1994), pp. 95–118.

[4] A. Greenbaum, V. Pták and Z. Strakoš, *Any Nonincreasing Convergence Curve is Possible for GMRES*, SIAM J. Matrix Anal. Appl., 17(1996), pp. 465–469.

[5] G. Meurant, *Notes on GMRES convergence (15): The matrix H and the Ritz values*, personal communication(2011).

[6] Y. Saad and M. H. Schultz, *GMRES: A generalized minimal residual algorithm for solving nonsymmetric linear systems*, SIAM J. Sci. Stat. Comput., 7(1986), pp. 856–869.

[7] E. Vecharynski and J. Langou, *Any admissible cycle-convergence behavior is possible for restarted GMRES at its initial cycles*, Numer. Linear Algebra Applications, 18(2011), pp. 499–511.